Impartial Selection with Additive Guarantees via Iterated Deletion

Javier Cembrano* Felix Fischer† David Hannon‡ Max Klimm§

Abstract

Impartial selection is the selection of an individual from a group based on nominations by other members of the group, in such a way that individuals cannot influence their own chance of selection. We give a deterministic mechanism with an additive performance guarantee of $O(n^{(1+\kappa)/2})$ in a setting with n individuals where each individual casts $O(n^{\kappa})$ nominations, where $\kappa \in [0, 1]$. For $\kappa = 0$, i.e., when each individual casts at most a constant number of nominations, this bound is $O(\sqrt{n})$. This matches the best-known guarantee for randomized mechanisms and a single nomination. For $\kappa = 1$ the bound is $O(n)$. This is trivial, as even a mechanism that never selects provides an additive guarantee of $n - 1$. We show, however, that it is also best possible: for every deterministic impartial mechanism there exists a situation in which some individual is nominated by every other individual and the mechanism either does not select or selects an individual not nominated by anyone.

1 Introduction

The selection of the 2021 Best FIFA Men's Player in early 2022 caused some controversy when it was revealed that Lionel Messi, who came in second with 44 votes, had not voted for Robert Lewandowski, who won the trophy with 48 votes. Lewandowski, on the other hand, had given votes to Jorginho, Messi, and Ronaldo, thus helping his closest contender and risking his own victory. Both Messi and Lewandowski were allowed to cast votes as the captains of their respective national teams. While we can only speculate about the reasons for the two players' voting behavior, it is evident that situations where there is at least a partial overlap between the set of voters and the set of candidates come with intrinsic incentive issues. Candidates who have a reasonable chance of winning may be motivated not to reveal their true opinion on who should win in order to increase their own chance of winning. This phenomenon is not limited to the selection of football players for an award but affects a wide range of areas from scientific peer review to the election of the pope.

Incentive issues that arise when members of a group are selected by other members were first studied in a systematic way by Holzman and Moulin [2013] and Alon et al. [2011], who formalized the problem in terms of a directed graph in which vertices correspond to voters and a directed edge from one voter to another indicates that the former nominates the latter. A (deterministic) selection mechanism then takes such a nomination graph as input and returns one of its vertices. In order to allow voters to express their true opinions about other voters without having to worry about their own chance of selection, an important property of a selection mechanism is its impartiality: a mechanism is impartial if, for all nomination graphs, a change of the outgoing edges of some vertex v does not change whether v is selected or not. It is easy to see that a mechanism that selects a vertex with maximum indegree and breaks ties in some

*Institut für Mathematik, Technische Universität Berlin, Germany
†School of Mathematical Sciences, Queen Mary University of London, UK
‡School of Mathematical Sciences, Queen Mary University of London, UK
§Institut für Mathematik, Technische Universität Berlin, Germany

consistent way is not impartial: if ties are broken in favor of greater index, for example, a vertex with maximum indegree that currently nominates another vertex with maximum indegree but greater index has an incentive to instead nominate a different vertex.

Holzman and Moulin have shown that impartial mechanisms are in fact much more limited even in a setting where each voter casts exactly one vote, i.e., where each vertex has outdegree one: for every impartial mechanism there is a nomination graph where it selects a vertex with indegree zero, or a nomination graph where it fails to select a vertex with indegree $n-1$, where n is the number of voters. This shows in particular that the best multiplicative approximation guarantee for impartial mechanisms, i.e., the worst case over all nomination graphs of the ratio between the maximum indegree and the indegree of the selected vertex is at least $n-1$. On the other hand, a multiplicative guarantee of $n-1$ is easy to obtain by always following the outgoing edge of a fixed vertex. As multiplicative guarantees do not allow for a meaningful distinction among deterministic impartial mechanisms, Caragiannis et al. [2019, 2021] proposed to instead consider an additive guarantee, i.e., the worst-case over all nomination graphs of the difference between the maximum indegree and the indegree of the selected vertex. A mechanism due to Holzman and Moulin, majority with default, achieves an additive guarantee of $\lceil n/2 \rceil$. Caragiannis et al. [2019, 2021] further propose a randomized mechanism with an additive guarantee of $O(\sqrt{n})$. It remains open, however, whether there exists a *deterministic* mechanism with a *sublinear* additive guarantee. A sublinear additive guarantee is significant because it implies asymptotic multiplicative optimality as the maximum indegree goes to infinity.

The setting studied by Holzman and Moulin, where each vertex has outdegree one, is commonly referred to as the plurality setting. The impossibility results of Holzman and Moulin regarding multiplicative guarantees carry over to the more general approval setting, where outdegrees can be arbitrary, but here even less is known about possible additive guarantees. While Caragiannis et al. [2019] have shown that deterministic impartial mechanisms cannot provide a better additive guarantee than 3, no mechanism is known that improves on the trivial guarantee of $n-1$ achieved by selecting a fixed vertex. Caragiannis et al. also gave a randomized mechanism for the approval setting with an additive guarantee of $\Theta(n^{2/3} \ln^{1/3} n)$.

1.1 Our Contribution

We develop a new deterministic mechanism for impartial selection that is parameterized by a pair of thresholds on the indegrees of vertices in the graph. The mechanism seeks to select a vertex with large indegree, and to achieve impartiality it iteratively deletes outgoing edges from vertices in decreasing order of their indegrees, until only the outgoing edges of vertices with indegrees below the lower threshold remain. It then selects a vertex with maximum remaining indegree if that indegree is above the higher threshold, and otherwise does not select. Any ties are broken according to a fixed ordering of the vertices. We give a sufficient condition for choices of thresholds that guarantee impartiality. The iterative nature of the deletions requires a fairly intricate analysis but is key to achieving impartiality. The additive guarantee is then obtained for a good choice of thresholds, and the worst case is the one where the mechanism does not select.

For instances with n vertices and maximum outdegree at most $O(n^\kappa)$, where $\kappa \in [0,1]$, the mechanism provides an additive guarantee of $O(n^{\frac{1+\kappa}{2}})$. This is the first sublinear bound for a deterministic mechanism and any $\kappa \in [0,1]$, and is sublinear for all $\kappa \in [0,1)$. For settings with constant maximum outdegree, which includes the setting of Holzman and Moulin where all outdegrees are equal to one, our bound matches the best known bound of $O(\sqrt{n})$ for randomized mechanisms and outdegree one, due to Caragiannis et al. [2019].

When the maximum outdegree is unbounded, the bound becomes $O(n)$. This is of course trivial, as even a mechanism that never selects a vertex provides an additive guarantee of $n-1$.

For a setting without abstentions, i.e., with minimum outdegree one, the guarantee can be improved slightly to $n-2$ by following the outgoing edge of a fixed vertex. We show that both of these bounds are best possible by giving matching lower bounds. This improves on the only lower bound known prior to our work, again due to Caragiannis et al., which is equal to 3 and applies to the setting with abstentions and mechanisms that select a vertex for every graph.

Just like the lower bounds regarding multiplicative guarantees for plurality, our lower bounds for approval are obtained through an axiomatic impossibility result. Holzman and Moulin have shown that in the case of plurality, impartiality is incompatible with positive and negative unanimity. Here, positive unanimity requires that a vertex with the maximum possible indegree of $n-1$ must be selected, and negative unanimity that a vertex with indegree zero cannot be selected. In the case of approval, this impossibility can be strengthened even further: call a selection mechanism weakly unanimous if it selects a vertex with positive indegree whenever there exists a vertex with the maximum possible indegree of $n-1$; then weak unanimity and impartiality are incompatible.

This result is obtained by analyzing the behavior of impartial mechanisms on a restricted class of graphs with a high degree of symmetry among vertices. Like Holzman and Moulin, we can assume that isomorphic vertices are selected with equal probabilities by a randomized relaxation of a mechanism. A suitable class of graphs for our purposes are those generated by partial orders on the set of ordered partitions. Our result is then obtained by combining counting results for ordered partitions and an argument similar to Farkas' Lemma.

1.2 Related Work

Impartiality as a formal property of social and economic mechanisms was first considered by De Clippel et al. [2008], for the distribution of a divisible commodity among a set of individuals according to the individuals' subjective claims. Holzman and Moulin [2013] and Alon et al. [2011] studied impartial selection in two different settings, plurality and approval, and established strong impossibility results regarding the ability of deterministic mechanisms to approximate the maximum indegree in a multiplicative sense. Alon et al. [2011] also proposed randomized mechanisms for the selection of one or more vertices. Fischer and Klimm [2015] then obtained a randomized mechanism with the best possible multiplicative guarantee of 2 for the selection of a single vertex in the approval setting, and Bjelde et al. [2017] gave improved deterministic and randomized mechanisms for the selection of more than one vertex.

Starting from the observation that impossibility results for randomized mechanisms in particular are obtained from graphs with very small indegrees, Bousquet et al. [2014] developed a randomized mechanism that is optimal in the large-indegree limit, i.e., that chooses a vertex with indegree arbitrarily close to the maximum indegree as the latter goes to infinity. Caragiannis et al. [2019, 2021] used the same observation as motivation to study mechanisms with additive rather than multiplicative guarantees. They developed new mechanisms that achieve such guarantees, established a relatively small but nontrivial lower bound of 3 for deterministic mechanisms in the approval setting, and gave improved deterministic mechanisms for a setting with prior information.

The axiomatic study of Holzman and Moulin has been refined and extended in a number of ways, for example with a focus on symmetric mechanisms [Mackenzie, 2015] and to the selection of more than one vertex [Tamura and Ohseto, 2014]. Mackenzie [2020] provided a detailed axiomatic analysis of mechanisms used in the papal conclave. Various selection mechanisms have also been proposed that are tailored to applications like peer review and exploit the particular preference and information structures of those applications [Kurokawa et al., 2015, Xu et al., 2018, Aziz et al., 2019, Mattei et al., 2020]. Impartial mechanisms have finally been considered for other objectives, specifically for the maximization of progeny [Babichenko et al., 2020, Zhang et al., 2021] and for rank aggregation [Kahng et al., 2018].

The proof of our impossibility result uses a class of graphs constructed from ordered partitions of the set of vertices. The class has been studied previously [e.g., Insko et al., 2017, Diagana and Maïga, 2017], and some of its known properties including its lattice structure and the number of graphs isomorphic to each graph within the class are relevant to us.

2 Preliminaries

For $n \in \mathbb{N}$, let $\mathcal{G}_n = \big\{(N,E) : N = \{1,2,\dots,n\}, E \subseteq (N \times N) \setminus \bigcup_{v\in N}\{(v,v)\}\big\}$ be the set of directed graphs with n vertices and no loops. Let $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$. For $G = (N,E) \in \mathcal{G}$ and $v \in N$, let $N^+(v,G) = \{u \in N : (v,u) \in E\}$ be the out-neighborhood and $N^-(v,G) = \{u \in N : (u,v) \in E\}$ the in-neighborhood of v in G. Let $\delta^+(v,G) = |N^+(v,G)|$ and $\delta^-(v,G) = |N^-(v,G)|$ denote the outdegree and indegree of v in G, $\delta^-_S(v,G) = |\{u \in S : (u,v) \in E\}|$ the indegree of v from a particular subset $S \subseteq N$ of the vertices, and $\Delta(G) = \max_{v\in N} \delta^-(v,G)$ the maximum indegree of any vertex in G. When the graph is clear from the context, we will sometimes drop G from the notation and write $N^+(v)$, $N^-(v)$, $\delta^+(v)$, $\delta^-(v)$, $\delta^-_S(v)$, and Δ. For $n,k \in \mathbb{N}$, let $\mathcal{G}^+_n = \{(N,E) \in \mathcal{G}_n : \delta^+(v) \geq 1 \text{ for every } v \in N\}$ be the set of graphs in $\mathcal{G}_n$ where all outdegrees are strictly positive, $\mathcal{G}_n(k) = \{(N,E) \in \mathcal{G}_n : \delta^+(v) \leq k \text{ for every } v \in N\}$ the set of graphs in $\mathcal{G}_n$ where outdegrees are at most k, and $\mathcal{G}^+_n(k) = \mathcal{G}^+_n \cap \mathcal{G}_n(k)$ for the set of graphs satisfying both conditions. Let $\mathcal{G}^+ = \bigcup_{n\in\mathbb{N}} \mathcal{G}^+_n$, $\mathcal{G}(k) = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n(k)$, and $\mathcal{G}^+(k) = \bigcup_{n\in\mathbb{N}} \mathcal{G}^+_n(k)$.

A (deterministic) *selection mechanism* is then given by a family of functions $f : \mathcal{G}_n \to 2^N$ that maps each graph to a subset of its vertices, where we require throughout that $|f(G)| \leq 1$ for all $G \in \mathcal{G}$. In a slight abuse of notation, we will use f to refer to both the mechanism and to individual functions from the family. Mechanism f is *impartial* on $\mathcal{G}' \subseteq \mathcal{G}$ if on this set of graphs the outgoing edges of a vertex have no influence on its selection, i.e., if for every pair of graphs $G = (N,E)$ and $G' = (N,E')$ in $\mathcal{G}'$ and every $v \in N$, $f(G) \cap \{v\} = f(G') \cap \{v\}$ whenever $E \setminus (\{v\} \times N) = E' \setminus (\{v\} \times N)$. Mechanism f is *α-additive* on $\mathcal{G}' \subseteq \mathcal{G}$, for $\alpha \geq 0$, if for every graph in $\mathcal{G}'$ the indegree of the choice of f differs from the maximum indegree by at most α, i.e., if

$$\sup_{G\in\mathcal{G}'} \Big\{ \Delta(G) - \delta^-(f(G),G) \Big\} \leq \alpha,$$

where, in a slight abuse of notation, $\delta^-(S,G) = \sum_{v\in S} \delta^-(v,G)$.

3 Iterated Deletion of Nominations

When outdegrees are at most one, the following simple mechanism is $\lfloor n/2 \rfloor$-additive: if there is a vertex with indegree at least $\lfloor n/2 \rfloor + 1$, select it; otherwise, do not select.[1] As there can be at most one vertex with degree $\lfloor n/2 \rfloor + 1$ or more and a vertex cannot influence its own indegree, the mechanism is clearly impartial. We will borrow from this mechanism the idea to impose a threshold on the minimum indegree a vertex needs to be selected, but will seek to lower the threshold in order to achieve a better additive guarantee and also to relax the constraint on the maximum outdegree. Of course, lower thresholds and larger outdegrees both mean that more and more vertices become eligible for selection, and we will no longer get impartiality for free.

As a first step, it is instructive to again consider the outdegree-one case but to use a lower threshold $t = \lfloor n/3 \rfloor + 1$. This choice of the threshold means that up to two vertices can have indegrees above the threshold and thus be eligible for selection. To achieve impartiality it makes sense to delete the outgoing edges of such vertices. Unfortunately, selecting a vertex with maximum indegree among those that remain above t after deletion, while breaking ties in

[1] This mechanism is a simpler version of a mechanism of Holzman and Moulin [2013], majority with default, which is required to always select and does so by singling out a default vertex whose outgoing edge is ignored and which is selected in the absence of a vertex with large indegree.

Figure 1: A mechanism that deletes the outgoing edges of vertices above t and selects a vertex with maximum remaining indegree above t is not impartial. This is illustrated by the two diagrams on the left for a mechanism that deletes edges iteratively, but is true also for simultaneous deletion. A mechanism that simultaneously deletes the outgoing edges of vertices above t and selects a vertex with maximum remaining indegree above $t+1$ is not impartial, as illustrated by the two diagrams on the right. All diagrams in this section place vertex x along the vertical axis according to $\delta^-(x)$ and along the horizontal axis according to $-x$. The selected vertex is the leftmost one among those that are highest, which corresponds to selecting a vertex with maximum indegree and breaking ties in favor of a greater index. A selected vertex is drawn in white. Edges incident to vertices not in the diagram are not shown.

favor of a greater index,[2] is not impartial: whether a vertex v ends up above or below t may depend on whether another vertex u does and thus whether the edge (u, v) is deleted or not; the latter may in turn depend on the existence of the edge (v, u). To make matters worse, which edges remain after deletion may also depend on whether edges are deleted simultaneously or in order, and on the particular order in which they are deleted. Both phenomena are illustrated in Figure 1. It turns out that for this particular mechanism impartiality can be restored if outgoing edges of the two vertices above the threshold are deleted iteratively, from large to small indegree and breaking ties in the same way as before, and a selection is only made if after deletion at least one vertex remains with indegree above a higher threshold of $T = t + 1$. In addition to being impartial, the resulting mechanism is $(\lfloor n/3 \rfloor + 2)$-additive.

It is natural to ask whether the threshold can be lowered further while maintaining impartiality, and whether guarantees can be obtained in a similar fashion for graphs with larger outdegrees. The answer to the first question is not obvious, as the number of graphs that need to be considered to establish impartiality grows very quickly in the number of vertices eligible for selection. The obvious generalization of the mechanism with threshold $\lfloor n/2 \rfloor + 1$ to a setting with outdegrees at most k, of selecting the unique vertex with indegree at least $\lfloor kn/2 \rfloor + 1$ if it exists and not selecting otherwise, is impartial and $\lfloor kn/2 \rfloor$-additive, but this guarantee is trivial when $k \geq 2$.

Our main result answers both questions in the affirmative. It applies to settings with outdegree at most $k = O(n^\kappa)$ for $\kappa \in [0, 1]$ and provides a nontrivial guarantee when $\kappa \in [0, 1)$. When k is constant the guarantee is $O(\sqrt{n})$, which matches the best guarantee known for randomized mechanisms and outdegree one [Caragiannis et al., 2019].

Theorem 1. *For every $n \in \mathbb{N}$, $\kappa \in [0, 1]$, and $k = O(n^\kappa)$, there exists an impartial and $O(n^{\frac{1+\kappa}{2}})$-additive mechanism on $\mathcal{G}_n(k)$. Specifically, for every $n \in \mathbb{N}$, there exists an impartial and $\sqrt{8n}$-additive mechanism on $\mathcal{G}_n(1)$.*

The result is achieved by a mechanism we call the *Twin Threshold Mechanism* and describe formally in Algorithm 1. The mechanism iteratively deletes the outgoing edges from vertices with indegree above a first threshold t from the highest to the lowest indegree and, in the end, selects the vertex with maximum remaining indegree as long as that indegree is above a second, higher threshold T. The parameters t and T will be chosen in order to achieve impartiality and obtain the desired bounds. Throughout the mechanism ties are broken as before, in favor of greater index.

[2]More formally, for a graph $G = (N, E)$ and $u, v \in N$ with $\delta^-(u)$, $\delta^-(v) \geq t$ and $\delta^-_{N\setminus\{v\}}(u) = \delta^-_{N\setminus\{u\}}(v)$, the mechanism selects $\max\{u, v\}$.

Algorithm 1: Twin Threshold Mechanism

Input: Digraph $G = (N, E) \in \mathcal{G}_n$, thresholds $T,\ t \in \{1, \dots, n-1\}$ with $t \leq T$.
Output: Set $S \subseteq N$ of selected vertices with $|S| \leq 1$.
Initialize $i \leftarrow 0$ and $d \leftarrow \Delta$;
$D^i \leftarrow \emptyset$; `// vertices with deleted outgoing edges in iteration` i `or before`
$\forall v \in N,\ \hat{\delta}^i(v) \leftarrow \delta^-(v)$; `// indegree of` v `omitting edges deleted up to` i
while $d \geq t$ **do**
  if $\{u \in N \setminus D^i : \hat{\delta}^i(u) = d\} = \emptyset$ **then**
    Update $d \leftarrow d - 1$ and **continue**
  end
  Let $v = \max\{u \in N \setminus D^i : \hat{\delta}^i(u) = d\}$;
  Update $\hat{\delta}^{i+1}(u) \leftarrow \hat{\delta}^i(u) - 1$ for every $u \in N^+(v)$ and $\hat{\delta}^{i+1}(u) \leftarrow \hat{\delta}^i(u)$ for every $u \in N \setminus N^+(v)$; `// outgoing edges of` v `are deleted`
  Update $D^{i+1} \leftarrow D^i \cup \{v\}$ and $i \leftarrow i + 1$
end
Let $I \leftarrow i$;
if $\hat{\Delta} := \max_{v \in N} \hat{\delta}^I(v) \geq T$ **then**
  Return $S = \{\max\{v \in N : \hat{\delta}^I(v) = \hat{\Delta}\}\}$
end
Return $S = \emptyset$

For any choice of the maximum outdegree k and the threshold parameters T and t, the mechanism achieves its worst additive performance guarantee in cases where it does not select, and this guarantee can be obtained in a straightforward way by bounding the maximum indegree. The proof of impartiality, on the other hand, uses a relatively subtle argument to show that for certain values of T and t, a vertex above the higher threshold T cannot influence whether another vertex ends up above or below the lower threshold t when edges have been deleted. Vertices above T then have no influence on the set of edges taken into account for selection, and since these are the only vertices that can potentially be selected impartiality follows.

In the following we will compare runs of the mechanism for different graphs, and denote by $\hat{\delta}^i(v, G)$, $D^i(G)$, and $I(G)$ the respective values of $\hat{\delta}^i(v)$, D^i, and I when the mechanism is invoked with input graph G. We use χ to denote the indicator function for logical propositions, i.e., $\chi(p) = 1$ when proposition p is true and $\chi(p) = 0$ otherwise. For a graph G and a vertex v in G whose outgoing edges are deleted by the mechanism, we use $i^\star(v, G)$ to denote the iteration in which this deletion takes place, such that $D^{i^\star(v,G)+1}(G) \setminus D^{i^\star(v,G)}(G) = \{v\}$. We use the convention that $i^\star(v, G) = I(G)$ if $v \notin D^{I(G)}(G)$ to extend the function to all vertices. For a graph G and vertex v, we write $\delta^\star(v, G) = \hat{\delta}^{i^\star(v,G)}(v, G)$ for the indegree of v, not taking into account any incoming edges deleted previously, at the last moment before the outgoing edges of v are deleted. When the graph G is clear from the context, we again drop G from the notation. It is clear from the definition of the mechanism that for any graph $G = (N, E)$ and vertex $v \in D^{I(G)}(G)$,

$$\delta^-(v, G) - \delta^\star(v, G) = |\{u \in N^-(v, G) : i^\star(u, G) < i^\star(v, G)\}|. \tag{1}$$

When comparing tuples of the form $(\delta^\star(v), v)$, we use regular inequalities to denote lexicographic order. These comparisons are relevant to our analysis because, for any graph $G = (N, E)$ and vertices $u, v \in D^{I(G)}(G)$,

$$i^\star(u, G) < i^\star(v, G) \quad \text{if and only if} \quad (\delta^\star(u, G), u) > (\delta^\star(v, G), v). \tag{2}$$

Figure 2 shows an example of the edge deletion process over four iterations of Algorithm 1. Observe that for $j \in \{1, 2\}$, $(\delta^-(v), v) > (\delta^-(u_j), u_j)$ but $(\delta^\star(v), v) < (\delta^\star(u_j), u_j)$. This is

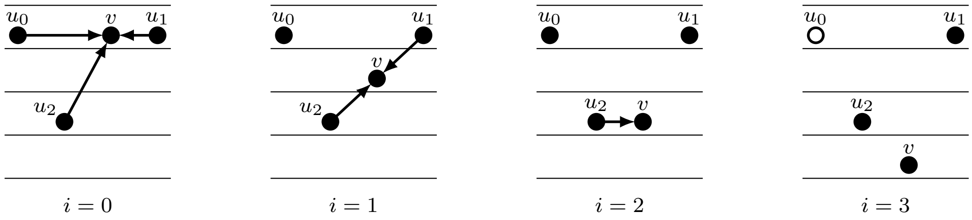

Figure 2: Illustration of the edge deletion process in Algorithm 1. We have assumed that all indegrees are above t.

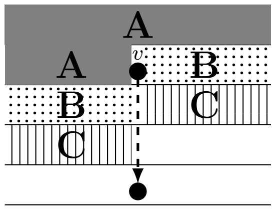

Figure 3: Illustration of Lemma 1 for $r = 3$. If the indegree of v drops as shown by the dashed arrow, there must be a vertex with an edge to v in A, another vertex with an edge to v in $A \cup B$, and a third vertex with an edge to v in $A \cup B \cup C$. Note that this exact condition is satisfied for the example of Figure 2.

caused by a drop in the indegree of v over the course of the algorithm, and this drop occurs before the algorithm considers possible outgoing edges of v for deletion. For our analysis, it will be important to bound how much the indegree of a vertex v can drop before v loses its outgoing edges. The following lemma, which we prove in Appendix A, characterizes this quantity in terms of the indegrees of the in-neighbors of v.

Lemma 1. *Let $G = (N, E) \in \mathcal{G}_n$, $v \in N$ and $(d, z) \in \mathbb{N}^2$ such that $(\delta^-(v), v) > (d, z) \geq (\delta^\star(v), v)$. Let $r = \delta^-(v) - d + \chi(v > z)$. Then there exist vertices $u_0, \dots, u_{r-1}$ such that for each $j \in \{0, 1, \dots, r-1\}$, $(u_j, v) \in E$ and $(\delta^\star(u_j), u_j) > (\delta^-(v) - j, v)$. Moreover, if $(d, z) = (\delta^\star(v), v)$, then for every vertex $u \in N^-(v) \setminus \{u_0, \dots, u_{r-1}\}$, $(\delta^\star(u), u) < (\delta^\star(v), v)$.*

If we take $(d, z) = (\delta^\star(v), v)$, the lemma implies that for any vertex $v \in D^I$,

$$\delta^-(v) - \delta^\star(v) = |\{u \in N^-(v) : (\delta^\star(u), u) > (\delta^\star(v) + 1, v)\}|. \tag{3}$$

In other words, if the indegree of a vertex v drops from $\delta^-(v)$ to $\delta^\star(v) = \delta^-(v) - r$ before the outgoing edges of v are deleted, there must be r vertices with edges to v that satisfy the following property: at least one of them, u_0, must have indegree high enough for its outgoing edges to be deleted before those of v, i.e., $(\delta^\star(u_0), u_0) > (\delta^-(v), v)$; another vertex, u_1, must have indegree high enough for its outgoing edges to be deleted before those of v after its indegree is reduced by one, i.e., $(\delta^\star(u_1), u_1) > (\delta^-(v) - 1, v)$; and so forth, up to u_{r-1} with $(\delta^\star(u_{r-1}), u_{r-1}) > (\delta^-(v) - (r-1), v) = (\delta^\star(v) + 1), v)$. The other vertices u with edges to v must satisfy $(\delta^\star(u), u) < (\delta^\star(v), v)$. This is illustrated in Figure 3 for the case where the indegree of v drops by $r = 3$. When $(d, z) > (\delta^\star(v), v)$ it is enough to carry out the analysis for the first $r = \delta^-(v) - d + \chi(v > z)$ vertices $u_0, \dots, u_{r-1}$.

To establish impartiality of the Twin Threshold Mechanism, we need to compare runs of the mechanism on graphs that differ in the outgoing edges of a single vertex. Intuitively, a change in the outgoing edges will make a difference to the outcome of the mechanism only if it affects the position of some other vertex relative to the lower threshold t at the time that vertex is considered: If at that time the vertex is above the threshold its outgoing edges are deleted, otherwise the edges remain and are used in the decision of which vertex to select. We are thus interested in pairs of graphs G_1 and G_2 that differ only in the outgoing edges of a vertex $\tilde{v}$, and which contain a second vertex $v \neq \tilde{v}$ such that $\delta^\star(v, G_1) > \delta^\star(v, G_2)$. Using Lemma 1, we can derive conditions in terms of the indegrees of the in-neighbors of v under which this can happen. Moreover, we can show that one of two additional conditions must be satisfied: either (i) $\tilde{v}$ has an edge to v in G_1, or (ii) there exists a vertex u_0 with an edge to v in both G_1 and G_2 such that $\delta^\star(u_0, G_1) < \delta^\star(u_0, G_2)$. We obtain the following lemma.

Lemma 2. *Let $G_1 = (N, E_1), G_2 = (N, E_2) \in \mathcal{G}_n$, v, $\tilde{v} \in N$ with $v \neq \tilde{v}$ be such that $E_1 \setminus (\{\tilde{v}\} \times N) = E_2 \setminus (\{\tilde{v}\} \times N)$, $\delta^\star(v, G_1) > \delta^\star(v, G_2)$, and $\delta^\star(v, G_1) \geq t$. Consider $(d, z) \in \mathbb{N}^2$ such that $(\delta^\star(v, G_1), v) > (d, z) \geq (\delta^\star(v, G_2), v)$, and let $r = \delta^\star(v, G_1) - d + \chi(v > z)$. Then, there exist vertices $u_0, \ldots, u_{r-1}$ such that, for every $j \in \{1, \ldots, r-1\}$,*

$$(u_j, v) \in E_1 \cap E_2, \quad (\delta^\star(u_j, G_2), u_j) > (\delta^\star(v, G_1) - j, v), \quad (\delta^\star(u_j, G_1), u_j) < (\delta^\star(v, G_1), v),$$

and one of the following holds:

(i) $(u_0, v) \in E_1 \setminus E_2$ and if $(\delta^\star(v, G_1), v) < (\delta^-(\tilde{v}, G_1), \tilde{v})$, taking $\tilde{r} = \delta^-(\tilde{v}, G_1) - \delta^\star(v, G_1) + \chi(\tilde{v} > v)$ we have that there are vertices $\tilde{u}_0, \ldots, \tilde{u}_{\tilde{r}-1}$, none of them equal to $\tilde{v}$, such that $(\delta^\star(\tilde{u}_j, G_1), \tilde{u}_j) > (\delta^-(\tilde{v}, G_1) - j, \tilde{v})$ for every $j \in \{0, \ldots, \tilde{r} - 1\}$; or

(ii) $(\delta^\star(u_0, G_2), u_0) > \cdots > (\delta^\star(u_{r-1}, G_2), u_{r-1})$, $(u_0, v) \in E_1 \cap E_2$, and $(\delta^\star(u_0, G_2), u_0) > (\delta^\star(v, G_1), v) > (\delta^\star(u_0, G_1), u_0)$.

We prove Lemma 2 in Appendix B but provide some intuition for its correctness here. Assume that the indegree of a vertex v drops from $\delta^-(v)$ to $\delta^\star(v) = \delta^-(v) - r$ before its outgoing edges are deleted. Then, by Lemma 1, there must be r in-neighbors of v whose indegrees are at least $\delta^\star(v) + 1$ when their outgoing edges are deleted. Thus, when v, G_1, and G_2 are as in the statement of Lemma 2, and defining $r_1 = \delta^-(v, G_1) - \delta^\star(v, G_1)$ and $r_2 = \delta^-(v, G_2) - \delta^\star(v, G_2)$, then r_1 in-neighbors of v must have indegree at least $\delta^\star(v, G_1) + 1$ in G_1 upon deletion of their outgoing edges, and r_2 in-neighbors of v must have indegree at least $\delta^\star(v, G_2) + 1$ in G_2 upon deletion of their outgoing edges. There are then two possible reasons for the difference between $\delta^\star(v, G_1)$ and $\delta^\star(v, G_2)$. The first, which can only occur when $r_1 = r_2$, is given in Condition (i) of the lemma: $\tilde{v}$ has an edge to v in G_1 but not in G_2, while all the other indegrees remain the same. However, for this difference to have an impact, the outgoing edges of $\tilde{v}$ must be deleted after those of v, and Lemma 1 implies the existence of in-neighbors of $\tilde{v}$ with indegrees as shown. The other reason, which can happen when $r_1 = r_2$ and necessarily happens otherwise, is that some in-neighbor u_0 of v in both G_1 and G_2 loses its outgoing edges after v when the input to the mechanism is G_1, but before v when the input is G_2. This must happen due to a change in the indegree of u_0 at the time its outgoing edges are deleted, i.e., $(\delta^\star(u_0, G_2), u_0) > (\delta^\star(v, G_1), v) > (\delta^\star(u_0, G_1), u_0)$. This is captured in Condition (ii). In both cases, Lemma 1 implies the existence of $\delta^\star(v, G_1) - \delta^\star(v, G_2) - 1$ further in-neighbors of v in both graphs, denoted as $u_1, \ldots, u_{r-1}$, which lose their outgoing edges after v in G_1 but before v in G_2. When $(d, z) > (\delta^\star(v, G_2), v)$, it again suffices to carry out the analysis for a smaller subset of the vertices with edges to v.

Lemma 2 implies that whenever G_1 and G_2 differ only in the outgoing edges of a single vertex $\tilde{v}$, and v is a different vertex with $\delta^\star(v, G_1) > \delta^\star(v, G_2)$, then either (i) $\tilde{v}$ has an edge to v in G_1, or (ii) there exists a vertex u_0 with $\delta^\star(u_0, G_1) < \delta^\star(u_0, G_2)$. The fact that this relationship is the opposite of that for v naturally suggests an iterative analysis, where the roles of G_1 and G_2 are exchanged in each iteration as long as Condition (ii) holds. Such an analysis leads to the following lemma, which establishes a sufficient condition for impartiality in terms of T, t, and k. Impartiality for a particular choice of T and t that guarantees the bound of Theorem 1 can then be obtained in a straightforward way.

Lemma 3. *For every $n, k \in \mathbb{N}$ with $k \leq n-1$, the Twin Threshold Mechanism with parameters T and t such that*

$$\frac{1}{2}(T^2 + 3T + t - t^2) > k(n+2)$$

is impartial on $\mathcal{G}_n(k)$.

Proof. Let f be the selection mechanism given by the Twin Threshold Mechanism with thresholds T and t. We suppose that f is not impartial and we want to see that the inequality in the statement of the lemma is thus violated.

Specifically, let $n \in \mathbb{N}$, $G = (N, E), G' = (N, E') \in \mathcal{G}_n$, and $\tilde{v} \in N$ such that $E \setminus (\{\tilde{v}\} \times N) = E' \setminus (\{\tilde{v}\} \times N)$ and $\tilde{v} \in f(G) \triangle f(G')$, i.e., $\tilde{v}$ is selected only for one of these graphs. In particular, $\delta^-(\tilde{v}, G) = \delta^-(\tilde{v}, G') \geq T$. For $\tilde{v}$ to be selected only for one of the graphs, there has to be a vertex $v \in N \setminus \{\tilde{v}\}$ whose vote is counted when the mechanism runs with input G but not when it runs with input G', or vice versa. Suppose w.l.o.g.that the former holds and denote $v^0 = v$. From Lemma 2 with $v = v^0$, $G_1 = G$, $G_2 = G'$, and $(d, z) = (t-1, v^0)$, we have that there are $r^0 = \delta^\star(v^0, G) - (t-1)$ vertices $u^0_0, \ldots, u^0_{r^0-1}$ for which $(u^0_j, v^0) \in E \cap E'$, $(\delta^\star(u^0_j, G'), u^0_j) > (\delta^\star(v^0, G) - j, v^0)$, $(\delta^\star(u^0_j, G), u^0_j) < (\delta^\star(v^0, G), v^0)$ for every $j \in \{1, \ldots, r^0 - 1\}$, and one of the conditions in the lemma holds. If Condition (i) holds, we denote $m = 0$. Otherwise, we have that $(\delta^\star(u^0_0, G'), u^0_0) > (\delta^\star(v^0, G), v^0) > (\delta^\star(u^0_0, G), u^0_0)$, thus we can define $v_1 = u^0_0$ and apply Lemma 2 with $v = v_1$, $G_1 = G'$, $G_2 = G$, and $(d, z) = (\delta^\star(v^0, G), v^0)$.

The argument can be repeated until Condition (i) holds at some iteration, which we denote m. This necessarily happens because n is finite, and we denote as $G^* \in \{G, G'\}$ the graph such that $(\tilde{v}, v_m) \in G^*$. In particular, $G^* = G$ if m is even and $G^* = G'$ if m is odd. For every iteration $\ell \in \{1, \ldots, m-1\}$ the following holds: There is a vertex $v^\ell = u^{\ell-1}_0$ and a strictly positive value $r^\ell \in \{\delta^\star(v^\ell, G) - \delta^\star(v^{\ell-1}, G') + 1, \delta^\star(v^\ell, G) - \delta^\star(v^{\ell-1}, G')\}$ (if ℓ is even) or $r^\ell \in \{\delta^\star(v^\ell, G') - \delta^\star(v^{\ell-1}, G) + 1, \delta^\star(v^\ell, G') - \delta^\star(v^{\ell-1}, G)\}$ (if ℓ is odd) such that there are vertices $u^\ell_0, \ldots, u^\ell_{r^\ell-1}$ such that for each $j \in \{1, \ldots, r^\ell - 1\}$,

$$(u^\ell_j, v^\ell) \in E \cap E',\ (\delta^\star(u^\ell_j, G'), u^\ell_j) > (\delta^\star(v^\ell, G) - j, v^\ell) \text{ and } (\delta^\star(u^\ell_j, G), u^\ell_j) < (\delta^\star(v^\ell, G), v^\ell)$$

if ℓ is even, and

$$(u^\ell_j, v^\ell) \in E \cap E',\ (\delta^\star(u^\ell_j, G), u^\ell_j) > (\delta^\star(v^\ell, G') - j, v^\ell) \text{ and } (\delta^\star(u^\ell_j, G'), u^\ell_j) < (\delta^\star(v^\ell, G'), v^\ell)$$

if ℓ is odd. Furthermore, we claim that for every $\ell, \ell' \in \{0, \ldots, m-1\}$ and every $j \in \{0, \ldots, r^\ell - 1\}$, $j' \in \{0, \ldots, r^{\ell'} - 1\}$ with $(\ell, j) \neq (\ell', j')$ it holds $u^\ell_j \neq u^{\ell'}_{j'}$. In order to see this, we actually show the following properties, that directly imply the previous one:

$$(\delta^\star(u^\ell_j, G'), u^\ell_j) > (\delta^\star(u^{\ell'}_{j'}, G'), u^{\ell'}_{j'}) \quad \begin{array}{l}\text{for every } \ell \in \{2, \ldots, m\} \text{ even }, \ell' \in \{0, \ldots, \ell - 1\},\\ j \in \{0, \ldots, r^\ell - 1\}, j' \in \{0, \ldots, r^{\ell'} - 1\} \text{ with } (\ell, j) \neq (\ell', j').\end{array}$$

$$(\delta^\star(u^\ell_j, G), u^\ell_j) > (\delta^\star(u^{\ell'}_{j'}, G), u^{\ell'}_{j'}) \quad \begin{array}{l}\text{for every } \ell \in \{1, \ldots, m\} \text{ odd }, \ell' \in \{0, \ldots, \ell - 1\},\\ j \in \{0, \ldots, r^\ell - 1\}, j' \in \{0, \ldots, r^{\ell'} - 1\} \text{ with } (\ell, j) \neq (\ell', j').\end{array}$$

We prove this by induction over ℓ, distinguishing whether this is an even or odd value. Let first $\ell \in \{2, \ldots, m\}$ be an even value and note that

$$(\delta^\star(u^\ell_j, G'), u^\ell_j) > (\delta^\star(v^\ell, G), v^\ell) > (\delta^\star(v^{\ell-1}, G'), v^{\ell-1}) \quad \text{for every } j \in \{0, \ldots, r^\ell - 1\}.$$

But from the definition of these vertices, $(\delta^\star(v^{\ell-1}, G'), v^{\ell-1}) > (\delta^\star(u^{\ell-1}_{j'}, G'), u^{\ell-1}_{j'})$ for each $j' \in \{0, \ldots, r^{\ell-1} - 1\}$. Moreover, we also have that $(\delta^\star(v^{\ell-1}, G'), v^{\ell-1}) > (\delta^\star(u^{\ell-2}_{j'}, G'), u^{\ell-2}_{j'})$ for every $j' \in \{0, \ldots, r^{\ell-2} - 1\}$ due to the chain of inequalities in Condition (ii) of Lemma 2. Therefore, we conclude that for every even ℓ we have both

$$(\delta^\star(u^\ell_j, G'), u^\ell_j) > (\delta^\star(u^{\ell-1}_{j'}, G'), u^{\ell-1}_{j'}) \quad \text{for every } j' \in \{0, \ldots, r^{\ell-1} - 1\},$$

and

$$(\delta^\star(u_j^\ell, G'), u_j^\ell) > (\delta^\star(u_{j'}^{\ell-2}, G'), u_{j'}^{\ell-2}) \quad \text{for every } j' \in \{0, \dots, r^{\ell-2} - 1\}.$$

This proves the claim for the base case $\ell = 2$, and also implies that if it holds for every $\ell \leq \hat{\ell}$ with $\hat{\ell} \geq 2$ then it holds directly for $\hat{\ell} + 2$ as well. For odd values of ℓ, the claim follows from a completely analogous reasoning using graph G instead of G', with the only difference that, for the base case, $\ell' = \ell - 1$ is the only possibility.

If $\delta^\star(v^m, G^*) \geq \delta^-(\tilde{v}, G)$, the indegrees (either in G or G') of the vertices u_j^ℓ for $\ell \in \{0, \dots, m-1\}$ and $j \in \{0, \dots, r^\ell - 1\}$, plus the indegree of $\tilde{v}$, sum up to at least

$$\begin{aligned}
&\sum_{\ell < m \text{ even}} \sum_{j=0}^{r^\ell - 1} \delta^\star(u_j^\ell, G') + \sum_{\ell < m \text{ odd}} \sum_{j=0}^{r^\ell - 1} \delta^\star(u_j^\ell, G) + \delta^-(\tilde{v}, G) \\
&\geq \sum_{\ell < m \text{ even}} \sum_{j=0}^{r^\ell - 1} (\delta^\star(v^\ell, G) - j) + \sum_{\ell < m \text{ odd}} \sum_{j=0}^{r^\ell - 1} (\delta^\star(v^\ell, G') - j) + \delta^-(\tilde{v}, G) \\
&\geq \sum_{j=0}^{\delta^\star(v^m, G) - t} (\delta^\star(v^m, G) - j) + \delta^-(\tilde{v}, G) \geq \sum_{j=t}^{T} j + T = \frac{1}{2}(T^2 + 3T + t - t^2),
\end{aligned}$$

where the second inequality uses that $\delta^\star(v^\ell, G) \geq \delta^\star(u_0^\ell, G) = \delta^\star(v^{\ell+1}, G') - r^{\ell+1}$ if ℓ is even, and $\delta^\star(v^\ell, G') \geq \delta^\star(u_0^\ell, G') = \delta^\star(v^{\ell+1}, G) - r^{\ell+1}$ if ℓ is odd, as stated in Lemma 2.

If $\delta^\star(v^m, G^*) < \delta^-(\tilde{v}, G)$, using Lemma 2 with $\tilde{r} = \delta^-(\tilde{v}, G) - \delta^\star(v^m, G^*) + \chi(\tilde{v} > v^m)$, we have that there are vertices $\tilde{u}_0, \dots, \tilde{u}_{\tilde{r}-1}$, none of them equal to $\tilde{v}$, such that $\delta^\star(\tilde{u}_j, G^*) \geq \delta^-(\tilde{v}, G) - j$ for every $j \in \{0, \dots, \tilde{r} - 1\}$. Therefore, even though the last inequality in the previous chain does not hold, we now have that the indegrees (either in G or G') of the vertices u_j^ℓ for $\ell \in \{0, \dots, m-1\}$ and $j \in \{0, \dots, r^\ell - 1\}$, plus the indegrees of the vertices $\tilde{u}_0, \dots, \tilde{u}_{\tilde{r}-1}$ and the indegree of $\tilde{v}$, sum up to at least

$$\begin{aligned}
&\sum_{\ell < m \text{ even}} \sum_{j=0}^{r^\ell - 1} \delta^\star(u_j^\ell, G') + \sum_{\ell < m \text{ odd}} \sum_{j=0}^{r^\ell - 1} \delta^\star(u_j^\ell, G) + \sum_{j=0}^{\tilde{r}-1} \delta^\star(\tilde{u}_j, G^*) + \delta^-(\tilde{v}, G) \\
&\geq \sum_{\ell < m \text{ even}} \sum_{j=0}^{r^\ell - 1} (\delta^\star(v^\ell, G) - j) + \sum_{\ell < m \text{ odd}} \sum_{j=0}^{r^\ell - 1} (\delta^\star(v^\ell, G') - j) + \sum_{j=\delta^\star(v^m, G^*)+1}^{\delta^-(\tilde{v}, G)} j + \delta^-(\tilde{v}, G) \\
&\geq \sum_{j=0}^{\delta^\star(v^m, G) - t} (\delta^\star(v^m, G) - j) + \sum_{j=\delta^\star(v^m, G)+1}^{\delta^-(\tilde{v}, G)} j + T \\
&\geq \sum_{j=t}^{T} j + T = \frac{1}{2}(T^2 + 3T + t - t^2).
\end{aligned}$$

Since the sum of the indegrees is at most the maximum number kn of edges, and since the indegrees of vertices summed over G and G' differ by at most $2k$, we conclude that

$$\frac{1}{2}(T^2 + 3T + t - t^2) \leq k(n + 2).$$

□

Figure 4 illustrates this result by showing a situation where the outgoing edge of a vertex $\tilde{v}$ with $\delta^-(\tilde{v}, G_1) = \delta^-(\tilde{v}, G_2) = T$ determines whether $\delta^\star(v) \geq t$ for another vertex v, and thus whether $\tilde{v}$ itself is selected or not. Note that in the example there exist vertices w_j such that $\delta^-(w_j, G) \geq t + j$ for every $j \in \{0, \dots, T - t\}$ and some $G \in \{G_1, G_2\}$. This property turns out to be universal and allows us to prove Lemma 3. Figure 4 in fact shows a worst-case situation, in the sense that with fewer vertices than the ones depicted there the same situation cannot occur. We are now ready to prove Theorem 1.

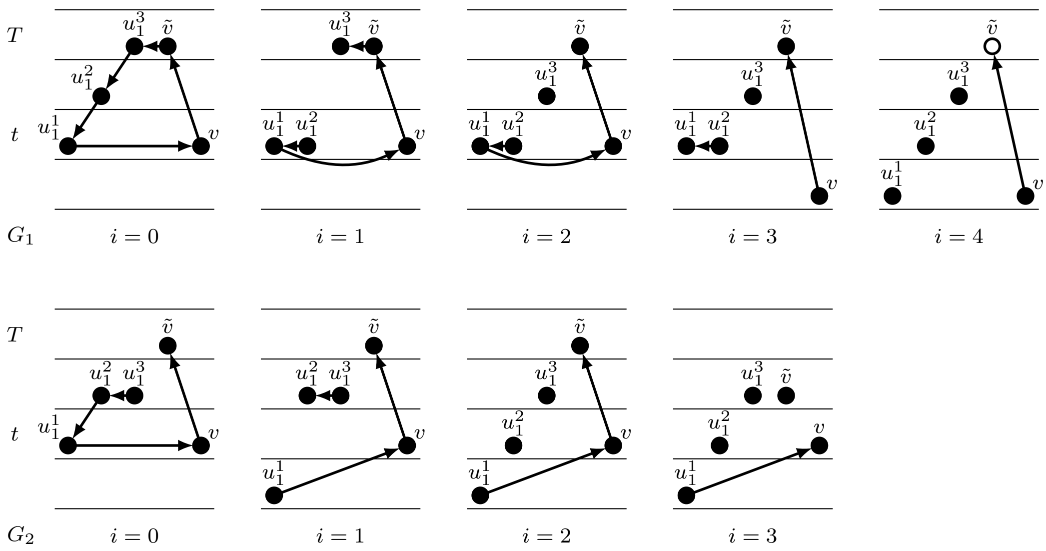

Figure 4: By changing its outgoing edge, $\tilde{v}$ is able to affect whether it is selected by the mechanism or not. Lemma 3 gives a condition over T, t, and k such that this cannot happen.

Proof of Theorem 1. Let f be the selection mechanism given by the Twin Threshold Mechanism with $T = \frac{5}{2}\sqrt{c}n^{\frac{1+\kappa}{2}} - 1$ and $t = \frac{1}{2}\sqrt{c}n^{\frac{1+\kappa}{2}}$. Let $n \in \mathbb{N}_+$, $G \in \mathcal{G}_n(k)$ and $\kappa, c > 0$ such that $k \leq cn^\kappa$. This implies that $k(n+2) \leq cn^{1+\kappa} + 2cn^\kappa \leq 3cn^{1+\kappa}$, thus from Lemma 3 we have that a sufficient condition for impartiality is that

$$\frac{1}{2}(T^2 + 3T + t - t^2) > 3cn^{1+\kappa}.$$

Replacing T and t yields

$$\frac{1}{2}(T^2 + 3T + t - t^2) = 3cn^{1+\kappa} + \frac{3}{2}\sqrt{c}n^{\frac{1+\kappa}{2}} - 1 > 3cn^{1+\kappa},$$

where the last equality uses that $cn^\kappa \geq k$ and that $n, k \geq 1$. We conclude that the mechanism is impartial for these values of T and t.

For obtaining the additive bound, first consider a graph $G = (N, E)$ such that the mechanism returns the empty set when run with this graph as input. Let v^* be such that $\delta^-(v^*) = \Delta(G)$ and note that necessarily $\hat{\delta}^I(v^*) \leq T - 1$. Since there are at most $\lfloor kn/t \rfloor$ vertices with indegree t or higher, a maximum of $\lfloor kn/t \rfloor - 1$ in-neighbors of v^* have their outgoing edges deleted during the algorithm. Therefore, we conclude that $\Delta(G) \leq T + \lfloor kn/t \rfloor - 2$.

Consider now $G = (N, E)$ such that the mechanism returns a set $\{v\}$, and let v^* be such that $\delta^-(v^*) = \Delta(G)$. Once again, a maximum of $\lfloor kn/t \rfloor - 1$ in-neighbors of v^* have their outgoing edges deleted during the algorithm. Using the fact that $\hat{\delta}^I(v^*) \leq \hat{\delta}^I(v)$ since v is selected, we conclude that

$$\Delta(G) - \delta^-(v) \leq \left(\delta^-(v) + \left\lfloor \frac{kn}{t} \right\rfloor - 1\right) - \delta^-(v) = \left\lfloor \frac{kn}{t} \right\rfloor - 1.$$

Since the value obtained in the former case is greater or equal than the one obtained in the latter for any values of T and t, we have that f is α-additive for $\alpha = T + \lfloor kn/t \rfloor - 2$. Therefore, for the specified values of T and t and given the upper bound on k, f is α-additive for

$$\alpha = \frac{5}{2}\sqrt{c}n^{\frac{1+\kappa}{2}} - 1 + \left\lfloor \frac{2cn^{1+\kappa}}{\sqrt{c}n^{\frac{1+\kappa}{2}}} \right\rfloor - 2 = \frac{5}{2}\sqrt{c}n^{\frac{1+\kappa}{2}} + \left\lfloor 2\sqrt{c}n^{\frac{1+\kappa}{2}} \right\rfloor - 3 = O(n^{\frac{1+\kappa}{2}}).$$

We conclude that f is $O(n^{\frac{1+\kappa}{2}})$-additive.

When $k = 1$, a more detailed analysis yields the bound of $\sqrt{8n}$. First observe that in this case, by Lemma 3, impartiality holds when

$$\frac{1}{2}(T^2 + 3T + t - t^2) - (n+2) > 0$$

and thus when

$$T^2 + 3T - (t^2 - t + 2n + 4) > 0.$$

The left-hand side is equal to zero if and only if

$$T = \frac{-3 \pm \sqrt{4(t^2 - t + 2n + 4) + 9}}{2} = \pm\sqrt{t^2 - t + 2n + \frac{25}{4}} - \frac{3}{2},$$

and since T has to be non-negative impartiality holds if

$$T > \max\left\{-\sqrt{t^2 - t + 2n + \frac{25}{4}} - \frac{3}{2}, \sqrt{t^2 - t + 2n + \frac{25}{4}} - \frac{3}{2}\right\}.$$

This is trivially satisfied if we take

$$t = \lceil\sqrt{n}\rceil, \quad T = \left\lfloor\sqrt{\lceil\sqrt{n}\rceil^2 - \lceil\sqrt{n}\rceil + 2n + \frac{25}{4}} - \frac{1}{2}\right\rfloor.$$

As before, we know that given the thresholds T and t, the mechanism is α-additive for any $\alpha \geq \alpha(n) := T + \lfloor n/t \rfloor - 2$. In order to obtain an upper bound on $\alpha(n)$, we start by bounding T from above:

$$\begin{aligned} T = \left\lfloor\sqrt{\lceil\sqrt{n}\rceil^2 - \lceil\sqrt{n}\rceil + 2n + \frac{25}{4}} - \frac{1}{2}\right\rfloor &\leq \sqrt{(\sqrt{n}+1)^2 - (\sqrt{n}+1) + 2n + \frac{25}{4}} - \frac{1}{2} \\ &= \sqrt{3n + \sqrt{n} + \frac{25}{4}} - \frac{1}{2}, \end{aligned}$$

where the first inequality holds because $a^2 - a$ is increasing for $a \geq 1/2$. Then

$$\alpha(n) = T + \lfloor n/t \rfloor - 2 \leq \sqrt{3n + \sqrt{n} + \frac{25}{4}} - \frac{1}{2} + \left\lfloor\frac{n}{\lceil\sqrt{n}\rceil}\right\rfloor - 2 \leq \sqrt{3n + \sqrt{n} + \frac{25}{4}} + \sqrt{n} - \frac{5}{2}.$$

To see that the last expression bounded from above by $\sqrt{8n}$, let $g(n) = \sqrt{8n} - \alpha(n)$ and observe that

$$g(1) \geq \sqrt{8} - \sqrt{3 + \sqrt{1} + \frac{25}{4}} - \sqrt{1} + \frac{5}{2} = \sqrt{8} - \sqrt{\frac{41}{4}} + \frac{3}{2} \approx 0.13 > 0.$$

Moreover,

$$g'(n) = \frac{\sqrt{2}}{\sqrt{n}} - \frac{6\sqrt{n}+1}{2\sqrt{n}\sqrt{12n + 4\sqrt{n} + 25}} - \frac{1}{2\sqrt{n}} = \frac{4(2\sqrt{2}-1)\sqrt{12n + 4\sqrt{n} + 25} - 24\sqrt{n} - 4}{8\sqrt{n}\sqrt{12n + 4\sqrt{n} + 25}},$$

which is non-negative when $n \geq 1$. To see this, note that the denominator of the last expression is positive and that $4(2\sqrt{2}-1) > 7$, so that $g'(n) \geq 0$ if

$$7\sqrt{12n + 4\sqrt{n} + 25} \geq 24\sqrt{n} + 4,$$

i.e., if

$$12n + 4\sqrt{n} + 1209 \geq 0.$$

This clearly holds when $n \geq 1$. We conclude that $\alpha(n) \leq \sqrt{8n}$ for every $n \geq 1$, thus f is $\sqrt{8n}$-additive. $\square$

4 A Tight Impossibility Result for Approval

So far, we have developed a new mechanism for impartial selection and have established an additive performance guarantee for the mechanism relative to the maximum outdegree in the graph. We will now take a closer look at the case where the maximum outdegree is unbounded, i.e., at the approval setting.

When applied to the approval setting, Theorem 1 provides a performance guarantee of $O(n)$. As the maximum indegree in a graph with n vertices is $n-1$, this bound is trivially achieved by any impartial mechanism including the mechanism that never selects. Caragiannis et al. [2019] have used a careful case analysis to show that deterministic impartial mechanisms cannot be better than 3-additive. We show that the trivial upper bound of $n-1$ is in fact tight for all n, which means that the mechanism that never selects provides the best possible additive performance guarantee among all deterministic impartial mechanisms. Our result is in fact more general and again holds relative to the maximum outdegree k.

Theorem 2. *Let $n \in \mathbb{N}$ and $k \leq n-1$. Let f be an impartial deterministic selection mechanism such that f is α-additive on $\mathcal{G}_n(k)$. Then $\alpha \geq k$. In particular, if f is α-additive on $\mathcal{G}_n$, then $\alpha \geq n-1$.*

In the practically relevant case where individuals are not allowed to abstain and the minimum outdegree is therefore at least 1, a small improvement can be obtained by selecting a vertex with an incoming edge from a fixed vertex, and again breaking ties by a fixed ordering of the vertices. The selected vertex then has indegree at least 1, which for the approval setting implies $(n-2)$-additivity. This guarantee is again best possible.

Theorem 3. *Let $n \in \mathbb{N}$ and $k \leq n-1$. Let f be an impartial deterministic selection mechanism such that f is α-additive on $\mathcal{G}^+_n(k)$. Then $\alpha \geq k-1$. In particular, if f is α-additive on $\mathcal{G}^+_n$, then $\alpha \geq n-2$.*

To prove both theorems we study the performance of impartial, but not necessarily deterministic, selection mechanisms on a particular class of graphs which in the case of n vertices we denote by $\mathcal{G}^T_n$. For each $n \in \mathbb{N}$, a graph $G = (N, E) \in \mathcal{G}_n$ belongs to $\mathcal{G}^T_n$ if and only if there exists an r-partition of N for some $r \geq 1$, which we denote by $(S_1, \dots, S_r)$, such that (i) $u < v$ for every $u \in S_i$, and $v \in S_j$ with $i < j$, and (ii) $E = \{(u, v) \in S_i \times S_j : i, j \in \{1, \dots, r\}, i \leq j, u \neq v\}$. In other words, $\mathcal{G}^T_n$ contains all graphs obtained by taking an ordered partition of a set of n unlabeled vertices and adding edges from each vertex to all other vertices in the same part and in greater parts. We will not be interested in isomorphic graphs within the class and thus only consider partitions of the vertices in increasing order. A graph in $\mathcal{G}^T_n$ is thus characterized by the partition $(S_1, \dots, S_r)$, or by the tuple $(s_1, \dots, s_r)$ where $s_i = |S_i|$ for each $i \in \{1, \dots, r\}$. For a given graph $G \in \mathcal{G}^T_n$, we denote the former by $S(G)$, latter by $s(G)$, and the length of $s(G)$ by $r(G)$. Finally, for $G \in \mathcal{G}^T_n$, let

$$\lambda_G = \frac{n!}{\prod_{i=1}^{r(G)} (s(G))_i!},$$

which is the number of graphs with n vertices isomorphic to G. Figure 5 shows the graphs in $\mathcal{G}^T_2$ and $\mathcal{G}^T_3$, along with their tuple representation $s(G)$ and associated values λ_G. The sums, for $n \in \mathbb{N}$, of the values λ_G for all graphs $G \in \mathcal{G}^T_n$ are known as Fubini numbers and count the number of weak orders on an n-element set. The following lemma establishes a property of Fubini numbers that is readily appreciated for the cases shown in Figure 5 but in fact holds for all n. The property was known previously [Diagana and Maïga, 2017], but we provide an alternative proof in Appendix C for the sake of completeness.

Lemma 4. *For every $n \in \mathbb{N}$, $n \geq 1$, $\sum_{G \in \mathcal{G}^T_n} \lambda_G$ is an odd number.*

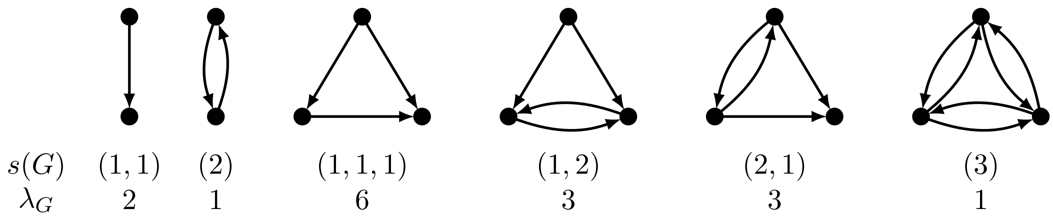

Figure 5: Graphs in $\mathcal{G}_2^T$ and $\mathcal{G}_3^T$.

For every pair of graphs $G, G' \in \mathcal{G}_n^T$ and $j \in \{2, \ldots, r(G)\}$, we say that there is a *j-transition* from G to G' if $r(G) = r(G') + 1$, $(s(G))_j = 1$, and

$$(s(G'))_i = \begin{cases} (s(G))_i & \text{if } i \leq j-2, \\ (s(G))_i + 1 & \text{if } i = j-1, \\ (s(G))_{i+1} & \text{if } i \geq j. \end{cases}$$

Intuitively, a j-transition can be obtained by changing the outgoing edges of the single vertex in the set $(S(G))_j$, including not only edges to vertices in $\bigcup_{i=j}^{r(G)} (S(G))_i$ but also to vertices in $(S(G))_{j-1}$. When the value of j is not relevant in a particular context, we simply say that there is a transition from G to G' if there exists some $j \in \{2, \ldots, r(G)\}$ such that there is a j-transition from G to G'. Observe that for every pair of graphs $G, G' \in \mathcal{G}_n^T$ there is at most one $j \in \{2, \ldots, r(G)\}$ such that there is a j-transition from G to G', and that if there is a transition from G to G', there cannot be a transition from G' to G. This kind of relation between ordered partitions has been exploited by Insko et al. [2017] for studying an expansion of the determinant, giving rise to a partial order on $\mathcal{G}_n^T$. In our context, it turns out to be relevant because whenever there is a j-transition from G to G', any impartial mechanism either selects the vertex in $(S(G))_j$ both in G and G', or in none of them.

In the case of plurality, impartiality was shown by Holzman and Moulin [2013] to be incompatible with two further axioms: *positive unanimity*, which requires for all $G = (N, E) \in \mathcal{G}(1)$ that $v \in f(G)$ if $\delta^-(v) = |N| - 1$; and *negative unanimity*, which requires for all $G = (N, E) \in \mathcal{G}(1)$ that $v \notin f(G)$ if $\delta^-(v) = 0$. This result holds even on a restricted class of graphs, consisting of a single cycle and additional vertices with edges onto that cycle, and has immediate and very strong implications on the best multiplicative approximation guarantee that an impartial mechanism can achieve. For additive performance guarantees, however, the incompatibility of impartiality with the other two axioms implies only a lower bound of 2. We will see in the following that on the class $\mathcal{G}_n^T$, impartiality is incompatible with a single axiom that weakens positive and negative unanimity. Strong lower bounds regarding additive performance guarantees for approval then follow immediately.

The class $\mathcal{G}_n^T$ is very different, and has to be very different, from the class of graphs used by Holzman and Moulin, and will ultimately require a new analysis. We can, however, follow the approach of Holzman and Moulin to consider randomized mechanisms rather than deterministic ones, which allows us without loss of generality to restrict attention to mechanisms that treat vertices symmetrically. A *randomized selection mechanism* for $\mathcal{G}$ is given by a family of functions $f : \mathcal{G}_n \to [0,1]^n$ that maps each graph to a probability distribution on the set of its vertices, such that $\sum_{i=1}^n (f(G))_i \leq 1$ for every graph $G \in \mathcal{G}_n$. Analogously to the case of deterministic mechanisms, we say that a randomized selection mechanism f is *impartial* on $\mathcal{G}' \subseteq \mathcal{G}$ if for every pair of graphs $G = (N, E)$ and $G' = (N, E')$ in $\mathcal{G}'$ and every $v \in N$, $(f(G))_v = (f(G'))_v$ whenever $E \setminus (\{v\} \times N) = E' \setminus (\{v\} \times N)$. We say that a randomized selection mechanism f satisfies *weak unanimity* on $\mathcal{G}_n$ if for every $G = (N, E) \in \mathcal{G}_n$ such that $\delta^-(v) = n - 1$ for some $v \in N$,

$$\sum_{u \in N : \delta^-(u) \geq 1} (f(G))_u \geq 1.$$

In other words, weak unanimity requires that a vertex with positive indegree is chosen with probability 1 whenever there exists a vertex with indegree $n-1$. We finally say that a randomized mechanism f is *symmetric* if it is invariant with respect to renaming of the vertices, i.e., if for every $G=(N,E)\in\mathcal{G}$, every $v\in N$ and every permutation $\pi=(\pi_1,\dots,\pi_{|N|})$ of N, $(f(G_\pi))_{\pi_v}=(f(G))_v$, where $G_\pi=(N,E_\pi)$ with $E_\pi=\{(\pi_u,\pi_v):(u,v)\in E\}$. For a given randomized mechanism f, we denote by f_s the mechanism obtained by applying a random permutation π to the vertices of the input graph, invoking f, and permuting the result by the inverse of π. Thus, for all $n\in\mathbb{N}$, $G\in\mathcal{G}_n$, and $v\in N$,

$$(f_s(G))_v=\frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}(f(G_\pi))_{\pi_v},$$

where $\mathcal{S}_n$ is the set of all permutations $\pi=(\pi_1,\dots,\pi_n)$ of a set of n elements.

The following lemma, which we prove in Appendix D, establishes that f_s is symmetric for every randomized mechanism f and inherits impartiality and weak unanimity from f. The lemma is a straightforward variant of a result of Holzman and Moulin and will allow us to restrict attention to symmetric randomized mechanisms.

Lemma 5. *Let f be a randomized selection mechanism that is impartial and weakly unanimous on $\mathcal{G}_n$. Then, f_s is symmetric, impartial, and weakly unanimous on $\mathcal{G}_n$.*

We are now ready to state our axiomatic impossibility result, which can be seen as a stronger version of that of Holzman and Moulin for the case of unbounded outdegree. Both lower bounds follow easily from this result.

Lemma 6. *For every $n\in\mathbb{N}$, $n\geq 2$, there exists no randomized selection mechanism f satisfying impartiality and weak unanimity on $\mathcal{G}_n^T$.*

Proof. Let $n\in\mathbb{N}$, $n\geq 2$ and suppose that there exists a randomized selection mechanism f satisfying impartiality and weak unanimity on $\mathcal{G}_n$. Since we can assume symmetry due to Lemma 5, for each graph $G\in\mathcal{G}_n^T$ we have that for every $i\in\{1,\dots,r(G)\}$ and every $u,v\in(S(G))_i$, $(f(G))_u=(f(G))_v$, and thus we denote this value simply as $(f(G))_i$. We consider for the proof an undirected graph $\mathcal{H}_n=(\mathcal{G}_n^T,\mathcal{F})$, such that for every pair of graphs $G,G'\in\mathcal{G}_n^T$ we have that $\{G,G'\}\in\mathcal{F}$ if and only if there is a transition from G to G' or from G' to G. By definition of a transition, for each $\{G,G'\}\in\mathcal{F}$ we have $|r(G)-r(G')|=1$. Therefore, $\mathcal{H}_n$ is bipartite with partition (L_n,R_n) where $L_n=\{G\in\mathcal{G}_n^T: r(G)\text{ is even}\}$ and $R_n=\{G\in\mathcal{G}_n^T: r(G)\text{ is odd}\}$. Figure 8 in Appendix E depicts the graphs $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. For each graph $G\in\mathcal{G}_n^T$, we define $i(G)=2$ if $(s(G))_1=1$ and $i(G)=1$ otherwise.

Let $G=(N,E)$ and $G'=(N,E')$ be two graphs in $\mathcal{G}_n^T$ such that there is a j-transition from G to G' for some $j\in\{2,\dots,r(G)\}$. Denoting by v the unique vertex in $(S(G))_j$, which is also in $S(G')_{j-1}$, we have that $E\setminus(\{v\}\times N)=E'\setminus(\{v\}\times N)$, and since f is impartial, $(f(G))_j=(f(G'))_{j-1}$. Therefore,

$$\begin{aligned}
\lambda_{G'}\cdot(s(G'))_{j-1}\cdot(f(G'))_{j-1} &= \frac{n!}{\prod_{i=1}^{r(G')}(s(G'))_i!}(s(G'))_{j-1}(f(G'))_{j-1}\\
&= \frac{n!}{((s(G'))_{j-1}-1)!\prod_{i\in\{1,\dots,r(G')\}\setminus\{j-1\}}(s(G'))_i!}(f(G'))_{j-1}\\
&= \frac{n!}{((s(G))_{j-1})!\prod_{i\in\{1,\dots,r(G)\}\setminus\{j-1,j\}}(s(G))_i!}(f(G))_j\\
&= \frac{n!}{\prod_{i\in\{1,\dots,r(G)\}}(s(G))_i!}(s(G))_j\cdot(f(G))_j\\
&= \lambda_G\cdot(s(G))_j\cdot(f(G))_j.
\end{aligned}$$

The first two and last two equalities are obtained by replacing known expressions and simple calculations. The third equality comes from the fact that $(s(G'))_i = (s(G))_i$ for every $i \leq j-2$, $(s(G'))_{j-1} = (s(G))_{j-1} + 1$, and $(s(G'))_i = (s(G))_{i+1}$ for every $i \geq j$. Moreover, observe that for each $G \in \mathcal{G}_n^T$ and for each $j \in \{i(G), \dots, r(G)\}$, there exists exactly one $G' \in \mathcal{G}_n^T$ such that there is a j-transition from G to G' (if $(s(G))_j = 1$) or a $(j+1)$-transition from G' to G (if $(s(G))_j \geq 2$). Therefore,

$$\sum_{G \in L_n} \sum_{i=i(G)}^{r(G)} \lambda_G \cdot (s(G))_i \cdot (f(G))_i = \sum_{G \in R_n} \sum_{i=i(G)}^{r(G)} \lambda_G \cdot (s(G))_i \cdot (f(G))_i. \tag{4}$$

We now derive two important sets of inequalities. From the fact that f is a selection mechanism, for each $G \in \mathcal{G}_n^T$ we have that

$$\sum_{i=i(G)}^{r(G)} (s(G))_i \cdot (f(G))_i \leq 1, \tag{5}$$

where replacing 1 by $i(G)$ on the left-hand side is possible since it can only make the sum smaller and thus keeps the inequality. On the other hand, from the fact that f satisfies weak unanimity, for each $G \in \mathcal{G}_n^T$ we have that

$$-\sum_{i=i(G)}^{r(G)} (s(G))_i \cdot (f(G))_i \leq -1, \tag{6}$$

where we can omit the term for $i = 1$ on the left-hand side whenever $(s(G))_1 = 1$, because the vertex in $(S(G))_1$ has indegree 0 in such case.

In order to cancel out the left-hand sides of the previous inequalities, we assign a sign to each part of the bipartition of $\mathcal{H}_n$. Let $\text{sign}(L_n), \text{sign}(R_n) \in \{-1, 1\}$ with $\text{sign}(L_n) \cdot \text{sign}(R_n) = -1$, and let $\text{sign}(G) = \text{sign}(L_n)$ for each $G \in L_n$ and $\text{sign}(G) = \text{sign}(R_n)$ for each $G \in R_n$. Summing up the inequalities (5) multiplied by λ_G for every $G \in \mathcal{G}_n^T$ with $\text{sign}(G) = 1$ and the inequalities (6) multiplied by λ_G for every $G \in \mathcal{G}_n^T$ with $\text{sign}(G) = -1$, we obtain

$$\sum_{G \in \mathcal{G}_n^T} \sum_{i=i(G)}^{r(G)} \text{sign}(G) \cdot \lambda_G \cdot (s(G))_i \cdot (f(G))_i \leq \sum_{G \in \mathcal{G}_n^T} \text{sign}(G) \cdot \lambda_G.$$

By Equation 4 the left-hand side is equal to 0. However, we know from Lemma 4 that $\sum_{G \in \mathcal{G}_n^T} \lambda_G$ is odd, so the right-hand side cannot be equal to 0. For one of the two possible choices of $\text{sign}(L_n)$ and $\text{sign}(R_n)$ the right-hand side is negative, and we obtain a contradiction. We conclude that a randomized selection mechanism f satisfying impartiality and weak unanimity on $\mathcal{G}_n$ cannot exist. □

As an illustration, the counterexamples constructed for $n = 3$ and $n = 4$ are shown in Figure 9 in Appendix E.

We are now ready to prove Theorems 2 and 3. In order to be able to apply Lemma 6 to deterministic mechanisms, we need a simple definition. For a given deterministic selection mechanism $f : \mathcal{G}_n \to 2^N$, let $f_{\text{rand}} : \mathcal{G}_n \to [0,1]^n$ be the randomized selection mechanism such that $(f_{\text{rand}}(G))_v = 1$ if $v \in f(G)$ and $(f_{\text{rand}}(G))_v = 0$ otherwise. It is then easy to see that whenever f is impartial, f_{rand} is impartial as well.

Proof of Theorem 2. The result is straightforward when $n = 1$. Let $n \geq 2$ and $k \leq n-1$, and suppose that there is an impartial deterministic selection mechanism f_k with

$$\Delta(G) - \delta^-(f_k(G), G) \leq k - 1$$

Algorithm 2: Selection mechanism f based on f_k.

Input: Digraph $G = (N, E) \in \mathcal{G}_{k+1}$, mechanism f_k and integer $n \geq k + 1$.
Output: Set S of selected vertices with $|S| \leq 1$.
Let $H = (N \cup N', E)$, where $N' = \bigcup_{j=1}^{n-k-1} \{u_j\}$;
Return $f_k(H)$

Algorithm 3: Selection mechanism f based on f_k^+.

Input: Digraph $G = (N, E) \in \mathcal{G}_k$, mechanism f_k^+ and integer $n \geq k + 1$.
Output: Set S of selected vertices with $|S| \leq 1$.
Let $H = (N \cup N', F)$, where $N' = \bigcup_{j=1}^{n-k} \{u_j\}$ and
$F = E \cup (N' \times N) \cup (\{v \in N : \delta^+(v, G) = 0\} \times \{u_1\})$;
Return $f_k^+(H)$

for every $G \in \mathcal{G}_n(k)$. We define the deterministic selection mechanism f based on f_k as specified in Algorithm 2. This mechanism receives a graph $G = (N, E)$ in $\mathcal{G}_{k+1}$ and adds new vertices, if necessary to complete n vertices. These vertices are isolated, in the sense that the set of edges in this new graph H remains the same. For every input graph G, the graph constructed belongs to $\mathcal{G}_n(k)$, so the mechanism finally applies f_k. We claim that f_{rand} is impartial and weakly unanimous on $\mathcal{G}_{k+1}^T$. To see that f_{rand} is weakly unanimous, observe that

$$\delta^-(v, H) = \delta^-(v, G) \text{ for every } v \in N, \text{ and } \delta^-(v, H) = 0 \text{ for every } v \notin N.$$

For each $G \in \mathcal{G}_{k+1}^T$, we have that $\Delta(G) = k$ and thus $\Delta(H) = k$. From the fact that f_k is $(k-1)$-additive on $\mathcal{G}_n(k)$, we conclude that f returns a vertex v^* of H with $\delta^-(v^*, H) \geq k-(k-1) = 1$. This implies, in the first place, that $v^* \in N$, thus f is indeed a selection mechanism on $\mathcal{G}_{k+1}^T$. Furthermore, we have $\delta^-(v^*, G) \geq 1$, thus

$$\sum_{v \in G : \delta^-(v) \geq 1} (f_{\text{rand}}(G))_v = 1,$$

i.e., f_{rand} is weakly unanimous on $\mathcal{G}_{k+1}^T$. Impartiality of f_{rand} is straightforward since f_k is impartial and the set of edges is not modified in the mechanism. This contradicts Lemma 6, so we conclude that mechanism f_k cannot exist. □

Proof of Theorem 3. The result is straightforward when $n \in \{1, 2\}$. Let $n \in \mathbb{N}, n \geq 3$ and suppose that there is an impartial deterministic selection mechanism f_k^+ with

$$\Delta(G) - \delta^-(f_k^+(G), G) \leq k - 2$$

for every $G \in \mathcal{G}_n^+(k)$. We define the deterministic selection mechanism f based on f_k^+ as specified in Algorithm 3. This mechanism requires a graph G in $\mathcal{G}_k$ and adds new vertices $N' = \{u_1, \ldots, u_{n-k}\}$, as well as edges from each of these vertices to every vertex of G, and edges from every vertex of G with outdegree zero to u_1. We first claim that for every input graph G, the graph H constructed in the mechanism belongs to $\mathcal{G}_n^+(k)$. Indeed, since $G \in \mathcal{G}_k$ every node $v \in N$ satisfies $\delta^+(v, G) \leq k - 1$. Moreover, an outgoing edge to u_1 is added for every v with $\delta^+(v, G) = 0$, thus $1 \leq \delta^-(v, H) \leq k - 1$ for each $v \in N$. On the other hand, each node in N' has outdegree k. Therefore, the mechanism is well defined, in the sense that in its last step it applies f_k^+ to a graph in $\mathcal{G}_n^+(k)$. We claim that f_{rand} is impartial and weakly unanimous on $\mathcal{G}_k^T$, which is a clear contradiction to Lemma 6 and thus implies that mechanism f_k^+ cannot exist. We prove the claim in what follows.

For each $G = (N, E) \in \mathcal{G}_k^T$, the set $\{v \in N : \delta^+(v) = 0\}$ is equal to $(S(G))_{r(G)}$ if $(s(G))_{r(G)} = 1$, or to the empty set, otherwise. Therefore, for $u_1, \dots, u_{n-k}$ and H as defined in the mechanism we have that $\delta^-(u_1, H) \leq 1$, $\delta^-(u_j, H) = 0$ for every $j \in \{2, \dots, n-k\}$, and $\delta^-(v, H) = \delta^-(v, G) + 1$ for every $v \in N$. Since $\Delta(G) = k - 1$ from the definition of the set $\mathcal{G}_k^T$, we have that $\Delta(H) = k$. Using that f_k^+ is $(k-2)$-additive on $\mathcal{G}_n^+(k)$, we conclude that f returns a vertex $v^* \in N \cup N'$ with $\delta^-(v^*, H) \geq k - (k-2) = 2$. This implies, in the first place, that $v^* \in N$, thus f is indeed a selection mechanism on $\mathcal{G}_k^T$. Furthermore, we have $\delta^-(v^*, G) \geq 1$, thus

$$\sum_{v \in G : \delta^-(v) \geq 1} (f_{\text{rand}}(G))_v = 1,$$

i.e., f_{rand} is weakly unanimous on $\mathcal{G}_k^T$.

To see that f is impartial on $\mathcal{G}_k^T$, let $G = (N, E), G' = (N, E') \in \mathcal{G}_k^T$ and $\tilde{v} \in N$ be such that $E \setminus (\{\tilde{v}\} \times N) = E' \setminus (\{\tilde{v}\} \times N)$. Denoting F and F' the edges of the graphs defined in the mechanism f when run with input G and G', respectively, it is enough to show that $F \setminus (\{\tilde{v}\} \times (N \cup N')) = F' \setminus (\{\tilde{v}\} \times (N \cup N'))$, because this would imply

$$f(G) \cap \{\tilde{v}\} = f_k^+(N \cup N', F) \cap \{\tilde{v}\} = f_k^+(N \cup N', F') \cap \{\tilde{v}\} = f(G') \cap \{\tilde{v}\}$$

where the second equality holds since f_k^+ is impartial on $\mathcal{G}_n^+(k)$ by hypothesis. Indeed,

$$\begin{aligned}
F \setminus (\{\tilde{v}\} \times (N \cup N')) &= (E \cup (N' \times N) \cup (\{v \in N : \delta^+(v, G) = 0\} \times \{u_1\})) \setminus (\{\tilde{v}\} \times (N \cup N')) \\
&= E \setminus (\{\tilde{v}\} \times N) \cup (N' \times N) \cup (\{v \in N \setminus \{\tilde{v}\} : \delta^+(v, G) = 0\} \times \{u_1\}) \\
&= E' \setminus (\{\tilde{v}\} \times N) \cup (N' \times N) \cup (\{v \in N \setminus \{\tilde{v}\} : \delta^+(v, G') = 0\} \times \{u_1\}) \\
&= (E' \cup (N' \times N) \cup (\{v \in N : \delta^+(v, G') = 0\} \times \{u_1\})) \setminus (\{\tilde{v}\} \times (N \cup N')) \\
&= F' \setminus (\{\tilde{v}\} \times (N \cup N')),
\end{aligned}$$

where the third equality comes the fact that the outgoing edges of every vertex in $(N \cup N') \setminus \{\tilde{v}\}$ are the same in G and G'. This implies that f_{rand} is impartial as well, concluding the proof of the claim and the proof of the theorem. $\square$

Theorems 2 and 3 provide tight bounds for the approval setting but have very weak implications for plurality. We end with a small but nontrivial lower bound for the latter, which applies to settings with and without abstentions. The proof of this result can be found in Appendix F

Theorem 4. *Let $n \in \mathbb{N}$ and let f be an impartial deterministic selection mechanism such that f is α-additive on $\mathcal{G}^+(1)$. Then, $\alpha \geq 3$.*

A Proof of Lemma 1

Let G and v be as defined in the statement of the lemma. We prove the existence of vertices as claimed by induction over j. For the base case, suppose that for every $u \in N^-(v)$ it holds $(\delta^\star(u), u) < (\delta^-(v), v)$. From the definition of the mechanism we thus have that $i^\star(u, G) > i^\star(v, G)$ for every $u \in N^-(v)$ and therefore $\delta^\star(v) = \delta^-(v)$, which contradicts the hypothesis of the lemma.

Now, let $j' \in \{0, \dots, r-2\}$ and assume that for every $j \in \{0, \dots, j'\}$ there is a vertex u_j such that $(u_j, v) \in E$ and $(\delta^\star(u_j), u_j) > (\delta^-(v) - j, v)$. Suppose that for every vertex $u \in N^-(v) \setminus \{u_1, \dots, u_{j'}\}$ it holds $(\delta^\star(u), u) < (\delta^-(v) - (j'+1), v)$. The expression $\delta^-(v) - (j'+1)$ is exactly the indegree of v after deleting the incoming edges from $u_0, \dots, u_{j'}$, thus from the definition of the mechanism we have that $i^\star(u) > i^\star(v)$ for every $u \in N^-(v) \setminus \{u_0, \dots, u_{j'}\}$. This yields

$$\delta^\star(v) \geq \delta^-(v) - (j'+1) \geq \delta^-(v) - (r-1),$$

implying $r \geq \delta^-(v) - \delta^\star(v) + 1$. In the case $v > z$, this is equivalent to $d \leq \delta^\star(v)$, but these two inequalities contradict $(d, z) \geq (\delta^\star(v), v)$. If $v < z$, on the other hand, it is equivalent to $d \leq \delta^\star(v) - 1$, which is again a contradiction. This concludes the existence of vertices $u_0, \dots, u_{r-1}$ as in the statement of the lemma.

To prove the last claim, we fix $(d, z) = (\delta^\star(v), v)$. We know that taking $r = \delta^-(v) - \delta^\star(v)$ there are vertices such that for each $j \in \{0, 1, \dots, r-1\}$, $(u_j, v) \in E$ and $(\delta^\star(u_j), u_j) > (\delta^-(v) - j, v)$. Suppose that there is a vertex $u \in N^-(v) \setminus \{u_0, \dots, u_{r-1}\}$ with $(\delta^\star(u), u) > (\delta^\star(v), v)$. It is clear that

$$|\{u \in N^-(v) : (\delta^\star(u), u) > (\delta^\star(v), v)\}| \geq r + 1,$$

thus by expressions (1) and (2) we conclude that $\delta^-(v) - \delta^\star(v) \geq r + 1$, which is a contradiction.

B Proof of Lemma 2

Since $E_1 \setminus (\{u\} \times N) = E_2 \setminus (\{u\} \times N)$, it is clear that

$$\delta^-(v, G_2) \geq \delta^-(v, G_1) - 1 = d + r + (\delta^-(v, G_1) - \delta^\star(v, G_1)) - 1 - \chi(v > z). \tag{7}$$

In the following, we denote $r' = r + (\delta^-(v, G_1) - \delta^\star(v, G_1))$ for ease of notation. If $\delta^-(v, G_2) = \delta^-(v, G_1)$, then $r' = \delta^-(v, G_2) - d + \chi(v > z)$. In addition,

$$(\delta^-(v, G_2), v) = (\delta^-(v, G_1), v) > (d, z) \geq (\delta^\star(v, G_2), v),$$

thus from Lemma 1 there are vertices $u'_0, \dots, u'_{r'-1}$ such that for every $j \in \{0, \dots, r'-1\}$ we have that $(u'_j, v) \in E_1 \cap E_2$—since $\delta^-(v, G_2) = \delta^-(v, G_1)$ we have $E_1 \cap N^-(v) = E_2 \cap N^-(v)$—and $(\delta^\star(u'_j, G_2), u'_j) > (\delta^-(v, G_2) - j, v)$. From Equation 3, we know that

$$|\{u \in \{u'_0, \dots, u'_{r'-1}\} : (\delta^\star(u, G_1), u) > (\delta^\star(v, G_1), v)\}| \leq \delta^-(v, G_1) - \delta^\star(v, G_1),$$

thus it is possible to take $\{u_0, \dots, u_{r-1}\} \subseteq \{u'_0, \dots, u'_{r'-1}\}$ such that $(\delta^\star(u_j, G_1), u_j) < (\delta^\star(v, G_1), v)$ also holds for $j \in \{0, \dots, r-1\}$. Moreover,

$$(\delta^\star(u_j, G_2), u_j) > (\delta^-(v, G_2) - j - ((\delta^-(v, G_1) - \delta^\star(v, G_1))), v) = (\delta^\star(v, G_1) - j, v). \tag{8}$$

Relabeling these vertices so that $(\delta^\star(u_0, G_2), u_0) > \cdots > (\delta^\star(u_{r-1}, G_2), u_{r-1})$ it is clear that the previous inequalities still hold, and in particular $(\delta^\star(u_0, G_2), u_0) > (\delta^\star(v, G_1), v) > (\delta^\star(u_0, G_1), u_0)$. We conclude that Condition (ii) holds.

Similarly, if $\delta^-(v, G_2) = \delta^-(v, G_1) + 1$, then $r' + 1 = \delta^-(v, G_2) - d + \chi(v > z)$. As before, $(\delta^-(v, G_2), v) > (d, z) \geq (\delta^\star(v, G_2), v)$, thus from Lemma 1 there are vertices $u'_0, \dots, u'_{r'}$ such that for every $j \in \{0, \dots, r'\}$ we have that $(u'_j, v) \in E_2$ and $(\delta^\star(u'_j, G_2), u'_j) > (\delta^-(v, G_2) - j, v)$. But $|(E_2 \cap N^-(v)) \setminus (E_1 \cap N^-(v))| = 1$, thus at least r' of these vertices u'_j are such that $(u'_j, v) \in E_1 \cap E_2$. As before, from Equation 3 we conclude that it is possible to take $\{u_0, \dots, u_{r-1}\} \subseteq \{u'_0, \dots, u'_{r'}\}$ such that $(u_j, v) \in E_1 \cap E_2$ and $(\delta^\star(u_j, G_1), u_j) < (\delta^\star(v, G_1), v)$ for every $j \in \{0, \dots, r-1\}$. Since Equation 8 still holds, relabeling the vertices such that $(\delta^\star(u_0, G_2), u_0) > \cdots > (\delta^\star(u_{r-1}, G_2), u_{r-1})$ we conclude the lemma with Condition (ii) as well. An example for each of the cases addressed so far are included in Figure 6.

In what follows, we suppose that the first inequality in Condition (7) is an equality, so $(\tilde{v}, v) \in E_1 \setminus E_2$ and $(E_2 \cap N^-(v)) \subset (E_1 \cap N^-(v))$. In this case, $r' - 1 = \delta^-(v, G_2) - d + \chi(v > z)$. Suppose first that $(\delta^-(v, G_1) - 1, v) > (d, z)$, thus

$$(\delta^-(v, G_2), v) = (\delta^-(v, G_1) - 1, v) > (d, z) \geq (\delta^\star(v, G_2), v).$$

In this case, from Lemma 1 there are vertices $u'_0, \dots, u'_{r'-2}$ such that for every $j \in \{0, \dots, r'-2\}$ we have that $(u'_j, v) \in E_1 \cap E_2$ and $(\delta^\star(u'_j, G_2), u'_j) > (\delta^-(v, G_2) - j, v)$. If $(\delta^\star(\tilde{v}, G_1), \tilde{v}) > (\delta^\star(v, G_1), v)$, from Equation 3 we have that

$$|\{u \in \{u'_0, \dots, u'_{r'-2}\} : (\delta^\star(u, G_1), u) > (\delta^\star(v, G_1), v)\}| \leq \delta^-(v, G_1) - \delta^\star(v, G_1) - 1,$$

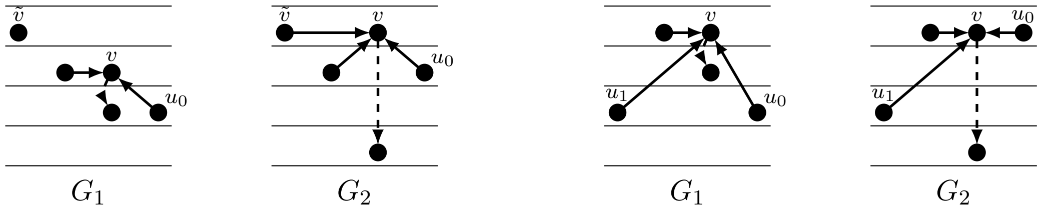

Figure 6: Illustration of Lemma 2 for the case $\delta^-(v, G_2) = \delta^-(v, G_1) + 1$, shown on the left, and for the case $\delta^-(v, G_2) = \delta^-(v, G_1)$, shown on the right. In contrast to Figure 2 only the initial iteration $i = 0$ is shown, and the overall drop in the indegree of v is illustrated by a dashed arrow. Observe that $(\delta^\star(u_0, G_2), u_0) > (\delta^\star(v, G_1), v) > (\delta^\star(u_0, G_1), u_0)$ and $(\delta^\star(u_1, G_2), u_1) > (\delta^\star(v, G_1) - 1, v)$, $(\delta^\star(u_1, G_1), u_1) < (\delta^\star(v, G_1), v)$..

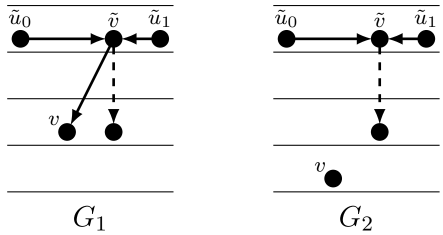

Figure 7: Illustration of Lemma 2 for the case where $\delta^-(v, G_2) = \delta^-(v, G_1) - 1$ with $(\delta^\star(v), v) < (\delta^-(\tilde{v}, G_1), \tilde{v})$. Only the initial iteration $i = 0$ is shown, and the overall drop in the indegree of $\tilde{v}$ is illustrated by a dashed arrow. Observe that $(\delta^\star(\tilde{u}_j, G_1), \tilde{u}_j) > (\delta^-(\tilde{v}, G_1) - j, \tilde{v})$ for $j \in \{0, 1\}$.

Therefore, we conclude once again that it is possible to take $\{u_0, \ldots, u_{r-1}\} \subseteq \{u'_0, \ldots, u'_{r'-2}\}$ such that $(\delta^\star(u_j, G_1), u_j) < (\delta^\star(v, G_1), v)$ for every $j \in \{0, \ldots, r-1\}$. We also have that

$$(\delta^\star(u_j, G_2), u_j) > (\delta^-(v, G_2) - j - (\delta^-(v, G_1) - \delta^\star(v, G_1) - 1), v) = (\delta^\star(v, G_1) - j, v),$$

so after relabeling the vertices with $(\delta^\star(u_0, G_2), u_0) > \cdots > (\delta^\star(u_{r-1}, G_2), u_{r-1})$ Condition (ii) follows again. On the other hand, if $(\delta^\star(\tilde{v}, G_1), \tilde{v}) < (\delta^\star(v, G_1), v)$, from Equation 3

$$|\{u \in \{u'_0, \ldots, u'_{r'-2}\} : (\delta^\star(u, G_1), u) > (\delta^\star(v, G_1), v)\}| \leq \delta^-(v, G_1) - \delta^\star(v, G_1).$$

Therefore, it is possible to take $\{u_1, \ldots, u_{r-1}\} \subseteq \{u'_0, \ldots, u'_{r'-2}\}$ such that both $(\delta^\star(u_j, G_2), u_j) > (\delta^\star(v, G_1) - j, v)$ and $(\delta^\star(u_j, G_1), u_j) < (\delta^\star(v, G_1), v)$ hold for every $j \in \{1, \ldots, r-1\}$. In addition, whenever $(\delta^\star(v, G_1), v) < (\delta^-(\tilde{v}, G_1), \tilde{v})$ we know from Lemma 1 that, taking $\tilde{r}' = \delta^-(\tilde{v}, G_1) - \delta^\star(\tilde{v}, G_1) + \chi(\tilde{v} > v)$, there exist vertices $\tilde{u}_0, \ldots, \tilde{u}_{\tilde{r}'-1}$, different than $\tilde{v}$, such that for every $j \in \{0, \ldots, \tilde{r}'-1\}$ it holds $(\tilde{u}_j, \tilde{v}) \in E$ and $(\delta^\star(\tilde{u}_j, G_1), \tilde{u}_j) > (\delta^-(\tilde{v}, G_1) - j, \tilde{v})$. Taking the first $\tilde{r} = \tilde{r}' - (\delta^\star(v, G_1) - \delta^\star(\tilde{v}, G_1))$ such vertices and $u_0 = \tilde{v}$, Condition (i) follows.

Finally, if $\delta^-(v, G_2) = \delta^-(v, G_1) - 1$ and $(\delta^-(v, G_1) - 1, v) < (d, z)$, we obtain from the inequality $(d, z) < (\delta^\star(v, G_1), v)$ that $\delta^\star(v, G_1) = \delta^-(v, G_1)$ and $r = 1$. Therefore, we must have $(\delta^\star(\tilde{v}, G_1), \tilde{v}) < (\delta^\star(v, G_1), v)$, and the same analysis above leads to the existence of vertices $\tilde{u}_0, \ldots, \tilde{u}_{\tilde{r}-1}$, different than $\tilde{v}$, such that for every $j \in \{0, \ldots, \tilde{r} - 1\}$ it holds $(\tilde{u}_j, \tilde{v}) \in E$ and $(\delta^\star(\tilde{u}_j, G_1), \tilde{u}_j) > (\delta^-(\tilde{v}, G_1) - j, \tilde{v})$. Condition (i) follows in this case as well.

C Proof of Lemma 4

Let $\mathcal{G}'_n = \{G \in \mathcal{G}_n^T : (s(G))_i = (s(G))_{r(G)+1-i}$ for every $i \in \{1, \ldots, r(G)\}\}$ be the set of graphs $G \in \mathcal{G}_n^T$ such that the tuple $s(G)$ is symmetric. From the definition of $\mathcal{G}_n^T$, it is clear that for every graph G with a non-symmetric tuple $s(G)$, i.e., for every graph $G \in \mathcal{G}_n^T \setminus \mathcal{G}'_n$, there exists a unique $H \in \mathcal{G}_n^T \setminus (\mathcal{G}_n' \cup \{G\})$ such that $r(G) = r(H) =: r$ and $(s(G))_i = (s(H))_{r+1-i}$ for every $i \in \{1, \dots, r\}$. Moreover, $\lambda_G = \lambda_H$ for such a pair of graphs. This implies that

$$\sum_{G \in \mathcal{G}_n^T \setminus \mathcal{G}_n'} \lambda_G$$

is even. In what follows we show how to conclude the lemma using the following claim: for every $G \in \mathcal{G}_n'$ with $r(G) \geq 2$, λ_G is even. Observe that

$$\sum_{G \in \mathcal{G}_n^T} \lambda_G = \sum_{G \in \mathcal{G}_n^T \setminus \mathcal{G}_n'} \lambda_G + \sum_{G \in \mathcal{G}_n' : r(G) \geq 2} \lambda_G + \sum_{G \in \mathcal{G}_n' : r(G) = 1} \lambda_G.$$

If the claim is true, we have that the first two sums on the right-hand side are even. The third sum only contains one term, namely λ_G for the complete graph G, for which $s(G) = (n)$ and thus $\lambda_G = n!/n! = 1$. Therefore, $\sum_{G \in \mathcal{G}_n^T} \lambda_G$ is the sum of an even term plus 1, and we conclude the result.

We now prove the claim. Let $G \in \mathcal{G}_n'$ with $r(G) \geq 2$. The multiplicity of the prime factor 2 in the numerator of λ_G, due to Legendre's formula, is simply

$$\sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{2^\ell} \right\rfloor,$$

while its multiplicity in the denominator is

$$\sum_{i=1}^{r(G)} \sum_{\ell=1}^{\infty} \left\lfloor \frac{(s(G))_i}{2^\ell} \right\rfloor.$$

Therefore, we need to prove that this last term is strictly lower than the former. It is easy to see that for every $\ell \in \mathbb{N},\ \ell \geq 1$ we have $\sum_{i=1}^{r(G)} \left\lfloor \frac{(s(G))_i}{2^\ell} \right\rfloor \leq \left\lfloor \frac{n}{2^\ell} \right\rfloor$, since without the floor functions we would have the equality, and the fractional parts of the terms $(s(G))_i/2^\ell$ must add up —when summing over i— to at least the fractional part of $n/2^\ell$. We now show that there exists some ℓ' for which this inequality is strict, by distinguishing two cases. If $(s(G))_i \leq n/2$ for every $i \in \{1, \dots, r(G)\}$, we let $\ell' \in \mathbb{N}$ such that $2^{\ell'} \leq n < 2^{\ell'+1}$, and then the following holds:

$$\sum_{i=1}^{r(G)} \left\lfloor \frac{(s(G))_i}{2^{\ell'}} \right\rfloor \leq \sum_{i=1}^{r(G)} \left\lfloor \frac{n/2}{2^{\ell'}} \right\rfloor = \sum_{i=1}^{r(G)} \left\lfloor \frac{n}{2^{\ell'+1}} \right\rfloor = 0 < 1 = \left\lfloor \frac{n}{2^{\ell'}} \right\rfloor.$$

On the other hand, if $(s(G))_{i'} > n/2$ for some $i' \in \{1, \dots, r(G)\}$, from the symmetry of $s(G)$ we have that $r(G)$ is odd and, moreover, $i' = (r(G)+1)/2$. We now define $\ell' \in \mathbb{N}$ such that $2^{\ell'} \leq n - (s(G))_{i'} < 2^{\ell'+1}$, which implies $(s(G))_i \leq \frac{n-(s(G))_{i'}}{2} < 2^{\ell'}$ for every $i \neq i'$. Therefore, the following holds:

$$\sum_{i=1}^{r(G)} \left\lfloor \frac{(s(G))_i}{2^{\ell'}} \right\rfloor = \left\lfloor \frac{(s(G))_{i'}}{2^{\ell'}} \right\rfloor \leq \left\lfloor \frac{n - 2^{\ell'}}{2^{\ell'}} \right\rfloor = \left\lfloor \frac{n}{2^{\ell'}} - 1 \right\rfloor < \left\lfloor \frac{n}{2^{\ell'}} \right\rfloor.$$

In either case, we obtain that

$$\sum_{i=1}^{r(G)} \sum_{\ell=1}^{\infty} \left\lfloor \frac{(s(G))_i}{2^\ell} \right\rfloor < \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{2^\ell} \right\rfloor,$$

and thus λ_G is even, which concludes the proof of the claim and the proof of the lemma.

D Proof of Lemma 5

To see that f_s is impartial, let $G = (N, E)$, $G' = (N, E') \in \mathcal{G}_n$ and $v \in N$ such that $E \setminus (\{v\} \times N) = E' \setminus (\{v\} \times N)$. Since f is impartial,

$$(f_s(G))_v = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} (f(G_\pi))_{\pi_v} = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} (f(G'_\pi))_{\pi_v} = (f_s(G'))_v,$$

and thus f_s is impartial.

To prove that f_s satisfies weak unanimity, let $G = (N, E) \in \mathcal{G}_n$ and $v \in N$ with $\delta^-(v) = n-1$. Since f satisfies weak unanimity,

$$\begin{aligned}\sum_{u \in N : \delta^-(u) \geq 1} (f_s(G))_u &= \sum_{u \in N : \delta^-(u) \geq 1} \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} (f(G_\pi))_{\pi_u} = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \sum_{u \in N : \delta^-(u) \geq 1} (f(G_\pi))_{\pi_u} \\ &\geq \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} 1 = 1.\end{aligned}$$

This concludes the proof of the lemma.

E Illustrations for the Proof of Lemma 6

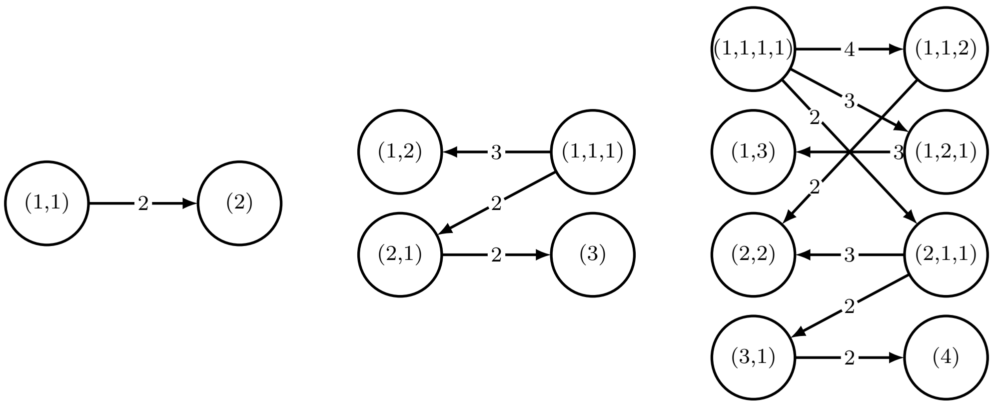

Figure 8: Directed versions of $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. For $n \in \{2, 3, 4\}$, each graph $G \in \mathcal{G}_n^T$ is represented by the tuple $s(G)$ and each edge (G, G') labeled with $j \in \mathbb{N}$ represents a j-transition from G to G'.

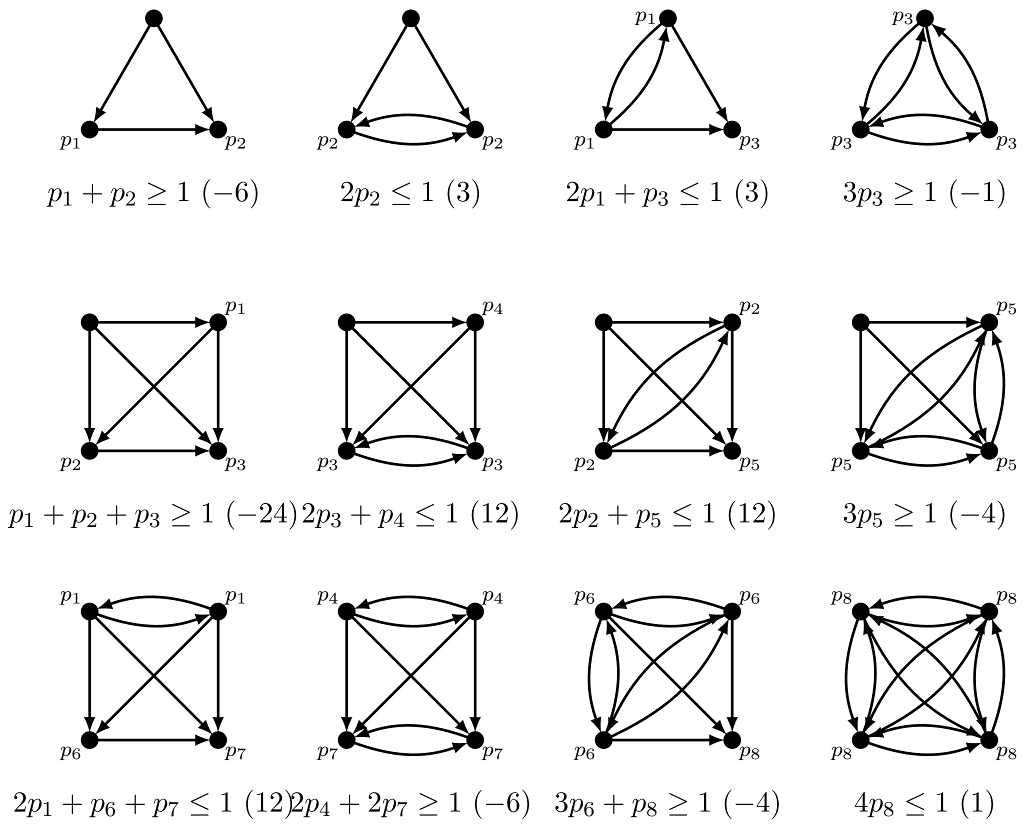

Figure 9: Counterexamples for the existence of a randomized selection mechanism satisfying impartiality and weak unanimity for $n = 3$ and for $n = 4$. The graphs are depicted with the impartial probabilities assigned to each vertex with indegree at least one. The inequalities included below each of them are obtained from imposing either that the probabilities sum up to at most 1—since at most one vertex is to be selected—or sum up to at least 1—due to weak unanimity. The values in parentheses show that the inequalities written for each set of graphs constitute an infeasible system because if one multiplies the corresponding inequality by them and sums the resulting inequalities, the contradiction $0 \leq -1$ is achieved.

F Proof of Theorem 4

Suppose that there is a randomized selection mechanism f that, with probability 1, selects a vertex with indegree at least $\Delta(G) - 2$ for every graph G with $\Delta(G) \geq 3$. Consider the seven graphs in Figure 10 with $n = 12$. It is easily verified that f must assign probabilities $(f(G))_v$ as in the figure. The inequalities below the graphs are obtained from imposing this additive bound and from the fact that the probabilities assigned to the vertices of a single graph have to add up to at most one. From the inequalities for the first three and last three graphs,

$$p_1 \leq \frac{1}{6}, \quad p_2 \leq \frac{1}{4}, \quad p_3 \leq \frac{1}{2}, \quad p_4 = p_5 = 0,$$

which implies that

$$p_1 + p_2 + p_3 + p_4 + 4p_5 \leq \frac{11}{12}.$$

This contradicts the inequality for the fourth graph. We conclude that mechanism f cannot exist.

Suppose now that there is an impartial deterministic selection mechanism f which is α-additive on $\mathcal{G}^+(1)$ for $\alpha \leq 2$. It is clear that f_{rand} is also impartial and, moreover, for every graph $G = (N, E) \in \mathcal{G}^+(1)$ with $\Delta(G) \geq 3$,

$$\sum_{v \in N : \delta^-(v) \geq \Delta - 2} (f_{\text{rand}}(G))_v = 1$$

since $f(G) = v^*$ with $\delta^-(v^*) \geq \Delta - 2$. But this contradicts the non-existence of such a randomized selection mechanism, thus we conclude that $\alpha \geq 3$.

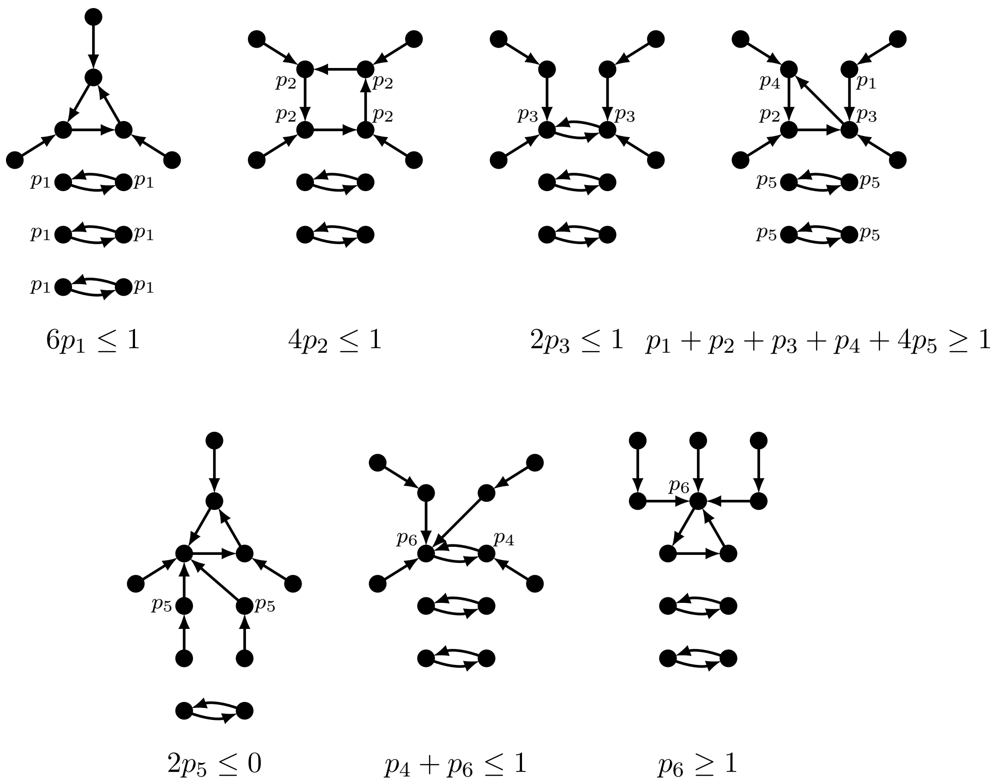

Figure 10: Counterexample to the existence of an impartial randomized selection mechanism with $\alpha \leq 2$ for the plurality setting and $n = 12$.

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