On the Hardness and Existence of Quasi-Strict Equilibria*

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Abstract. This paper investigates the computational properties of quasi-strict equilibrium, an attractive equilibrium refinement proposed by Harsanyi, which was recently shown to always exist in bimatrix games. We prove that deciding the existence of a quasi-strict equilibrium in games with more than two players is NP-complete. We further show that, in contrast to Nash equilibrium, the support of quasi-strict equilibrium in zero-sum games is unique and propose a linear program to compute quasi-strict equilibria in these games. Finally, we prove that every symmetric multi-player game where each player has two actions at his disposal contains an efficiently computable quasi-strict equilibrium which may itself be asymmetric.

1 Introduction

Perhaps the most ubiquitous solution concept in non-cooperative game theory is Nash equilibrium—a strategy profile that does not permit beneficial unilateral deviation. One of the main drawbacks of this concept is its potential multiplicity: While Nash equilibria are guaranteed to exist in finite games, there may be many of them, which causes uncertainty among the players which one to choose. For this reason, a number of concepts that single out particularly reasonable Nash equilibria—so-called equilibrium refinements—have been proposed over the years.

An important result by Norde et al. [23] has cast doubt upon this strand of research. Norde et al. [23] have shown that Nash equilibrium can be completely characterized by *utility maximization in one-player games*, *consistency*,[1] and *existence*. As a consequence, all common equilibrium refinements either violate consistency or existence. In particular, all refinements that are guaranteed to

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[1] Consistency as introduced by Peleg and Tijs [25] is defined as follows. Let S be a solution of game G and let G' be a reduced game where a subset of players are assumed to invariably play the strategies prescribed by S. A solution concept is *consistent* if the solution S' in which all of the remaining players still play according to S is a solution of G'.

exist such as *perfect*, *proper*, and *persistent* equilibria suffer from inconsistency while other refinements such as *quasi-strict*, *strong*, and *coalition-proof* equilibria may not exist. Since consistency is a very intuitive and appealing property, its failure may be considered more severe than possible non-existence. Harsanyi's quasi-strict equilibrium, which refines the Nash equilibrium concept by requiring that every action in the support yields strictly more payoff than actions not in the support, has been shown to always exist in bimatrix games [22] and generic n-player games (and thus in "almost every" game) [12]. Furthermore, Squires [28] has shown that quasi-strict equilibrium is very attractive from an axiomatic perspective as it satisfies the Cubitt and Sugden axioms [6], a strengthening of a similar set of axioms by Samuelson. This result can be interpreted so that the existence of quasi-strict equilibrium is sufficient to justify the assumption of common knowledge of rationality. In fact, Quesada [26] even poses the question whether the existence of quasi-strict equilibrium is sufficient for *any* reasonable justification theory. Finally, isolated quasi-strict equilibria satisfy almost all desirable properties defined in the refinements literature. They are essential, strongly stable, regular, proper, and strictly perfect [see, *e.g.*, 14, 29, 30].[2]

In recent years, the computational complexity of a number of equilibrium refinements in various classes of games such as extensive-form games, congestion games, or graphical games has come under increasing scrutiny [see, *e.g.*, 11, 19, 20, 27]. In this paper, we study the computational properties of quasi-strict equilibrium in zero-sum games, general normal-form games, and certain classes of symmetric or anonymous games. The remainder of the paper is structured as follows. In the next section, we introduce classes of strategic games and the solution concepts of Nash equilibrium and quasi-strict equilibrium. Section 3 focuses on two-player games. We show that quasi-strict equilibria of zero-sum games, unlike Nash equilibria, possess a unique support, and propose linear programs that characterize the quasi-strict equilibria in non-symmetric and symmetric zero-sum games. In Section 4 we turn to games with more than two players. We first distinguish multi-player games where a quasi-strict equilibrium is guaranteed to exist and can be found efficiently from those where existence is not guaranteed. An example of the former class are symmetric games where every player has two actions. We then move on to show that deciding the existence of a quasi-strict equilibrium in games with more than two players is NP-complete in general. This is in contrast to the two-player case, where the decision problem is trivial due to the existence result by Norde [22].

2 Preliminaries

An accepted way to model situations of strategic interaction is by means of a normal-form game [see, *e.g.*, 17].

[2] Using the framework of Peleg and Tijs [25] and Norde et al. [23], quasi-strict equilibria could easily be axiomatically characterized by consistency and *strict* utility maximization in one-player games.

Definition 1 (normal-form game). *A* game in normal-form *is a tuple $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ where $N = \{1, \ldots, n\}$ is a set of* players *and for each player $i \in N$, A_i is a nonempty set of* actions *available to player i, and $p_i : (\times_{i\in N} A_i) \to \mathbb{R}$ is a function mapping each action profile of the game (*i.e., *combination of actions) to a real-valued* payoff *for player i.*

A vector $s \in \times_{i\in N} A_i$ of actions is also called a profile of *pure strategies*. This concept can be generalized to *(mixed) strategy profiles* $s \in S = \times_{i\in N} S_i$ by letting players randomize over their actions. We have S_i denote the set of probability distributions over player i's actions, or *(mixed) strategies* available to player i. We further write s_i and s_{-i}, respectively, for the strategy of player i and the strategy profile for all players but i. For $a \in A_i$, we denote by $s(a)$ the probability with which player i plays a in strategy profile s. A game with two players is often called a bimatrix game. A bimatrix game is a *zero-sum game* if $p_1(a)+p_2(a) = 0$ for all $a \in A$.

Our results on symmetric and anonymous games will be based on the taxonomy introduced by Brandt et al. [4].[3] A common aspect of games in all classes of the taxonomy is that players cannot, or need not, distinguish between the other players. A lattice of four classes of symmetric games is then defined by considering two additional properties: *identical payoff functions* for all players and the ability to *distinguish oneself* from the other players.

Definition 2 (symmetries). *Let $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ be a normal-form game and A a set of actions such that $A_i = A$ for all $i \in N$. For any permutation $\pi : N \to N$ of the set of players, let $\pi' : A^N \to A^N$ be the permutation of the set of action profiles given by $\pi'((a_1, \ldots, a_n)) = (a_{\pi(1)}, \ldots, a_{\pi(n)})$. Γ is called*

- anonymous *if $p_i(s) = p_i(\pi'(s))$ for all $s \in A^N$, $i \in N$ and all π with $\pi(i) = i$,*
- symmetric *if $p_i(s) = p_j(\pi'(s))$ for all $s \in A^N$, $i, j \in N$ and all π with $\pi(j) = i$,*
- self-anonymous *if $p_i(s) = p_i(\pi'(s))$ for all $s \in A^N$, $i \in N$, and*
- self-symmetric *if $p_i(s) = p_j(\pi'(s))$ for all $s \in A^N$, $i, j \in N$.*

It is easily verified that the class of self-symmetric games equals the intersection of symmetric and self-anonymous games, and that both of these are strictly contained in the class of anonymous games. Anonymous multi-player games admit a compact representation when the number of actions is bounded.

One of the best-known solution concepts in game theory is Nash equilibrium [21]. In a Nash equilibrium, no player is able to increase his payoff by unilaterally changing his strategy.

Definition 3 (Nash equilibrium). *Let $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ be a normal-form game. A strategy profile $s \in S$ is called* Nash equilibrium *if for all $i \in N$, $a \in A_i$,*

$$p_i(s) \geq p_i(s_{-i}, a).$$

[3] However, the terminology has been adjusted to coincide with that of Daskalakis and Papadimitriou [7].

maximize v
subject to

$$\begin{array}{lll}
\sum_{a\in A_1} s_1(a)\ p(a,b) & \geq v & \forall b\in A_2\\
s_1(a) & \geq 0 & \forall a\in A_1\\
\sum_{a\in A_1} s_1(a) & = 1 &
\end{array}$$

minimize v
subject to

$$\begin{array}{lll}
\sum_{b\in A_2} s_2(j)\ p(a,b) & \leq v & \forall a\in A_1\\
s_2(b) & \geq 0 & \forall b\in A_2\\
\sum_{b\in A_2} s_2(b) & = 1 &
\end{array}$$

Fig. 1. Primal/dual linear programs for computing minimax strategies in zero-sum games

The solution concept of quasi-strict equilibrium proposed by Harsanyi [12] refines the Nash equilibrium concept by requiring that actions played with positive probability must yield *strictly* more payoff than actions played with probability zero.[4]

Definition 4 (quasi-strict equilibrium). *Let* $\Gamma = (N, (A_i)_{i\in N}, (p_i)_{i\in N})$ *be a normal-form game. A Nash equilibrium* $s \in S$ *is called* quasi-strict *if for all* $i \in N$ *and all* $a, b \in A_i$ *with* $s_i(a) > 0$ *and* $s_i(b) = 0$, $p_i(s_{-i}, a) > p_i(s_{-i}, b)$.

Quasi-strict equilibrium is a very natural concept in that it requires *all* best responses to be played with positive probability.

3 Two-Player Zero-Sum Games

It has been shown by a rather elaborate construction using Brouwer's fixed point theorem that quasi-strict equilibrium always exists in two-player games [22]. Since every quasi-strict equilibrium is also a Nash equilibrium, the problem of finding a quasi-strict equilibrium is intractable unless $P = PPAD$ [5]. The same is true for symmetric two-player games, because the symmetrization of Gale et al. [9] preserves quasi-strictness [15]. For the restricted class of zero-sum games, however, quasi-strict equilibria, like Nash equilibria, can be found efficiently by linear programming. In contrast to Nash equilibria, the support of quasi-strict equilibria in zero-sum games turns out to be unique.

Theorem 1. *Quasi-strict equilibria in two-player zero-sum games possess a unique support and can be computed using the linear program given in Figure 2.*

Proof. It is known from the work of Jansen [13] that every bimatrix game with a convex equilibrium set, and thus every two-player zero-sum game, possesses a quasi-strict equilibrium. We first establish that the support of a quasi-strict equilibrium must contain every action that is played with positive probability

[4] Harsanyi originally referred to quasi-strict equilibrium as "quasi-strong". However, this term has been dropped to distinguish the concept from Aumann's strong equilibrium [1].

$$
\begin{array}{llll}
\multicolumn{4}{l}{\textbf{maximize } \varepsilon} \\
\multicolumn{4}{l}{\textbf{subject to}} \\
\sum_{b \in A_2} s_2(b)\, p(a,b) & \leq v & \forall a \in A_1 \\
s_2(b) & \geq 0 & \forall b \in A_2 \\
\sum_{b \in A_2} s_2(b) & = 1 & \\
s_1(a) + v - \sum_{b \in A_2} s_2(b)\, p(a,b) - \varepsilon & \geq 0 & \forall a \in A_1 \\
\sum_{a \in A_1} s_1(a)\, p(a,b) & \geq v & \forall b \in A_2 \\
s_1(a) & \geq 0 & \forall a \in A_1 \\
\sum_{a \in A_1} s_1(a) & = 1 & \\
s_2(b) + v - \sum_{a \in A_1} s_1(a)\, p(a,b) - \varepsilon & \geq 0 & \forall b \in A_2
\end{array}
$$

Fig. 2. Linear program for computing quasi-strict equilibria in zero-sum games

in some equilibrium of the game. Assume for contradiction that (s_1, s_2) is a quasi-strict equilibrium with value v and $a \in A_1$ is an action with $s_1(a) = 0$. It follows from the definition of quasi-strict equilibrium that $p_1(a, s_2) < v$. Now, if a is in the support of some Nash equilibrium, the exchangeability of equilibrium strategies in zero-sum games requires that $p_1(a, s_2) = v$, a contradiction. As a consequence, the support of any quasi-strict equilibrium is unique and consists precisely of those actions that are played with positive probability in some equilibrium.

In order to compute quasi-strict equilibria, consider the two standard linear programs for finding the minimax strategies for player 1 and 2, respectively, given in Figure 1 [see, *e.g.*, 17]. It is well-known from the minimax theorem [31], and also follows from LP duality, that the value v of the game is identical and unique in both cases. We can thus construct a linear feasibility program that computes equilibrium strategies for both players by simply merging the sets of constraints and omitting the minimization and maximization objectives. Now, quasi-strict equilibrium requires that action a yields strictly more payoff than action b if and only if a is in the support and b is not. For a zero-sum game with value v this can be achieved by requiring that for every action $a \in A_1$ of player 1, $s_1(a) + v > \sum_{b \in A_2} s_2(a)\ p(a, b)$. If $s_1(a) = 0$ (a is not in the support), action a yields strictly less payoff than the game's value. If, on other hand, $s_1(a) > 0$ (a is in the support), these constraints do not impose any restrictions if the strategy profile is indeed an equilibrium with value v, which is ensured by the remaining constraints. Since strict inequalities are not allowed in linear programs, we introduce another variable ε to be maximized. Due to the existence of at least one quasi-strict equilibrium, we will always find a solution with positive ε, turning the weak inequality into a strict one. The resulting linear program is given in Figure 2. □

We proceed by showing that every symmetric zero-sum game contains a symmetric quasi-strict equilibrium. This result should be contrasted to Theorem 3 in Section 4 which establishes that this is *not* the case for symmetric two-player games in general.

$$
\begin{array}{lll}
\textbf{maximize } \varepsilon & & \\
\textbf{subject to} & & \\
\sum_{b \in A_2} s(b)\ p(a,b) & \leq 0 & \forall a \in A_1 \\
s(b) & \geq 0 & \forall b \in A_2 \\
\sum_{b \in A_2} s(b) & = 1 & \\
s(a) - \sum_{b \in A_2} s(b)\ p(a,b) - \varepsilon & \geq 0 & \forall a \in A_1
\end{array}
$$

Fig. 3. Linear program for computing quasi-strict equilibria in symmetric zero-sum games

Theorem 2. *Every symmetric two-player zero-sum game contains a symmetric quasi-strict equilibrium that can be computed using the linear program given in Figure 3.*

Proof. According to Theorem 1, the support of any quasi-strict equilibrium contains all actions that are played with positive probability in some equilibrium. Clearly, in symmetric games, these actions coincide for both players and any minimax probability distribution over these actions constitutes a symmetric equilibrium. Since both players can enforce their minimax value using the same strategy in a symmetric zero-sum game, the value of the game has to be zero. Given that the equilibrium strategies (s, s) have to be symmetric and that the value of the game is known, the linear program in Figure 2 can be significantly simplified, resulting in the linear program given in Figure 3. □

The linear program in Figure 3 can be used to directly compute the *essential set* of a dominance graph. The essential set is defined as the set of actions played with positive probability in some Nash equilibrium of the zero-sum game given by the (symmetric) adjacency matrix of a directed graph [8]. It follows from Theorem 1 that this is exactly the unique support of all quasi-strict equilibria.

4 Multi-Player Games

In games with three or more players the existence of a quasi-strict equilibrium is no longer guaranteed. However, there are very few examples in the literature for games without quasi-strict equilibria.[5]. An important question is of course which natural classes of games always contain a quasi-strict equilibrium. It has already been shown that this is not the case for the class of *single-winner* games which require that all outcomes are permutations of the payoff vector $(1, 0, \ldots, 0)$ [3].

In the following, we will analyze whether *symmetric* and *anonymous* games always admit a quasi-strict equilibrium. It turns out that self-anonymous games, and thus also anonymous games, need *not* possess a quasi-strict equilibrium. For this, consider the following three-player single-loser game where players Alice,

[5] To the best of our knowledge, there are four examples, all of which involve three players with two actions each [29, 16, 6, 3].

Bob, and Charlie independently and simultaneously are to decide whether to raise their hand or not (for instance, in order to decide who has to take out the garbage). Alice loses if exactly one player raises his hand, whereas Bob loses if exactly two players raise their hands, and Charlie loses if either all or no players raise their hand. The matrix form of this self-anonymous game is depicted in Figure 4. The game exhibits some peculiar phenomena, some of which may be attributed to the absence of quasi-strict equilibrium. For example, the security level of all players is 0.5 and the expected payoff in the *only* Nash equilibrium (which has Alice raise her hand and Charlie randomize with equal probability) is $(0.5, 0.5, 1)$. However, the minimax strategies of Alice and Bob are different from their equilibrium strategies, *i.e.*, they can *guarantee* their equilibrium payoff by *not* playing their respective equilibrium strategies.[6] Furthermore, the unique equilibrium is not quasi-strict, *i.e.*, Alice and Bob could as well play any other action without jeopardizing their payoff.

$(1,1,0)$	$(0,1,1)$
$(0,1,1)$	$(1,0,1)$

$(0,1,1)$	$(1,0,1)$
$(1,0,1)$	$(1,1,0)$

Fig. 4. Self-anonymous game without a quasi-strict equilibrium. Players 1, 2, and 3 choose rows, columns, and matrices, respectively. In the only Nash equilibrium of the game player 1 plays his second action, player 2 plays his first action, and player 3 randomizes over both his actions.

For symmetric and self-symmetric games, on the other hand, the picture appears to be different. Self-symmmetric games are a subclass of common-payoff games, where the payoff of all players is identical in every outcome. Starting from an outcome with maximum payoff p for all players, a quasi-strict equilibrium can be found by iteratively adding actions to the support by which a player, and thus all players, can obtain the same payoff p. We will extend this result by showing that the existence of quasi-strict equilibria also holds for *symmetric* games where each player has only two actions at his disposal. It follows from a theorem by Nash [21] that every symmetric game has a *symmetric* Nash equilibrium, *i.e.*, a Nash equilibrium where all players play the same strategy. Perhaps surprisingly, it may be the case that all quasi-strict equilibria of a symmetric game are *asymmetric*.

Theorem 3. *Every symmetric game with two actions for each player has a quasi-strict equilibrium. Such an equilibrium can be found in polynomial time.*

Proof. Let $\Gamma = (N, \{0,1\}^N, (p_i)_{i \in N})$ be a symmetric game. By Definition 2, there exist $2(n-1)$ numbers $p_{m\ell} \in \mathbb{R}$ such that for all i, $p_i(s) = p_{m\ell}$ whenever $s_i = \ell$ and m is the number of players playing action 1 in s_{-i}. We can further assume w.l.o.g. that $p_{00} = p_{01}$ and $p_{n-1,0} \geq p_{n-1,1}$, and that $p_{m0} \neq p_{m1}$ for

[6] Similar phenomena were also observed by Aumann [2].

some m. To see this, observe that Γ must possess a symmetric equilibrium s [21], which we can assume to be the pure strategy profile where all players play action 0 with probability 1. If all players played both of their actions with positive probability, this would directly imply quasi-strictness of s. Now, if one of the former two equations was not satisfied, then one of the two symmetric pure strategy profiles would be a quasi-strict equilibrium. If the latter condition were not to hold, there would exist a quasi-strict equilibrium where all players randomize between their actions.

We will now distinguish two different cases according to the relationship between p_{m0} and p_{m1} for the different values of m. First assume that there exists m such that $p_{m0} > p_{m1}$ and for all $m' < m$, $p_{m'0} = p_{m'1}$. We claim that in this case any strategy profile s in which $m-1$ players randomize between both actions and the remaining $n-m+1$ players play action 0 is a quasi-strict equilibrium. To see this, consider first any player $i \in N$ who randomizes between both of his actions. It is easily verified that for every action profile a which is played with positive probability in s_{-i} and in which exactly m' players play action 1, it must hold that $p_{m'0} = p_{m'1}$. On the other hand, consider any player $i \in N$ who plays action 0 with probability 1. Then, for any action profile a which is played with positive probability in s_{-i} and in which exactly m' players play action 1, it must hold that $p_{m'0} \geq p_{m'1}$, and there exists one such action profile for which the inequality is strict.

Now assume that there exists m such that $p_{m0} < p_{m1}$, and choose m' such that for all m'', $m < m'' < m'$, $p_{m''0} = p_{m''1}$, and either $p_{m'0} > p_{m'1}$ or $m' = n$. We claim that in this case any strategy profile where $n-m'$ players play action 0, m players play action 1, and the remaining $m'-m$ players randomize between both of their actions is a quasi-strict equilibrium of Γ. For this, again consider any player $i \in N$ who plays both actions with positive probability. It is easily verified that for every action profile a which is played with positive probability in s_{-i} and in which exactly m' players play action 1, it must hold that $p_{m'0} = p_{m'1}$. On the other hand, for any player $i \in N$ who plays action 0 with probability 1 and any action profile a which is played with positive probability in s_{-i} and in which exactly m' players play action 1, it must hold that $p_{m'0} \geq p_{m'1}$, and there exists one such action profile for which the inequality is strict. Finally, for any player $i \in N$ who plays action 1 with probability 1 and any action profile a which is played with positive probability in s_{-i} and in which exactly m' players play action 1, it must hold that $p_{m'0} \leq p_{m'1}$, and there exists one such action profile for which the inequality is strict.

Since a symmetric equilibrium of a symmetric game with a constant number of actions can be found in polynomial time [24], and since the proof of the first part of the theorem is constructive, the second part follows immediately. □

We leave it as an open problem whether all symmetric games contain a quasi-strict equilibrium. If the symmetrization procedure due to Gale et al. [9] can be extended to multi-player games while still preserving quasi-strictness, a counter-example could be constructed from one of the known examples of games without quasi-strict equilibria. Of course, in light of Theorem 3, the number of actions

$$
\begin{array}{c|ccc|c|}
 & b_1 & \cdots & b_{|V|} & b_0 \\
\hline
a_1 & & & & (0,0,0) \\
\vdots & & (m_{ij}, e_{ij}, m_{ij})_{i,j \in V} & & \vdots \\
a_{|V|} & & & & (0,0,0) \\
\hline
a_0 & (0,0,0) & \cdots & (0,0,0) & (0,1,0) \\
\hline
\end{array}
\qquad
\begin{array}{c|ccc|c|}
 & b_1 & \cdots & b_{|V|} & b_0 \\
\hline
a_1 & (0,0,K) & \cdots & (0,0,K) & (0,0,0) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{|V|} & (0,0,K) & \cdots & (0,0,K) & (0,0,0) \\
\hline
a_0 & (1,0,0) & \cdots & (1,0,0) & (0,0,0) \\
\hline
\end{array}
$$

$$c_1 \qquad\qquad\qquad\qquad c_2$$

Fig. 5. Three-player game Γ used in the proof of Theorem 4. Players 1, 2, and 3 choose rows columns, and matrices, respectively.

per player in such a counter-example has to be greater than two (and may very well be substantially greater than that).

We conclude by showing that deciding whether a given normal-form game contains a quasi-strict equilibrium is NP-complete.

Theorem 4. *Deciding whether a game in normal-form possesses a quasi-strict equilibrium is NP-complete, even if there are just three players and a constant number of payoffs.*

Proof. Membership in NP is obvious. We can simply guess a strategy profile and verify that it is an equilibrium and that the payoff is strictly lower for all actions that are not played.

For *hardness*, we provide a reduction from the NP-complete problem CLIQUE [see, *e.g.*, 10] reminiscent to a construction used by McLennan and Tourky [18] to give simplified NP-hardness proofs for various problems related to Nash equilibria in bimatrix games. Given an undirected graph $G = (V, E)$ and a positive integer $k \leq |E|$, CLIQUE asks whether G contains a clique of size at least k, *i.e.*, a subset $V' \subseteq V$ such that $|V'| \geq k$ and for all $v, w \in V'$, $(v, w) \in E$. Given a particular CLIQUE instance $((V, E), k)$ with $V = \{1, \ldots, m\}$, we construct a game Γ with three players, actions $A_1 = \{\, a_i \mid i \in V \,\} \cup \{a_0\}$, $A_2 = \{\, b_i \mid i \in V \,\} \cup \{b_0\}$ and $A_3 = \{c_1, c_2\}$, and payoffs p_i illustrated in Figure 5. If player 3 plays c_1 and players 1 and 2 play a_i and b_j, respectively, for $i, j \in V$, payoffs are given by a matrix $(m_{ij})_{i,j \in V}$ defined according to G, and by the identity matrix $(e_{ij})_{i,j \in V}$, where

$$
m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } i = j \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}
$$

If player 3 instead plays c_2, he obtains a payoff of $K = (2k - 3)/2k$. We claim that Γ possesses a quasi-strict equilibrium if and only if there exists a clique of size at least k in G.

Assume there exists a clique $V' \subseteq V$, $|V'| \geq k$, and consider the strategy profile s with $s(c_1) = 1$, and $s(a_i) = s(b_i) = 1/|V'|$ if $i \in V'$ and $s(a_i) = s(b_i) = 0$

otherwise. By construction of Γ, for all $i \in V \cup \{0\}$, $p_2(s_{-2}, b_i) < p_2(s)$ whenever $s(a_i) = 0$. Furthermore, by maximality of V', $p_1(s_{-1}, a_i) < p_1(s)$ for all $i \notin V'$. Finally, $p_3(s) = (k-1)/k > (2k-3)/2k = p_3(s_{-3}, c_2)$. Thus, s is a quasi-strict equilibrium of Γ.

Now assume for contradiction that there is no clique of size at least k in G, and that s is a quasi-strict equilibrium of Γ. In equilibrium, for all $b, b' \in A_2$, we must have $p_2(s_{-2}, b) = p_2(s_{-2}, b')$ whenever $s(b) > 0$ and $s(b') > 0$, and thus, for all $a, a' \in A_1$, $s(a) = s(a')$ whenever $s(a) > 0$ and $s(a') > 0$. As a consequence, for s to be quasi-strict, $s(b_i) > 0$ whenever $s(a_i)$ for all $i \in V \cup \{0\}$. First consider the case where $s(c_1) > 0$. If $s(a_0) = s(b_0) = 1$, s cannot be quasi-strict for player 1. If on the other hand $s(a_i) > 0$ or $s(b_i) > 0$ for some $i \in V$, then there would have to be a set $V' \subseteq V$, $|V'| \geq k$, such that for all $i \in V$ with $s(a_i) > 0$ and all $j \in V'$, $j \neq i$, $p_1(a_i, b_j, c_1) = 1$. By construction of Γ, V' would be a clique of size at least k in G, a contradiction. Now consider the case where $s(c_2) = 1$. If $s(a_0) = 1$ or $s(b_0) = 1$, s is not quasi-strict for player 3. If, on the other hand, $s(a_i) > 0$ and $s(b_j) > 0$ for some $i, j \in V$, then player 1 could deviate to a_0 to get a higher payoff. This completes the proof. $\square$

It follows that the problem of *finding* a quasi-strict equilibrium in games with more than two players is NP-hard (under polynomial-time Turing reductions), whereas no such statement is known for Nash equilibrium.

5 Conclusion

We investigated the computational properties of an attractive equilibrium refinement known as quasi-strict equilibrium. It turned out that quasi-strict equilibria in zero-sum games have a unique support and can be computed efficiently via linear programming. In games with more than two players, finding a quasi-strict equilibrium is NP-hard.

As pointed out in Section 1, classes of games that always admit a quasi-strict equilibrium, such as bimatrix games, are of vital importance to justify rational play based on elementary assumptions. We specifically looked at symmetric and anonymous games and found that self-anonymous games (and thus also anonymous games) may not contain a quasi-strict equilibrium while symmetric games with two actions for each player always possess a quasi-strict equilibrium. Other classes of multi-player games for which this question might be of interest include unilaterally competitive games, potential games, single-winner games where all players have positive security levels, and graphical games with bounded neighborhood.

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