The Computational Complexity of Choice Sets

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Social choice rules are often evaluated and compared by inquiring whether they satisfy certain desirable criteria such as the *Condorcet criterion*, which states that an alternative should always be chosen when more than half of the voters prefer it over any other alternative. Many of these criteria can be formulated in terms of choice sets that single out reasonable alternatives based on the preferences of the voters. In this paper, we consider choice sets whose definition merely relies on the pairwise majority relation. These sets include the *Copeland set*, the *Smith set*, the *Schwartz set*, *von Neumann-Morgenstern stable sets*, the *Banks set*, and the *Slater set*. We investigate the relationships between these sets and completely characterize their computational complexity, which allows us to obtain hardness results for entire classes of social choice rules.

1 Introduction

Given a profile of individual preferences over a number of alternatives, the simple majority rule—choosing the alternative which the majority of agents prefer over the other alternative—is an attractive way of aggregating social preferences over any pair of alternatives. It has an intuitive appeal to democratic principles, is easy to understand and, most importantly, satisfies some attractive formal properties. May's theorem shows that a number of rather weak and intuitively acceptable principles completely characterize the majority rule in settings with two alternatives [1]. Moreover, almost all common social choice rules satisfy May's axioms and thus coincide with the majority rule in the two-alternative case. Thus, it would seem that the existence of a majority of individuals preferring alternative a to alternative b signifies something fundamental and generic about the group's preferences over a and b. We will say that alternative *a dominates* alternative b in such a case.

Based on the simple majority rule, this dominance relation is obviously *asymmetric* in the strong sense that a dominating b implies that b does not dominate a. Consequently, the dominance relation is also irreflexive, i.e., no alternative dominates itself. Conversely, any asymmetric binary relation on the set of alternatives is the dominance relation of some preference profile, provided that the number of voters is large enough compared to the number of alternatives [2]. In particular, as is well known from Condorcet's paradox [3], the dominance relation may contain cycles. This implies that the dominance relation need not have a maximum, and in fact not even a maximal, element, even if the underlying individual preferences do. Hence, the concept of maximality is rendered untenable in most cases.

There are several ways to get around this problem. One of them is, of course, to abandon the simple majority rule altogether. We will not consider such attempts here. Another would be to take more structure of the underlying individual preference profiles into account. We will not consider these here either. A third way out is to take the dominance relation as given and define alternative concepts to take over the role of the maximality. We will therefore be concerned with criteria for social choice correspondences that are based on the dominance relation only, i.e., those that Fishburn [4] called $C1$ functions. Formally, by a $C1$ social choice concept we will understand a concept that is invariant under changes to preference profiles that do not affect the dominance relation. Examples of such concepts are *the Condorcet winner*, defined as the alternative, if any, that dominates all other alternatives. Other examples are:

- the *Copeland set*, i.e., the set of all alternatives for which the difference between the number of alternatives it dominates and the number of alternatives that it is dominated by is maximal,

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- the *Smith set*, i.e., the smallest set of alternatives that dominate any alternative not in the set,
- the *Schwartz set*, i.e., the union of all minimal sets of alternatives that are not dominated by any alternative outside that set,
- *von Neumann-Morgenstern stable sets*, i.e., any set U consisting precisely of those alternatives that are not dominated by any alternative in U,
- the *Banks set*, i.e., the set of maximal elements of inclusion-maximal subsets of alternatives for which the dominance relations is transitive and complete,
- the *Slater set*, i.e., the set of undominated elements of those acyclic relations that share as many tuples with the original dominance relation as possible.

The literature on social choice theory often mentions that one choice rule is "more difficult to compute" than another. The main goal of this paper is to provide formal grounds for such statements and, in particular, to obtain lower bounds for the computational complexity of entire classes of choice functions. This approach is inspired by Bartholdi, III et al. [5], who proved the NP-hardness of any social welfare functional that is neutral, consistent, and satisfies the Condorcet criteron. They admit that "since only the Kemeny rule satisfies the hypotheses, this corollary is not entirely satisfying" [5]. Over the last few years, the computational complexity of various existing voting rules, such as Dodgson's, Kemeny's, or Young's, has been completely characterized [see, e.g., 6]. We are, however, unaware of hardness results regarding broader classes of rules.[1]

It is interesting to note that the literature on pairwise majorities in social choice theory strongly focuses on *tournaments*, i.e., asymmetric and complete relations over a set of alternatives. For any odd number of *linear* individual preferences, the dominance relation is indeed a tournament. From a social choice perspective these could be taken as relatively mild and technically convenient restrictions. For one, a tournament is transitive if and only if it is acyclic. Secondly, there can be at most one maximal element in a tournament, and if there is one it is the *Condorcet winner*. From the perspective of computational complexity, however, the restriction to tournaments is not as harmless. We will find that some problems we consider are computationally significantly easier in tournaments than for the general case. Furthermore, in many settings of computational interest, such as webpage ranking, there is usually a large number of alternatives, only a fraction of which is covered by the voters' preferences.

The remainder of this paper is structured as follows. The social choice setting we consider is introduced in Section 2. Section 3 motivates, introduces, and analyzes six choice sets, whose computational complexity is then investigated in Section 4. Section 5 concludes with an overview and interpretation of the results.

2 Preliminaries

We consider a finite set of agents N who choose among a finite set A of alternatives. The cardinalities of these sets will be denoted by n and m, respectively. For each agent $i \in N$ there is a binary preference relation $\succsim_i$ over the alternatives in A. We have $a \succsim_i b$ denote that agent i values alternative a at least as much as alternative b. As usual, we write $\succ_i$ for the strict part of $\succsim_i$, i.e., $a \succ_i b$ if $a \succsim_i b$ but not $b \succsim_i a$. We make no specific structural assumptions about individual preferences, apart from the indifference relation being reflexive and symmetric. Possible preferences thus include all *linear orders*—i.e., reflexive, transitive, complete and anti-symmetric relations—over the alternatives. On the other end of the spectrum, the definition also allows for *incomplete* or *quasi-transitive* preferences.[2]

Given a *preference profile* $(\succsim_i)_{i \in N}$, we say that alternative a *dominates* alternative b, in symbols $a \succ b$, whenever the number of voters i for which $a \succsim_i b$ exceeds the number of voters i for which $b \succsim_i a$. We will sometimes also refer to the weak part of the dominance relation and say that a *weakly dominates* b, in symbols $a \succsim b$, whenever b does not dominate a. Obviously, the (strict) dominance relation is *asymmetric*. Despite the fact that most

[1] Due to the possibility of ties, many common choice *rules* do not always select only one alternative. Thus the distinction between a choice rule and a choice *set*, as a criterion for a choice rule to select its alternatives from, is merely a gradual one.

[2] We say relation $\geq$ is *asymmetric* whenever $x \geq y$ implies $y \not\geq x$. We say $\geq$ is *anti-symmetric* whenever $x \geq y$ and $y \geq x$ imply $x = y$. The relation $\geq$ is *quasi-transitive* if $>$ (the strict part of $\geq$) is transitive.

of the existing literature has focused on *tournaments* [see, e.g., 7, 8, 9, 10], i.e., complete dominance relations, the dominance relation need not in general be *complete*.[3] In fact, McGarvey [2] shows that *any* dominance relation can be realized by a particular preference profile for a number of voters polynomial in m, even if individual preferences are required to be transitive, complete and anti-symmetric. In the presence of *incomplete* or *quasi-transitive* preferences, incomplete dominance relations are the norm rather than just a theoretical possibility. In the remainder of this paper, we will mainly be concerned with dominance relations and tacitly assume appropriate underlying individual preferences.

3 Choice sets

In this section, we motivate and introduce six choice sets based on the pairwise majority dominance relation and analyze the relationships between these sets. We will use the term *choice set* both for the function from the preference profile to a subset of the alternatives as well as the subset itself.

We say that an alternative $a \in A$ is *undominated* relative to $\succ$ whenever there are no alternatives $b \in A$ with $b \succ a$. A *Condorcet winner* is an alternative that dominates every other alternative. The concept of a *maximal element* we reserve in this paper for transitive (and possibly reflexive) relations $\geq$. An alternative $a \in A$ is said to be *maximal* in such a transitive relation if for all $b \in A$ such that $b \geq a$ also $a \geq b$. Equivalently, the maximal elements of $\geq$ can be defined as the undominated elements in the strict (i.e., asymmetric) part of $\geq$.

Given its asymmetry, transitivity of the dominance relation implies its acyclicity. The implication in the other direction holds for tournaments but not for general dominance relations. Failure of transitivity or completeness means that a Condorcet winner need not exist; failure of acyclicity, moreover, that the dominance relation need not even contain maximal elements. As such, the obvious notion of maximality is no longer available to single out the "best" alternatives among which the social choice should be selected. Other concepts had to be devised to take over its role. We will be concerned with six of these concepts: the Copeland set, the Smith set, the Schwartz set, von Neumann-Morgenstern stable sets, the Banks set, and the Slater set.

3.1 Definitions

If a Condorcet winner exists, it is obviously the alternative that dominates the greatest number of alternatives, viz. all but itself, and is dominated by the smallest number, viz. by none. The *Copeland set* varies on this theme by singling out those alternatives that maximize the difference between the number of alternatives they dominate and the number of alternatives they are dominated by [11].

Definition 1 (Copeland score and Copeland set)**.** Given a dominance relation $\succ$ on a set of alternatives A, the *Copeland score* $c(a)$ of an alternative a equals $|\{b \in A : a \succ b\}| - |\{b \in A : b \succ a\}|$. The *Copeland set* C is given by $\{a \in A : c(a) \geq c(b), \text{ for all } b \in A\}$, i.e., the set of alternatives with maximal Copeland score.

Obviously, the Copeland set is always nonempty and contains the Condorcet winner as its only element if there is one.

A nonempty set of alternatives X is a *dominating set* if every alternative in X dominates every alternative not in X, i.e., if $x \succ y$ holds for all $x \in X$ and all $y \notin X$. Note that the set of all alternatives satisfies this property, and hence the existence of at least one dominating set is trivially guaranteed. Dominating sets are, moreover, totally ordered by set inclusion. This is an immediate consequence of the following lemma.

Lemma 1. *Let X and Y be two dominating sets. Then, $X \subseteq Y$ or $Y \subseteq X$.*

Proof. Assume for contradiction that neither $X \subseteq Y$ nor $Y \subseteq X$. Then there is some $x \in X \setminus Y$ and some $y \in Y \setminus X$. As by assumption both X and Y are dominating sets, both $x \succ y$ and $y \succ x$, contradicting the asymmetry of the dominance relation. □

[3] Obviously, one is guaranteed to obtain a complete dominance relation if the number of voters is odd and individual preferences are linear.

As there always is at least one dominating set and the intersection of any set of dominating sets is also a dominating set, the *Smith set*[4] can be conveniently defined as the unique smallest dominating set [12, 13].[5]

Definition 2 (Smith set)**.** The *Smith set* S is the smallest dominating set, i.e., the smallest nonempty set of alternatives such that $a > b$, for all $a \in S$ and all $b \notin S$.

Alternatively, the Smith set can be defined as the set of maximal elements of the transitive closure of the weak dominance relation (see Lemma 2 below). If the Smith set contains only one element, this alternative is the Condorcet winner. Numerous choice rules always pick alternatives from the Smith set, such as the rules due to Nanson, Kemeny, or Fishburn [see, e.g., 4].

We say that a nonempty subset X of alternatives is *undominated* whenever no alternative in X is dominated by some alternative not in X, i.e., for no $a \in X$ there exists $b \notin X$ with $b > a$. Vacuously the set of all alternatives is undominated and so the existence of an undominated set is guaranteed. In contradistinction to dominating sets, however, there need not be in general a *unique* minimal undominated set. With the set of alternatives having been assumed to be finite, we can single out those undominated sets that are minimal with respect to set inclusion. We say that an alternative is in the *Schwartz set*, whenever it is an alternative of some such minimal undominated set [14].

Definition 3 (Schwartz set)**.** The *Schwartz set* $T \subseteq A$ is the union of all sets $X \subseteq A$ such that

(*i*) there is no $b \notin X$ and no $a \in X$ with $b > a$, and

(*ii*) there is no nonempty proper subset of X that satisfies property (*i*).

Alternatively, the Schwartz set can be defined as the set of maximal elements of the asymmetric part of the transitive closure of the dominance relation (see Lemma 4 below). It is also worth observing that, if the dominance relation is acyclic, the Schwartz set consists precisely of all undominated alternatives. Moreover, unlike the Smith set, the Schwartz set can contain a single alternative without this alternative being the Condorcet winner (see Figure 2). If there is a Condorcet winner, however, it will invariably be the only element of the Schwartz set. The Schwartz set coincides with the Smith set if the dominance relation is complete, i.e., in the case of tournaments. Well-known choice rules that always pick alternatives from the Schwartz set are Schulze's rule and ranked pairs [see, e.g., 15].

The intuition behind *stable sets* can perhaps best be understood by thinking of the social choice situation as one in which the voters have to settle upon a selection of alternatives from which the eventual social choice is to be selected by lot or some other mechanism beyond their control. One could argue that any such selection should at least satisfy two properties. No majority can be found in favor of restricting the selection by excluding some alternative from it. In a similar vein, it must be possible to find a majority against the inclusion of an outside alternative in the selection. Formally, stable sets are defined as follows.

Definition 4 (Stable set)**.** A set of alternatives $U \subseteq A$ is *stable* if it satisfies the following two properties, also known as *internal* and *external* stability:

(*i*) $a > b$, for no $a, b \in U$, and

(*ii*) for all $a \notin U$ there is some $b \in U$ with $b > a$.

Equivalently, stable sets can be given a single fixed point characterization: the alternatives in a *stable* set U are precisely those that are undominated by any alternative in U. Observe that this definition does not exclude the possibility that an alternative outside a stable set dominates an alternative inside it.

Stable sets were proposed by von Neumann and Morgenstern [16] to deal with intransitive dominance relations on imputations. Originally, they were introduced as a solution concept for cooperative games and as such they have been studied extensively [see, e.g., 17]. In the context of social choice, stable sets have received considerably less attention [see, e.g., 18, 19, 20]. One reason for this might be that in tournaments, a stable set exists if and only

[4]The Smith set is sometimes also referred to as the *Condorcet set* or the *Good set*.

[5]Had the set of alternatives been infinite this would have been unwarranted. E.g., for the natural numbers with the greater-than relation as dominance relation, no smallest dominating set exists.

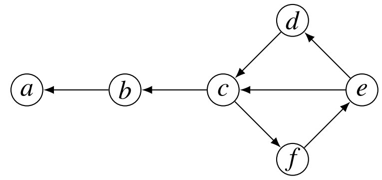

Fig. 1 Dominance graph over a set of six alternatives and with Copeland set $C = \{e\}$, Smith set $S = \{a, b, c, d, e, f\}$, Schwartz set $T = \{c, d, e, f\}$, the unique stable set $U = \{b, d, f\}$, Banks set $B = \{b, c, e, f\}$, and Slater set $L = \{e, f\}$

if there is a Condorcet winner, which is then the only element in this set. In the general case, however, neither uniqueness nor existence of stable sets is guaranteed. For transitive dominance relations, there is a unique stable set, which consists precisely of the relation's maximal elements, and thus equals the Schwartz set. Moreover, there is a unique singleton stable set if and only if there is Condorcet winner.

An alternative is in the *Banks set* if it is the maximal element of a subset of the alternatives for which the dominance relation is complete and transitive, and which is itself maximal with respect to set inclusion [21].[6]

Definition 5 (Banks set)**.** An alternative $a_1 \in A$ is in the *Banks set* if there exists a subset $\{a_1, a_2, \ldots, a_k\} \subseteq A$, such that

(*i*) $a_i > a_j$ for all $1 \leq i < j \leq k$, and

(*ii*) there is no $b \in A$ such that $b > a_i$ for all $1 \leq i \leq k$.

It is not very hard to show that the Banks set is a singleton if and only if there is a Condorcet winner.

The last set we consider is the Slater set which contains the maximal alternatives of those acyclic relations that disagree with the original dominance relation for a minimal number of tuples [23].

Definition 6 (Slater set)**.** An alternative $a \in A$ is in the *Slater set* if a is undominated in an acyclic subrelation of $>$ with maximal cardinality.

The Slater set contains the Condorcet winner as its only element if it exists. However, the Slater set may also be a singleton in case no Condorcet winner exists (see Figure 2). We conclude this section by stating without proof that none of the considered sets may contain the Condorcet loser, i.e., an alternative that is dominated by all other alternatives.

3.2 Dominance and Directed Graphs

It is very convenient to view the dominance relation derived from the voters' preferences as a directed graph $G = (V, E)$ where the set V of vertices equals the set A of alternatives and there is a directed edge $(a, b) \in E$ for $a, b \in V$ if and only if $a > b$ [see, e.g., 10]. Figure 1 shows the directed graph obtained for a set of six alternatives and the following profile of partial preferences for five voters (to improve readability, we only give the strict part of the preference ordering $\succsim_i$ for each voter $i \in N$).

Voter 1: $e \succ_1 d \succ_1 c \succ_1 b \succ_1 a$
Voter 2: $b \succ_2 a \succ_2 e, \quad d \succ_2 c \succ_2 f$
Voter 3: $a \succ_3 c, \quad f \succ_3 e \succ_3 d$
Voter 4: $a \succ_4 c \succ_4 e, \quad a \succ_4 b \succ_4 d$
Voter 5: $e \succ_5 c \succ_5 a$

Since all choice sets considered in this paper are defined in terms of the dominance relation only, we will henceforth restrict our attention to dominance graphs.

[6] There are various possible extensions of the Banks set, which was originally defined for complete dominance relations, to general dominance graphs. We consider the one referred to as B_1 by Banks and Bordes [22].

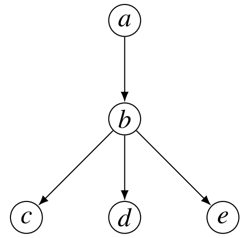

Fig. 2 The intersection between the Copeland set and the Schwartz set, the Slater set, or a unique stable set may be empty. The Copeland set consists of alternative b, the Schwartz set and the Slater set consist of alternative a, and the unique stable set is $\{a, c, d, e\}$.

3.3 Relationships Between Choice Sets

Laffond et al. [8] have conducted a thorough comparison of choice sets and derived various set-theoretic inclusions. However, their study is restricted to tournaments, where many of the following observations are not possible since the Smith set and the Schwartz set coincide, and does not cover stable sets. For these reasons, we provide an exhaustive set-theoretic analysis of the concepts defined in Section 3.1. We start by observing that all sets we consider are contained in the Smith set.

Theorem 1. *The Copeland set, the Schwartz set, every stable set, the Banks set, and the Slater set are contained in the Smith set.*

Proof. *(i) Copeland set:* Consider an arbitrary alternative a in the Copeland set and assume for a contradiction that a is not in the Smith set. We may assume the Smith set to include at least one alternative, say b. We show that the Copeland score of b is strictly greater than that of a. First observe that any alternative x with $a > x$ cannot be in the Smith set. Otherwise, also $x > a$, which would be at variance with the asymmetry of the dominance relation. As b is in the Smith set, it follows that all alternatives dominated by a are also dominated by b. As, moreover, b dominates a, whereas a does not dominate itself, the number of alternatives dominated by b is at least one greater than the number of the alternatives dominated by a. The second thing to observe is that b dominates all alternatives that are not in the Smith set. By asymmetry of the dominance relation, b can only be dominated by alternatives that are in the Smith set. Moreover, a is dominated by at least *all* alternatives in the Smith set, including b. As b does not dominate itself, the number of alternatives that dominate a is at least one greater than the number of alternatives that dominate b. These two observations taken together allow us to conclude that the Copeland score of b is at least two greater than that of a.

(ii) Schwartz set: Consider an arbitrary undominated subset X of A. Assume that X is not contained in the Smith set S, i.e., $x \notin S$, for some $x \in X$. We show that $X \cap S$ is undominated. Since $X \cap S$ is a subset of S as well as a proper subset of X, it follows that each minimal undominated subset is contained in the Smith set. Hence, the Schwartz set itself is contained in the Smith set as well. We may assume S to contain an alternative a. Then, $a \in X$, for otherwise $a > x$, which would be at variance with X being undominated. Hence, $X \cap S$ is nonempty. Finally, consider an arbitrary $x' \in X \cap S$ along with an equally arbitrary $y \notin X \cap S$. So $y \notin X$ or $y \notin S$. If the former, $y \not> x'$, as $x' \in X$ and X is undominated. If the latter, $x' > y$, because x' is in S and y is not. Then, $y \not> x'$ follows by asymmetry of $>$. We may conclude that $X \cap S$ is undominated.

(iii) Stable set: Assume for contradiction that some alternative u_1 in some stable set U, is not in the Smith set S. Consider an alternative $v \in S$. By definition of the Smith set, u_1 has to be dominated by v. Furthermore, v cannot be in U due to the internal stability of U and, by external stability, has to be dominated by at least one alternative $u_2 \in U$. Because of the asymmetry of the dominance relation, we have $u_2 \neq u_1$. It follows that u_2 is contained in S, violating the internal stability of U.

(iv) Banks set: An alternative a is in the Banks set if and only if it is the maximal element of an inclusion-maximal subset X for which the dominance relation is complete and transitive. Assume for contradiction that alternative a is in the Banks set but not in the Smith set. It follows that no alternative in X can be in the Smith set because it is dominated by a. However, according to Definition 2, there must an alternative b in the Smith set that dominates all alternatives in X which contradicts the assumption that X is inclusion-maximal.

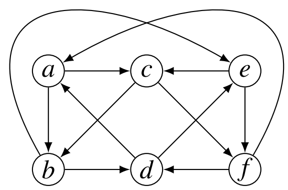

Fig. 3 The intersection between the Banks set and all stable sets may be empty. The set of alternatives is partitioned into the Banks set $B = \{a, b, e, f\}$ and the unique stable set $U = \{c, d\}$.

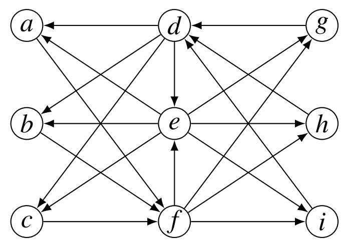

Fig. 4 The intersection between the Banks set and the Copeland set may be empty. The set of all alternatives is partitioned into the Copeland set $C = \{e\}$ and the Banks set $C = A \setminus \{e\}$.

(v) Slater set: Assume for contradiction that a is in the Slater set but not in the Smith set S. Consider the acyclic set X of maximal cardinality in which a is an undominated alternative. It follows that $X \cap S = \emptyset$, otherwise a would be dominated by any alternative in $X \cap S$. For any $b \in S$, however, we then have that $X \cup \{b\}$ is also acyclic, because b dominates each x in X. As S is nonempty, X cannot be an acyclic set of maximal cardinality, a contradiction. □

We leave it to the reader to verify that no other inclusion relationships between the discussed sets hold. It turns out that all considered sets except the Copeland set always intersect with the Schwartz set (see Figure 2 for an example where the Copeland set and the Schwartz set are disjoint).

Theorem 2. *Every stable set, the Banks set, and the Slater set always have a nonempty intersection with the Schwartz set.*

Proof. (i) Stable sets: Let X be a set satisfying external stability, Y an undominated set. Assume for a contradiction that X and Y are disjoint. As Y is nonempty, $y \in Y$, for some alternative y. Moreover, $y \notin X$, by disjointness of X and Y. Because of external stability of X, we have $x > y$, for some $x \in X$. As, due to disjointness of X and Y, $x \notin Y$, this contradicts Y being undominated. More precisely, every set satisfying external stability intersects with every undominated set.

(ii) Banks set: Let a be an element of the Schwartz set that is not in the Banks set. By definition of the Banks set, there is no inclusion-maximal subset of alternatives for which the dominance relation is transitive and complete and that has a as its maximal element. On the other hand, a must be *contained* in an inclusion-maximal subset for which the dominance relation is transitive and complete. Let b be the maximal element of this subset. By definition, b is in the Banks set and $b > a$, and since a is an element of the Schwartz set, b must be in the Schwartz set as well.

(iii) Slater set: Undominated alternatives are obviously contained in both the Schwartz set and the Slater set. If there is no undominated alternative, the Schwartz set consists of cycles and at least one edge from these cycles has to be removed to obtain the Slater set. This yields at least one undominated alternative that is contained in both the Schwartz and the Slater set. □

The intersections of all remaining pairs of choice sets may be empty.

Theorem 3. *Any stable set, the Banks set, the Slater set, and the Copeland set may be pairwise disjoint.*

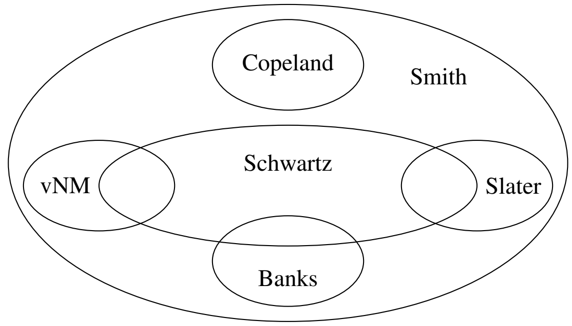

Fig. 5 Relationships between the choice sets considered in this paper. Sets that intersect in the diagram *always* intersect. Sets that are disjoint in the diagram *may* have an empty intersection, i.e., there exist instances where these sets do not intersect.

Proof. The Copeland set and the (unique) stable set of the graph depicted in Figure 1 are disjoint (see also Figure 2 for another example). When adding a directed edge from vertex a to vertex f in this graph, $\{b, d, f\}$ remains the only stable set. This modified graph contains three cycles with one common edge (f, e) whose removal yields the Slater set as the maximal alternatives of the resulting graph. Alternative e is the only undominated alternative in this case and is not contained in the stable set $\{b, d, f\}$. Figure 3 shows an example graph where the vertices can be partitioned into the Banks set and a unique stable set. An example graph where the Banks set consists of all alternatives except the single element of the Copeland set is given in Figure 4. Laffond and Laslier [24] have shown that the Slater set and the Banks set may be disjoint (even for the more restricted case of tournaments). An example where the Slater set and the Copeland set do not intersect is given in Figure 2. □

This completes our picture of set-theoretic relationships between the considered choice sets. Figure 5 combines all results of this section in one diagram.

4 Complexity Results

In the remainder of the paper, we investigate the computational complexity of the considered choice sets. We assume the reader to be familiar with the well-known chain of complexity classes

$$\mathrm{AC}^0 \subset \mathrm{TC}^0 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{NC} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{P}_{\|}^{\mathrm{NP}},$$

and with the notions of constant-depth and polynomial-time reducibility [see, e.g., 25]. AC^0 is the class of problems solvable by uniform constant-depth Boolean circuits with unbounded fan-in and a polynomial number of gates, and TC^0 adds so-called threshold gates which output true if and only if the number of true inputs exceeds a certain threshold. Here, uniformity means that there is an "efficient" algorithm for constructing, for each input length n, the circuit C_n from the circuit family $C = (C_n)_{n\geq 0}$. Different notions of efficiency give rise to different notions of uniformity [26]. We will consider log-space uniform circuit families, where the mapping $n \mapsto C_n$ is computable in deterministic logarithmic space. In some of our constructions we use sub-circuits described by Chandra et al. [27] as basic building blocks, and it is easy to see that all these sub-circuits are log-space uniform. L and NL are the classes of problems solvable by deterministic and nondeterministic Turing machines using only logarithmic space, and P and NP are the classes of problems that can be solved in polynomial time by deterministic and nondeterministic Turing machines. Finally, $\mathrm{P}_{\|}^{\mathrm{NP}}$ is the class of problems solvable on a deterministic Turing machine with parallel, i.e., non-adaptive, access to a non-deterministic Turing machine.

It is not hard to see that the computation of the adjacency matrix of the dominance graph for a given preference profile requires the invocation of the majority function on a string of bits for each pair of alternatives, and is in fact complete for the complexity class TC^0. As a consequence, every choice rule that chooses the Condorcet winner whenever one exists is TC^0-hard. In order to obtain a finer differentiation of the complexity of choice sets, we therefore assume that the input for the problems we consider is given explicitly in terms of the adjacency matrix of the dominance graph. More precisely, the decision problems for each of the six choice sets defined in Section 3.1 are defined as follows: Given a set A of alternatives, a particular alternative $a \in A$, and an asymmetric dominance relation $\succ$, IN-COPELAND, IN-SMITH, IN-SCHWARTZ, IN-STABLE, IN-BANKS, and IN-SLATER ask whether a is contained in the Copeland set, the Smith set, the Schwartz set, a stable set, the Banks set, and the Slater set of $(A, \succ)$, respectively. Clearly, deciding whether an alternative is contained in a choice set is computationally equivalent (via AC^0 reductions) to actually finding the set.

4.1 Copeland Set

Membership of a given alternative in the Copeland set can be decided by checking whether it has maximum Copeland score. We show that this problem is in fact TC^0-complete.

Theorem 4. *IN-COPELAND is TC^0-complete under AC^0 Turing reductions. Hardness holds even for tournaments.*

Proof. Using circuits described by Chandra et al. [27] as basic building blocks, *membership* in TC^0 is straightforward. The number of alternatives an alternative dominates and the number of alternatives it is dominated by can be computed using BINARY-COUNT sub-circuits. The Copeland score of each alternative can then be determined by subtracting the latter from the former. A MAXIMUM and a COMPARISON circuit can then be used to compute the maximum Copeland score and compare it with that of the vertex in question. Clearly, the resulting overall circuit is log-space uniform.

For *hardness*, we provide a reduction from the TC^0-complete problem MAJORITY of deciding whether the majority of a string of bits is set to 1. For a particular string $x = (x_1, x_2, \dots, x_m) \in \{0,1\}^m$, define a tournament $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_{m+1}\}$ such that all vertices in the induced subgraph of G with vertex set $V \setminus \{v_{m+1}\}$ dominate $\lfloor \frac{m}{2} \rfloor$ or $\lceil \frac{m}{2} \rceil$ other vertices, and for all i with $1 \leq i \leq m$, $(v_{m+i}, v_i) \in E$ if and only if $x_i = 1$. A graph with the former property can be obtained by arranging the vertices on a cycle and letting each vertex dominate the next $\lfloor \frac{m}{2} \rfloor$ or $\lceil \frac{m}{2} \rceil$ vertices on the cycle. It is readily appreciated that this reduction can be computed in AC^0, and that v_{m+1} is in the Copeland set of G if and only if at least $\lceil \frac{m}{2} \rceil$ bits of x are 1. □

4.2 Smith Set

In graph-theoretic terms, the Smith set is the unique minimal undominated strongly connected component of the weak dominance graph, while the Schwartz set consists of all minimal undominated strongly connected components of the dominance graph. With the help of these characterizations, we can reduce the computation of these sets to reachability problems. In order to simplify the following exposition, we fix some notation. Let a be an alternative in A and $\geq$ a binary relation over A, e.g., the dominance relation $\succ$ or the weak dominance relation $\succsim$. We define $D_{\geq}(a) = \{\, b \in A \mid a \geq b\}$ and $\overline{D}_{\geq}(a) = \{\, b \in A \mid b \geq a\}$. Moreover, for $k \geq 0$, let

$$D_{\geq}^{k+1}(a) = D_{\geq}^{k}(a) \cup \bigcup_{b \in D_{\geq}^{k}(a)} D_{\geq}(b) \qquad \text{and} \qquad \overline{D}_{\geq}^{k+1}(a) = \overline{D}_{\geq}^{k}(a) \cup \bigcup_{b \in \overline{D}_{\geq}^{k}(a)} \overline{D}_{\geq}(b)$$

be defined inductively such that $D_{\geq}^{0}(a) = \overline{D}_{\geq}^{0}(a) = \{a\}$. Finally, set $D_{\geq}^{*}(a) = \bigcup_{k \geq 0} D_{\geq}^{k}(a)$ and $\overline{D}_{\geq}^{*}(a) = \bigcup_{k \geq 0} \overline{D}_{\geq}^{k}(a)$. $D_{\geq}^{*}(a)$ thus contains the alternatives reachable from a, and $\overline{D}_{\geq}^{*}(a)$ the alternatives from which a can be reached.

Lemma 2. *An alternative $a \in A$ is in the Smith set of $(A, \succ)$ if and only if for every $b \in A$, there is a path from a to b in the weak dominance graph.*

Proof. The statement of the lemma can be rephrased using the notation introduced above: Alternative $a \in A$ is in the Smith set S if and only if $D_{\succsim}^{*}(a) = A$.

For the implication from left to right, let $a \in S$. We show that for any $b \in A$, $\overline{D}_{\succsim}^{*}(b)$ is a dominating set. Assume for contradiction that there are alternatives $c \in \overline{D}_{\succsim}^{*}(b)$ and $d \notin \overline{D}_{\succsim}^{*}(b)$ such that c does not dominate d. This implies that d weakly dominates c and consequently that $d \in \overline{D}_{\succsim}^{*}(b)$, a contradiction. Since a is contained in all dominating sets, $a \in \overline{D}_{\succsim}^{*}(b)$ for any $b \in A$. As a consequence, $D_{\succsim}^{*}(a) = A$.

For the converse, assume that $a \notin S$ and let $b \in S$. Since all alternatives in S dominate all alternatives outside S, no alternative in S, including b, can be contained in $D_{\succsim}^{*}(a)$. □

Tantau [28] has shown that reachability in tournaments can be decided in AC^0, which implies that IN-SMITH is in AC^0 for tournaments. In the following we slightly extend Tantau's result to graphs induced by complete relations that are not necessarily anti-symmetric, such as the weak dominance relation.

Lemma 3. *Let $G = (V, E)$ be directed graph such that for all $v, w \in V$, $(v, w) \in E$ or $(w, v) \in E$. Then, reachability in G can be decided in AC^0.*

Proof. Let $G = (V, E)$ be a directed graph as required and define $\succsim$ so that $v \succsim w$ if and only if $(v, w) \in E$. Further, let $s, t \in V$ be two vertices and call a set $V' \subseteq V$ *closed* if $D_{\succsim}(v) \subseteq V'$ for all $v \in V'$. We claim that $t \in V$ is *not* reachable from $s \in V$ if and only if there exists $v \in V$ such that $D^2_{\succsim}(v)$ is closed, $s \in D^2_{\succsim}(v)$, and $t \notin D^2_{\succsim}(v)$. It is readily appreciated that the latter property can be decided in log-space uniform AC^0. In particular, $D^2_{\succsim}(v)$ is closed if and only if $D^2_{\succsim}(v) = D^3_{\succsim}(v)$.

The direction from right to left is straightforward. If s is contained in a closed set, and t is not contained in this set, then t cannot be reachable from s.

For the direction from left to right, assume that t is not reachable from s, and let $V' = D^*_{\succsim}(s)$. Consider a vertex $v \in V'$ for which $|D_{\succsim}(v)|$ is maximal. We claim that $D^2_{\succsim}(v) = V'$.[7] Assume for contradiction that there is $v' \in V' \setminus D^2_{\succsim}(v)$. Then, $v \not\succsim v'$ and for every $w \in D_{\succsim}(v)$, $w \not\succsim v'$. Since any pair of vertices is connected by at least one edge, this implies that $D_{\succsim}(v) \subset D_{\succsim}(v')$ and consequently that $|D_{\succsim}(v')| > |D_{\succsim}(v)|$, a contradiction. Hence, $D^2_{\succsim}(v)$ is closed, $s \in D^2(v)$, and $t \notin D^2(v)$ as required. □

By combining Lemma 2 and Lemma 3, we have the following.

Theorem 5. *IN-SMITH is in AC^0.*

Proof. Obviously, the weak dominance relation can easily be obtained from the dominance relation in log-space uniform AC^0. According to Lemma 2, a given alternative is contained in the Smith set if and only if it reaches every alternative in the weak dominance graph. The latter can be checked in parallel using the construction given in Lemma 3. □

4.3 Schwartz Set

As noted earlier, the Smith set and the Schwartz set differ only by their treatment of ties in pairwise comparisons. Nevertheless, deciding membership in the Schwartz set turns out to be computationally more difficult. Similar to the Smith set, the Schwartz set is characterized by a graph-theoretic property [30].[8]

Lemma 4. *An alternative $a \in A$ is in the Schwartz set of $(A, \succ)$ if and only if for every $b \in A$, there is a path from a to b whenever there is a path from b to a in the dominance graph.*

Proof. The statement of the lemma can be rephrased as follows: Alternative $a \in A$ is in the Schwartz set T if and only if $\overline{D}^*_{\succ}(a) \subseteq D^*_{\succ}(a)$. Since the statement is trivially satisfied for alternatives that are undominated, we only need to consider alternatives for which $\overline{D}^*_{\succ}(a) \setminus \{a\} \neq \emptyset$.

To see the implication from left to right, assume for contradiction that $a \in T$, and that $\overline{D}^*_{\succ}(a) \setminus D^*_{\succ}(a) \neq \emptyset$. Clearly, $\overline{D}^*_{\succ}(a)$ is undominated. If $D^*_{\succ}(a)$ is removed from this set, it remains undominated since, by definition of $D^*_{\succ}(a)$, there can be no $b \in D^*_{\succ}(a)$ and $c \notin D^*(a)$ such that $b \succ c$. Thus, $\overline{D}^*_{\succ}(a) \setminus D^*_{\succ}(a)$ is undominated and a proper subset of $\overline{D}^*_{\succ}(a)$. This implies that $a \notin T$, which yields a contradiction.

Conversely assume that $a \notin T$ and that $\overline{D}^*_{\succ}(a) \subseteq D^*_{\succ}(a)$. Again, we only consider the case where a is dominated by at least one other alternative, hence $\overline{D}^*_{\succ}(a) \setminus \{a\} \neq \emptyset$. Then, however, $\overline{D}^*_{\succ}(a)$ is an inclusion-minimal undominated set, contradicting the assumption that a is not in the Schwartz set. □

---

[7] A vertex that reaches every other vertex on a path of length at most two is sometimes called a *king* [29].

[8] A minor variation of the proof of Lemma 2 shows that the Smith set can be characterized equivalently simply by replacing the dominance relation with the weak dominance relation in Lemma 4. These characterizations also provide an alternative argument for the inclusion of the Schwartz set in the Smith set (Theorem 1).

Using Lemma 4, IN-SCHWARTZ for a given alternative $a \in A$ can be decided by checking whether the set of alternatives reachable from a contains at least the alternatives from which a is reachable. Since reachability in a directed graph can be checked in NL, it is not hard to see that IN-SCHWARTZ is in NL as well. We show that it is in fact NL-complete.

Theorem 6. *IN-SCHWARTZ is NL-complete under AC^0 many-one reductions.*

Proof. It is well-known that the directed graph reachability problem is NL-complete [see, e.g., 25], i.e., both in NL and NL-hard. Given a dominance graph and using Lemma 4, membership of an alternative $a \in A$ in the Schwartz set can be shown by checking for every other alternative $b \in A$ that b is reachable from a if a is reachable from b. Membership in the Schwartz set can then be decided using an additional pointer into the input to store alternative b currently being checked.

For *hardness*, we provide a reduction from directed graph reachability, which is NL-complete, even when restricted to graphs that contain no symmetric edges (symmetric edges can be avoided by introducing additional vertices). Given a particular directed graph $G = (V, E)$ and two designated vertices $s, t \in V$, we construct a dominance graph $G' = (V', E')$ by adding two additional vertices s' and t', an edge from s' to s, an edge from t to t', edges from any original vertex but s to s', and edges from t' to any original vertex but t, i.e.,

$$\begin{aligned} V' &= V \cup \{s', t'\} \quad \text{and} \\ E' &= E \cup \{(s', s)\} \cup \{\,(v, s') : v \in V \text{ and } v \neq s\,\} \cup \{(t, t')\} \cup \{\,(t', v) : v \in V \text{ and } v \neq t\,\}. \end{aligned}$$

It is easily verified that G' can be computed from G by an AC^0 circuit. We claim that s is contained in the Schwartz set for G' if and only if there exists a path from s to t in G. First of all, observe that we have added no additional paths from s to vertices in V or from vertices in V to t, so a path from s to t in G' exists if and only if such a path already existed in G. By construction, every vertex of G', including s, can now be reached from t. Hence, by Lemma 4, s cannot be contained in the Schwartz set if t cannot be reached from s. Conversely assume that t is reachable from s. Then this property holds for every vertex of G' as well, particularly for those from which s can be reached. In virtue of Lemma 4, we may conclude that s is in the Schwartz set. Furthermore, since s is reachable from every vertex via s', *all* vertices are contained in the Schwartz set if and only if there is path from s to t. □

Naturally, hardness of the membership problem for a particular set does not automatically imply hardness of all choice rules that always yield an alternative from that set. We will for example see later that finding an arbitrary alternative from the Banks set is actually *easier* than deciding whether a given alternative is contained in it, unless P equals NP. However, in the case of the Schwartz set, under a mild tie-breaking condition, we can prove the hardness of all *Schwartz-consistent* choice rules, i.e., choice rules that always select an alternative from the Schwartz set. Consider a dominance relation with several minimal undominated sets. A choice rule with *fixed tie-breaking order* may arbitrarily select a "candidate" from each of these sets. However, which alternative is ultimately chosen from these candidates only depends on a predefined order that is independent of the voters' preferences. Thus, we may assume that alternatives possess consecutive indices and that the alternative with the smallest index within the set of candidates is selected.

Theorem 7. *Consider a choice rule that selects an alternative from the Schwartz set using a fixed tie-breaking order. This choice rule cannot be executed on a deterministic Turing machine with logarithmic space, unless L = NL.*

Proof. As pointed out in the proof of Theorem 6, the problem of deciding whether *all* alternatives are contained in the Schwartz set is NL-hard. This can be used to show via an AC^0 Turing reduction that the problem SCHWARTZ-SINGLETON of deciding whether the Schwartz set contains only one element, is also NL-hard. More precisely, we provide a Boolean circuit with constant-depth, unbounded fan-in, and access to a SCHWARTZ-SINGLETON oracle that decides, for a given dominance graph $G = (V, E)$, whether the Schwartz set contains all vertices. For every vertex $v \in V$, we let the oracle decide SCHWARTZ-SINGLETON for $G_v = (V \cup \{u\}, E \cup \{(u, v)\})$. The Schwartz set of G contains all vertices if and only if the oracle yields a positive answer for every G_v.

We proceed to show the hardness of every Schwartz-consistent choice rule with a fixed tie-breaking order using an AC^0 Turing reduction from SCHWARTZ-SINGLETON. Let f be a Schwartz-consistent choice rule, $G = (V, E)$

an arbitrary dominance graph, and v the alternative chosen by f for graph G. If v is dominated by some other vertex, we can easily decide SCHWARTZ-SINGLETON because this other vertex has to be in the Schwartz set as well. If, on the other hand, v is undominated, we invoke f on the modified graph $G_v = (V \cup \{u\}, E \cup \{(u, v)\})$, where the tie-breaking index of u is chosen to be greater than all existing indices. Now, if f yields u, then there cannot be another minimal undominated set, which implies that v was the unique element in the Schwartz set of G. Whenever there exists another minimal undominated set, f must choose a vertex from such a set, because all of them have a lower tie-breaking index than u. In this case, SCHWARTZ-SINGLETON can be decided in the negative. □

4.4 Von Neumann-Morgenstern Stable Sets

For all choice sets considered so far, we can check in polynomial time whether they contain a particular alternative or not. Unfortunately, this is not case for stable sets, unless P=NP.

Theorem 8. *IN-STABLE is NP-complete, even if a stable set is guaranteed to exist.*

Proof. Membership in NP is straightforward. Given a dominance graph over a set A of alternatives and a particular alternative $a \in A$, we can simply guess a subset $U \subseteq A$ such that $a \in U$, and verify that for every $b \notin U$ there is an edge from some element of U to b and that there are no edges between vertices of U.

For *hardness*, we provide a reduction from satisfiability of a Boolean formula φ in conjunctive normal form (SAT) to the problem of deciding whether a designated alternative $a \in A$ is contained in a stable set (or the union of all stable sets). The reduction relies on a reduction used by Chvátal [31] to show NP-hardness of the problem of deciding whether a directed graph has a kernel. Let $B = \bigwedge_{1 \le i \le m} \bigvee_{1 \le j \le k_i} p_{ij}$ be a SAT instance over variables X. We construct an asymmetric dominance graph $G = (V, E)$ with three vertices c_{i1}, c_{i2}, and c_{i3} for each clause of φ, four vertices x_i, $\bar{x}_i$, x'_i, and $\bar{x}'_i$ for each variable of φ, and four additional vertices d_1, d_2, d_3, and d_4, such that d_1 is contained in a stable set if and only if φ has a satisfying assignment. Vertices c_{ij} will henceforth be called clause vertices, x_i and $\bar{x}_i$ will be referred to as positive and negative literal vertices, respectively. Edges are such that the vertices of each clause form a directed cycle of length three, and the vertices of each variable as well as the decision vertices form a cycle of length four according to the sequences given above. Furthermore, there is an edge from a positive or negative literal vertex to all clause vertices of a clause in which the respective literal appears. Finally, there is an edge from d_2 to every clause vertex. More formally, we have

$$
\begin{aligned}
E = \; & \{(d_1, d_2), (d_2, d_3), (d_3, d_4), (d_4, d_1)\} \\
& \cup \; \{(c_{i1}, c_{i2}), (c_{i2}, c_{i3}), (c_{i3}, c_{i1}) : 1 \le i \le m\} \\
& \cup \; \{(x_i, \bar{x}_i), (\bar{x}_i, x'_i), (x'_i, \bar{x}'_i), (\bar{x}'_i, x_i) : 1 \le i \le |X|\} \\
& \cup \; \textstyle\bigcup_{1 \le \ell \le k_j} \{(x_i, c_{j1}), (x_i, c_{j2}), (x_i, c_{j3}) : p_{j\ell} = x_i\} \\
& \cup \; \textstyle\bigcup_{1 \le \ell \le k_j} \{(\bar{x}_i, c_{j1}), (\bar{x}_i, c_{j2}), (\bar{x}_i, c_{j3}) : p_{j\ell} = \bar{x}_i\} \\
& \cup \; \{(d_2, c_{i1}), (d_2, c_{i2}), (d_2, c_{i3}) : 1 \le i \le m\}.
\end{aligned}
$$

Figure 6 illustrates this construction for a particular Boolean formula. We observe the following facts:

- G can be constructed from φ in polynomial time.
- $\{x_i, x'_i : 1 \le i \le m\} \cup \{d_2, d_4\}$ is a stable set of G irrespective of the structure of φ.
- Every stable set of G must either contain d_1 and d_3 or d_2 and d_4, but not both. For each i, every stable set must either contain x_i and x'_i or $\bar{x}_i$ and $\bar{x}'_i$, but not both.
- A stable set of G cannot contain a pair of clause vertices for the same clause. In turn, a stable set must contain vertices with outgoing edges to at least two of the three vertices for every clause. However, every vertex that has an outgoing edge to any vertex for some clause has an outgoing vertex to all three vertices for that clause. Hence, a stable set cannot contain any clause vertices.
- A stable set must contain either d_2 or a subset of the literal vertices containing at least one vertex for a literal in every clause. Since a stable set cannot contain both x_i and $\bar{x}_i$, the latter corresponds to a satisfying assignment φ. Hence, a stable set containing d_1 exists if and only if φ is satisfiable. □

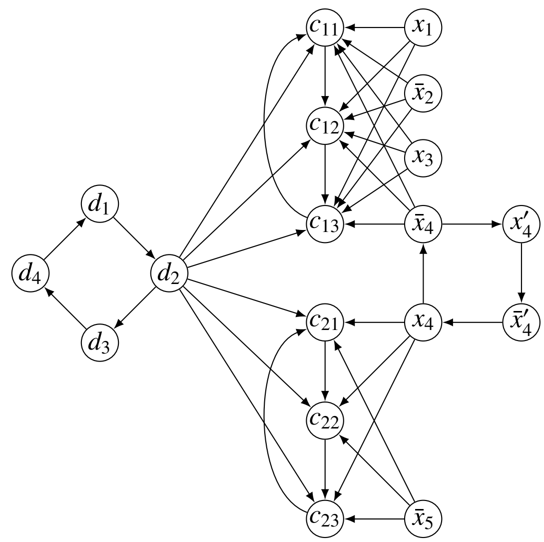

Fig. 6 Dominance graph for the Boolean formula $(x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_4 \vee \bar{x}_5)$ according to the construction used in the proof of Theorem 8. If a certain variable appears exclusively as either a positive or negative literal, the other three vertices for the variable are omitted.

As in the case of the Schwartz set, we can derive a stronger result, concerning the computational complexity of *any* choice rule that is guaranteed to select an alternative from a stable set, if such an alternative exists.

Theorem 9. *Consider a choice rule that selects an alternative from a stable set if one exists and an arbitrary alternative otherwise. This choice rule cannot be executed in worst-case polynomial time, unless P=NP.*

Proof. Again consider the construction used in the proof of Theorem 8 and illustrated in Figure 6. In this construction, four designated vertices d_1 to d_4 have been used to guarantee the existence of a stable set, no matter whether the underlying Boolean formula φ has a satisfying assignment or not. This guarantee also means that finding *some* alternative that belongs to a stable set is trivial. It is easily verified that if we remove vertices d_1 to d_4, a stable set in graph G exists if and only if φ has a satisfying assignment, and the vertices in such a stable set are those corresponding to the literals set to true in a particular satisfying assignments.

Now consider a Turing machine with an oracle that computes a single alternative belonging to a stable set, if such a set exists, and an arbitrary alternative otherwise. Using this machine, the existence of a satisfying assignment for a particular Boolean formula φ can be decided as follows. First, compute the dominance graph $G = (V, E)$ corresponding to φ. Then, iteratively reduce the graph by requesting a vertex v from the oracle and removing vertices as follows: if $v = x_i$ or $v = x'_i$ for some $1 \leq i \leq |X|$, remove x_i, x'_i, $\bar{x}_i$, $\bar{x}'_i$ and all c_{ij} such that $(x_i, c_{ij}) \in E$; if $v = \bar{x}_i$ or $v = \bar{x}'_i$ for some $1 \leq i \leq |X|$, remove x_i, x'_i, $\bar{x}_i$, $\bar{x}'_i$ and all c_{ij} such that $(\bar{x}_i, c_{ij}) \in E$. If at some point there no longer exists any vertex c_{ij}, let the machine halt and accept. If at some point there no longer exists any x_i or $\bar{x}_i$ but there still is some c_{ij}, or if the oracle returns c_{ij} for some $1 \leq i \leq m$, $j \in \{1, 2, 3\}$, let the machine halt and reject.

As already pointed out in the proof of Theorem 8, the graph G can be computed from φ in polynomial time. In every later step, the machine either halts or removes at least one vertex, of which there are only polynomially many. Hence, the machine is guaranteed to halt after a polynomial number of steps. Furthermore, if the machine accepts, the set of all vertices returned by the oracle form a stable set of G, which can only exist if φ has a satisfying assignment. We have thus provided a Turing reduction from SAT to the problem of selecting an arbitrary element of a stable set, showing that a polynomial-time algorithm for the latter would imply P=NP. □

4.5 Banks Set and Slater Set

For the Banks and the Slater set, the complexity of the membership problems is well-studied. We just briefly list the results for completeness. Woeginger [32] has shown that deciding whether a given alternative is contained

Choice set	Membership in tournaments	Membership in general graphs	Hardness of choice rules
Copeland	TC^0-complete	TC^0-complete	TC^0-hard
Smith	in AC^0	in AC^0	TC^0-hard
Schwartz	in AC^0	NL-complete	NL-hard[a]
vNM	in AC^0	NP-complete	NP-hard
Banks	NP-complete[b]	NP-complete	at most P-hard[c]
Slater	in $P_{\|}^{NP}$	$P_{\|}^{NP}$-complete[d]	$P_{\|}^{NP}$-hard

[a] for fixed tie-breaking order, see Theorem 7

[b] Woeginger [32]

[c] Hudry [34] pointed out that an arbitrary element of the Banks set can be found in polynomial time, which implies that the greatest lower bound for *all* choice rules choosing from the Banks set cannot be greater than P.

[d] Hemaspaandra et al. [35]

Table 1 Computational complexity of choice sets. The first two columns show the computational complexity of deciding whether a given alternative is contained in a particular choice set when given the *dominance graph* as input. The last column contains lower bounds for choice rules choosing from a particular choice set when given the *individual preferences* as input.

in the Banks set of a tournament is NP-complete via a reduction from graph 3-colorability.[9] Interestingly, an arbitrary element of the Banks set can be found in polynomial time [34].

A feedback arc set of a directed graph is a subset of the edges containing at least one edge from every cycle. Hemaspaandra et al. [35] show that deciding whether there exists a minimum size feedback arc set containing all edges entering a particular vertex in a directed graph is complete for $P_{\|}^{NP}$. As this problem is equivalent to the membership problem for the Slater set, deciding whether a particular alternative is contained in the Slater set is also $P_{\|}^{NP}$-complete. The related problem of deciding the existence of an arc feedback set with at most k edges is NP-complete, even when the graph is a tournament [36]. It is not known whether the membership problem for the Slater set is as hard in tournaments as it is in general directed graphs.

It is not hard to see that choice rules guaranteed to select an alternative from the Slater set cannot be computed in worst-case polynomial time by a nondeterministic Turing machine, unless $NP=P_{\|}^{NP}$. Let $G = (V, E)$ be a directed graph, f a function mapping each graph to the size of a minimum size feedback arc set. Denote by $d(v) = |\{u \in V : u > v\}|$ the number of alternatives dominating v, and by $G|_{V\setminus\{v\}}$ the graph obtained by removing v from G. Then, $f(G) = d(v) + f(G|_{V\setminus\{v\}})$ if and only if v is an element of the Slater set of G. By using this observation twice, to recursively compute $f(G)$ and to compare $f(G)$ with $d(w) + f(G|_{V\setminus\{v\}})$ for a specific alternative w, one obtains a polynomial-time Turing reduction from the problem of deciding membership in the Slater set to the problem of finding some element of the Slater set.

5 Conclusion

We have investigated the set-theoretic interrelationships and the computational complexity of various choice sets based on the pairwise majority relation. Table 1 summarizes the complexity-theoretic results, which can be interpreted as follows. All considered problems except IN-STABLE, IN-BANKS, and IN-SLATER are computationally tractable and in fact contained in the complexity class NC of problems amenable to parallel computation. Moreover, all problems except IN-SCHWARTZ, IN-STABLE, IN-BANKS, and IN-SLATER can be solved on a deterministic Turing machine using only logarithmic space. These results can be used to make statements regarding the complexity of entire classes of choice rules, e.g., the NL-hardness of every choice rule that picks an alternative from the Schwartz set or the NP-hardness of every choice rule that picks an alternative from a stable set. There are also implications concerning voting trees [see, e.g., 9]. For example, there is no polynomial-size voting tree for the Slater set, unless the polynomial hierarchy collapses.

In addition, Table 1 underlines the significant difference between tournaments and general dominance graphs. Perhaps surprisingly, the Smith set turned out to be computationally easier than the Schwartz set in general

[9] An alternative, arguably simpler, proof has been given by Brandt et al. [33].

dominance graphs, while both concepts coincide in tournaments. Deciding whether an alternative is included in a stable set is NP-complete in general dominance graphs, while in tournaments the same problem is equivalent to the very simple problem of checking whether the alternative is the Condorcet winner. The computational complexity of other dominance-based choice sets such as the tournament equilibrium set and variations of the uncovered set and the minimal covering set has been analyzed by Brandt et al. [33] and Brandt and Fischer [37], respectively.

Finally, it should be noted that our results are fairly general in the sense that they rely only on the *asymmetry* of the dominance relation. As a matter of fact, all considered sets are reasonable substitutes for maximality in the face of non-transitive relations, no matter whether these relations stem from aggregated preferences or not.

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