Computing the Minimal Covering Set

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Abstract

We present the first polynomial-time algorithm for computing the minimal covering set of a (weak) tournament. The algorithm draws upon a linear programming formulation of a subset of the minimal covering set known as the essential set. On the other hand, we show that no efficient algorithm exists for two variants of the minimal covering set, the minimal upward covering set and the minimal downward covering set, unless P equals NP. Finally, we observe a strong relationship between von Neumann-Morgenstern stable sets and upward covering on the one hand, and the Banks set and downward covering on the other.

Keywords: Social Choice Theory, Minimal Covering Set, Essential Set, Uncovered Set, Computational Complexity

JEL classification codes: D71 (Social Choice), C63 (Computational Techniques)

1 Introduction

Various problems in the mathematical social sciences can be approached by identifying the "most desirable" elements of a set of alternatives according to some binary dominance relation. Examples are diverse and include selecting socially preferred candidates in social choice settings (*e.g.*, Fishburn, 1977; Laslier, 1997), finding valid arguments in argumentation theory (*e.g.*, Dung, 1995), determining the winners of a sports tournament (*e.g.*, Dutta and Laslier, 1999), making decisions based on multiple criteria (*e.g.*, Bouyssou et al., 2006), choosing the optimal strategy in a symmetric two-player zero-sum game (*e.g.*, Duggan and Le Breton, 1996), and investigating which coalitions will form in cooperative game theory (Gillies, 1959; Brandt and Harrenstein, 2007). In

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social choice theory, where dominance-based solutions are most prevalent, the dominance relation is usually defined as the pairwise majority relation, *i.e.*, an alternative a is said to dominate another alternative b if the number of individuals preferring a to b exceeds the number of individuals preferring b to a. McGarvey (1953) has shown that *any* asymmetric dominance relation can be realized by a particular preference profile, even if individual preferences are required to be linear. As is well known from Condorcet's paradox (de Condorcet, 1785), the dominance relation may thus contain cycles. This implies that the dominance relation need not have a maximum, or even a maximal, element, even if the underlying individual preferences do. As a consequence, the concept of maximality is rendered untenable in most cases, and various so-called *solution concepts* that take over the role of maximality in non-transitive relations have been suggested (*e.g.*, Fishburn, 1977; Miller, 1980; Banks, 1985; Dutta, 1988). Some particularly attractive solution concepts are defined in terms of a *covering* relation—a transitive subrelation of the dominance relation (Gillies, 1959; Fishburn, 1977; Miller, 1980; Dutta, 1988). There are three natural conceptions of covering:

- *upward covering*, where an alternative a is said to cover another alternative b if a dominates b and the alternatives dominating a form a subset of those dominating b,
- *downward covering*, where a covers b if a dominates b and the alternatives dominated by b form a subset of those dominated by a, and
- *bidirectional covering*, where a covers b if a covers b upward and downward.

In tournaments, *i.e.*, complete dominance relations, all three notions of covering coincide. [1] Tournaments have received particular attention in social choice theory because the pairwise majority relation is guaranteed to be complete given an odd number of voters with linear preferences.

Since each of the covering relations is transitive, maximal (*i.e.*, uncovered) elements are guaranteed to exist if the set of alternatives is finite. Consequently, the set of uncovered alternatives for a given covering relation constitutes a natural solution concept. In tournaments, the resulting *uncovered set* turns out to consist precisely of those alternatives that dominate any other alternative along a domination path of length one or two, and is the finest solution concept satisfying the expansion property γ (Moulin, 1986). Dutta and Laslier (1999) generalize Moulin's result and provide an appealing axiomatic characterization of the bidirectional uncovered set for incomplete dominance relations.

[1] Additional covering relations due to Fishburn (1977) and Miller (1980), which do not require that a dominates b for a to cover b, were defined only in the context of tournaments where they again coincide with all other covering relations. Since they possess some undesirable properties for *incomplete* dominance relations (see, *e.g.*, Dutta and Laslier, 1999), we will not consider them here.

Uncovered sets tend to be rather large and are not idempotent as solution concepts. Thus, a natural refinement of the uncovered set can be obtained by repeatedly computing the uncovered set until no more alternatives can be removed. This solution is called the *iterated uncovered set* (see Laslier, 1997). Unfortunately, the iterated uncovered set does not satisfy some criteria that are considered essential for any solution concept. To overcome this problem, Dutta (1988) proposed the *minimal covering set*, which is the smallest set of alternatives (with respect to set inclusion) that satisfies specific notions of internal and external stability (with respect to the underlying covering relation). Minimal covering sets are always contained in their corresponding iterated uncovered set. The minimal *bidirectional* covering set of a dominance relation is regarded as particularly attractive because it is unique and satisfies a large number of desirable criteria (Laslier, 1997; Dutta and Laslier, 1999; Peris and Subiza, 1999). Minimal *upward* and *downward* covering sets are considered for the first time in this paper.

The computational effort required to determine a solution is obviously a very important property of any solution concept. If computing a solution is intractable, the solution concept is rendered virtually useless for large problem instances that do not exhibit additional structure. The importance of this aspect has by no means escaped the attention of economists. For example, Robert Aumann proclaimed in an interview with Eric van Damme (1998): "My own viewpoint is that *inter alia*, a solution concept must be calculable, otherwise you are not going to use it." This paper uses the well-established framework of *computational complexity theory* (see, *e.g.*, Papadimitriou, 1994, for an excellent introduction). Complexity theory deals with *complexity classes* of problems that are computationally equivalent in a certain well-defined way. Typically, problems that can be solved by an algorithm whose running time is polynomial in the size of the problem instance are considered *tractable*, whereas problems that do not admit such an algorithm are deemed *intractable*. The class of decision problems that can be solved in polynomial time is denoted by P, whereas NP (for "nondeterministic polynomial time") refers to the class of decision problems whose solutions can be *verified* in polynomial time. The famous P≠NP conjecture states that the hardest problems in NP do not admit polynomial-time algorithms and are thus not contained in P. Although this statement remains unproven, it is widely believed to be true. A third complexity class we will encounter in this paper is the class coNP of decision problems whose complement is in NP, *i.e.*, problems for which nonexistence of a solution can be verified efficiently. *Hardness* of a problem for a particular class intuitively means that the problem is no easier than any other problem in that class. Both membership and hardness are established in terms of *reductions* that transform instances of one problem into instances of another problem using computational means appropriate for the complexity class under consideration. In the context of this paper, we will be interested in reductions that can be computed in time polynomial in the size of the problem instances. Finally, a problem is said to be *complete* for a complexity class if it is both contained in and hard for that class. Given the current state of complexity theory, we cannot prove the *actual* intractability of most algorithmic problems, but merely give *evidence* for their intractability. Showing NP-hardness of a problem is commonly regarded as very strong evidence for computational intractability because it relates the problem to a large class of problems for which no efficient, *i.e.*, polynomial-time, algorithm is known, despite enormous efforts to find such algorithms. To some extent, the same reasoning can be applied to coNP-hardness.

In the context of this paper, the definition of any solution concept induces a naive algorithm, which exhaustively enumerates all subsets of alternatives and checks which of them comply with the conditions stated in the definition. Not surprisingly, such an algorithm is very inefficient. Yet, proving the intractability of a solution concept essentially means that *any* algorithm that implements this concept is asymptotically as bad as the naive algorithm! While for some solution concepts either efficient algorithms or hardness results have been put forward (see, *e.g.*, Bartholdi, III et al., 1989; Woeginger, 2003; Brandt et al., 2007), very little is known about the computational complexity of solution concepts based on covering relations. In fact, Laslier states that the "computational needs for the different methods to be applied also vary a lot. [...] Unfortunately, no algorithm has yet been published for finding the minimal covering set or the tournament equilibrium set of large tournaments. For tournaments of order 10 or more, it is almost impossible to find (in the general case) these sets at hand" (Laslier, 1997, p. 8).[2] Brandt et al. (2008) have recently shown that Laslier was right about the tournament equilibrium set as computing this set is NP-hard. In contrast, we provide polynomial-time algorithms for finding the minimal bidirectional covering set (the other set Laslier was referring to), the essential set (an attractive subset of the minimal bidirectional covering set), and iterated uncovered sets in this paper. Moreover, we show that deciding whether an alternative is in a minimal upward or downward covering set is NP-hard and that deciding whether an alternative is contained in *all* minimal upward or downward covering sets is coNP-complete. These results imply that there exist no polynomial-time algorithms for computing the minimal upward or downward covering set unless P equals NP. In addition, we derive various set-theoretic inclusions that reveal a strong connection between von Neumann-Morgenstern stable sets and upward covering on the one hand, and the Banks set and downward covering on the other hand. In particular, we show that every stable set is also a minimal upward covering set.

[2] Dutta (1988) also notes the computational difficulty of computing the minimal covering set as he was unable to find an algorithm based on multi-stage elimination trees.

2 Preliminaries

Let A be a finite set of *alternatives* and let $\succ \subseteq A \times A$ be an *asymmetric* and *irreflexive* relation on A, the *dominance relation*. The fact that an alternative a dominates another alternative b, denoted $a \succ b$, means that a is "strictly better than" b or "beats" b in a pairwise comparison. We do *not* in general assume *completeness* or *transitivity* of $\succ$ but allow for ties among alternatives and cyclical dominance. A dominance relation that does satisfy completeness is called a *tournament*. In the literature, the more general case of an incomplete dominance relation as studied in this paper is often referred to as a *weak tournament*. For $B \subseteq A$ and $a \in A$, we will denote by $\overline{D}_B(a)$ and $D_B(a)$ the set of alternatives in B dominating a and dominated by a, respectively, *i.e.*, $\overline{D}_B(a) = \{\, b \in B \mid b \succ a \}$ and $D_B(a) = \{\, b \in B \mid a \succ b \}$. We will sometimes find it convenient to view $\succ$ as a directed *dominance graph* (V, E) with vertex set $V = A$ and $(a, b) \in E$ if and only if $a \succ b$, or as a (skew-symmetric) *adjacency matrix* $M_{A,\succ} = (m_{ij})_{i,j \in A}$ where

$$m_{ij} = \begin{cases} 1 & \text{if } i \succ j, \\ -1 & \text{if } j \succ i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We say that an alternative $a \in A$ is *undominated* relative to $\succ$ whenever $\overline{D}_A(a) = \emptyset$. A special type of undominated alternative is the *Condorcet winner*, an alternative that dominates every other alternative. The concept of a *maximal element* we reserve in this paper to denote an undominated element of a transitive relation. Given its asymmetry, transitivity of the dominance relation implies its acyclicity. The implication in the other direction holds for tournaments but not for the general case. Failure of transitivity or completeness makes that a Condorcet winner need not exist; failure of acyclicity, moreover, that the dominance relation need not even contain maximal elements. As such, the obvious notion of maximality is no longer available to single out the "best" alternatives, and other concepts have been devised to take over its role. In the context of this paper, a *solution concept* is a function f from the set of ordered pairs $(A, \succ)$ to the set of non-empty subsets of A. The value of f for a particular input is commonly referred to as a choice set. While choice sets are always computed for a pair $(A, \succ)$, we will often omit $\succ$ where the meaning is obvious from the context.

3 Covering Relations and Solution Concepts

In this paper we focus on solution concepts based on transitive subrelations of the dominance relation called covering relations.

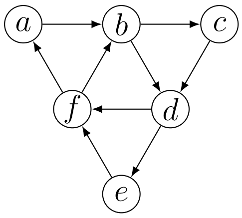

Fig. 1. Example of upward, downward, and bidirectional covering. The set A of alternatives is partitioned into the upward uncovered set $UC_u(A) = \{a, c, e\}$ and the downward uncovered set $UC_d(A) = \{b, d, f\}$, whereas $UC_b(A) = A$.

Definition 1 (covering) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, for any $x, y \in A$,*

- x upward covers y, *denoted xC_uy, if $x \succ y$ and $\overline{D}_A(x) \subseteq \overline{D}_A(y)$;*
- x downward covers y, *denoted xC_dy, if $x \succ y$ and $D_A(y) \subseteq D_A(x)$; and*
- x bidirectionally covers y, *denoted xC_by, if xC_uy and xC_dy.*

It is easily verified that each of these covering relations is asymmetric and transitive, and thus a *strict partial order* on A. The set of maximal elements of such an ordering is referred to as the *uncovered set*.

Definition 2 (uncovered set) *Let A be a set of alternatives, C a covering relation on A. Then, the* uncovered set *of A with respect to C is defined as*

$$UC_C(A) = \{\, x \in A \mid yCx \textit{ for } \text{no } y \in A \,\}.$$

In particular, we will write $UC_u = UC_{C_u}$ for the upward uncovered set, $UC_d = UC_{C_d}$ for the downward uncovered set, and $UC_b = UC_{C_b}$ for the bidirectional uncovered set.

For an example of uncovered sets according to the different covering relations, consider the dominance graph of Figure 1. Here, a upward covers b because f, the only alternative that dominates a, also dominates b. Alternative a itself is not upward covered by f because d and e dominate f but not a. On the other hand, f downward covers a because it dominates b, the only alternative dominated by a. Neither a nor f downward covers b, because the latter is the only alternative that dominates c. By symmetry of the graph, we have $UC_u(A) = \{a, c, e\}$, $UC_d(A) = \{b, d, f\}$, and $UC_b(A) = A$.

The uncovered set is not idempotent as a solution concept and may be applied iteratively to obtain finer solutions. We write $UC_C^k(A) = UC_C(UC_C^{k-1}(A))$ for the kth iteration of UC_C on A and define the iterated uncovered set as the fixed point $UC_C^\infty(A) = UC_C^m(A)$ for some m such that $UC_C^m(A) = UC_C^{m+1}(A)$.

Dutta (1988) proposes a further refinement of the iterated uncovered set in tournaments, which is based on the notion of a *covering set*.

Fig. 2. Minimal upward and downward covering sets need not be unique. There are two minimal upward covering sets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ in the dominance graph on the left, and two minimal downward covering sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ in the dominance graph on the right.

Definition 3 (covering set) *Let A be a set of alternatives, $\succ$ a dominance relation on A, and C a covering relation based on $\succ$. Then, $B \subseteq A$ is a* covering set *for A under C if*

(i) $UC_C(B) = B$, *and*
(ii) *for all* $x \in A \setminus B$, $x \notin UC_C(B \cup \{x\})$.

Properties (i) and (ii) are referred to as *internal* and *external stability* of a covering set, respectively.

For tournaments, where the different notions of covering and uncovered sets coincide, Dutta (1988) proves the existence of a unique *minimal* covering set with respect to set inclusion.[3] Peris and Subiza (1999) and Dutta and Laslier (1999) extend this result to incomplete dominance graphs by showing that there is always a unique minimal *bidirectional* covering set. We will denote the corresponding solution concept by MC. The minimal bidirectional covering set is regarded as particularly attractive because it satisfies a large number of desirable criteria (Laslier, 1997; Dutta and Laslier, 1999; Peris and Subiza, 1999). Furthermore, Duggan and Le Breton (1996) have pointed out that the minimal bidirectional covering set of a tournament coincides with the *weak saddle* of the corresponding adjacency matrix—a game-theoretic solution concept that was proposed independently and much earlier by Shapley (1964).

Figure 2 illustrates that uniqueness of a minimal covering set is not guaranteed for upward or downward covering. $\{x_1, x_2\}$ is a minimal upward covering set for the dominance graph on the left because $x_1 C_u y_1$ in $\{x_1, x_2, y_1\}$ and $x_2 C_u y_2$ in $\{x_1, x_2, y_2\}$, while no single alternative can cover the remaining three alternatives. By a symmetric argument, the same holds for $\{y_1, y_2\}$. All other sets of alternatives either contain one of the former two sets as a proper subset or fail to cover some alternative that is not in the set. For the dominance

[3] Laslier (1997) has shown that, in tournaments, every *minimal* externally stable set automatically satisfies internal stability, thus simplifying the definition of the minimal *bidirectional* covering set (Dutta and Laslier, 1999). The same is not true for the unidirectional variants of the minimal covering set.

graph on the right, x_i is downward uncovered in $\{x_1, x_2, x_3\}$ and y_i is downward covered in $\{x_1, x_2, x_3, y_i\}$ for all $i \in \{1, 2, 3\}$. Since any proper subset of $\{x_1, x_2, x_3\}$ fails to cover y_i for some $i \in \{1, 2, 3\}$, we have actually found a *minimal* downward covering set. We leave it to the reader to verify that apart from the symmetric set $\{y_1, y_2, y_3\}$ there are no additional minimal downward covering sets.

A serious defect of downward covering sets is that they may fail to exist already for very simple instances. For example, no subset of $A = \{a, b, c\}$ with $a \succ b \succ c$ satisfies both internal and external stability under the downward covering relation. The same phenomenon may occur in more complex dominance graphs, even if they are strongly connected, and we will see later that deciding the existence of a downward covering set is NP-complete. [4] Minimal upward covering sets, on the other hand, are guaranteed to exist.

Theorem 1 *There always exists a minimal upward covering set.*

Proof: We show by induction on k that $UC_u^k(A)$ is externally stable for all k, hence $UC_u^\infty(A)$ is a covering set for A. Obviously, $UC_u^1(A) = UC_u(A)$ is externally stable. Now assume for contradiction that for some k, $UC_u^k(A)$ is externally stable and $UC_u^{k+1}(A)$ is not. More precisely, there must be some $x \in A \setminus UC^k(A)$ that is covered in $UC_u^k(A) \cup \{x\}$ and uncovered in $UC_u^{k+1}(A) \cup \{x\}$ and hence some $y \in UC_u^k(A) \setminus UC_u^{k+1}(A)$ such that $y \succ x$ and $z \succ y$ implies $z \succ x$ for all $z \in UC_u^k(A)$. Furthermore, since $y \notin UC_u^{k+1}(A)$, there must be a particular $z \in UC_u^{k+1}$ such that $z \succ y$ and $z' \succ z$ implies $z' \succ y$ for all $z' \in UC_u^{k+1}$. We thus have $z \succ x$, and $z' \succ z$ implies $z' \succ x$ for all $z' \in UC_u^{k+1}$, *i.e.*, z covers x in $UC_u^{k+1} \cup \{x\}$, a contradiction. $\square$

A refinement of the minimal bidirectional covering set can be obtained by considering the *adjacency game* (called tournament game by Dutta and Laslier, 1999) in which two parties propose alternatives $x, y \in A$. The first party wins if $x \succ y$, the second party wins if $y \succ x$, and the game ends in a tie if neither of the two alternatives dominates the other. In other words, the adjacency game $\Gamma(A, \succ)$ is a symmetric two-player zero-sum game where actions correspond to elements of A and the payoff of the first player is given by the adjacency matrix $M_{A,\succ}$ of the dominance graph for A and $\succ$. A (mixed) *strategy* in such a game consists of a probability distribution over the different actions. A pair of strategies is called a *Nash equilibrium* if neither of the two players can increase his (expected) payoff by changing his strategy, given that the strategy of the other player remains the same (see, *e.g.*, Luce and Raiffa, 1957). Dutta and Laslier (1999) define the *essential set* as the set of alternatives for which the corresponding action is played with positive probability

[4] One way to guarantee the existence of a minimal downward covering is to neglect internal stability.

in some Nash equilibrium of the adjacency game. It suffices to restrict attention to symmetric equilibria because the set of equilibria in zero-sum games is convex.

Definition 4 (essential set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, the* essential set *of A is defined as*

$$ES(A) = \{\, a \in A \mid s_a > 0 \textit{ for some } (s,s) \in N(\Gamma(A,\succ)) \,\},$$

where $N(\Gamma)$ denotes the set of Nash equilibria of game Γ and s_a the probability of action a under strategy s.

The essential set generalizes the *bipartisan set*, which is defined in terms of the unique Nash equilibrium of the adjacency game of a tournament (Laffond et al., 1993). The essential set and the solution concepts based on bidirectional covering can be ordered linearly with respect to set inclusion (Dutta and Laslier, 1999): For every dominance graph $(A,\succ)$,

$$ES(A) \subseteq MC(A) \subseteq UC_b^\infty(A) \subseteq UC_b(A).$$

4 Set-Theoretic Relationships

By analyzing set-theoretic relationships between choice sets, one can gain additional insights into the reasons why, and the extent to which, particular solution concepts are different. An almost complete characterization of the relationships between various solution concepts in tournaments is given by Laffond et al. (1995). Bordes (1983) investigates relationships between the different variants of the uncovered set in general dominance graphs. We extend these results for the three variants of the minimal covering set.

It is straightforward to show that every minimal covering set has to be contained in the iterated uncovered set for the same dominance relation. Surprisingly, there exist dominance graphs containing a minimal upward or downward covering set that does *not* intersect with MC. Consider the two dominance graphs shown in Figure 3, and let $\succ_1$ and $\succ_2$ denote the corresponding dominance relations. Under both $\succ_1$ and $\succ_2$, the minimal bidirectional covering set is $\{x_i : 1 \leq i \leq 5\}$. Given $\succ_1$, $Y = \{y_i : 1 \leq i \leq 5\}$ is a minimal upward covering set. Given $\succ_2$, Y is a minimal downward covering set.

Figure 1 illustrates that upward and downward uncovered sets, and hence the corresponding minimal covering sets, can have an empty intersection. This example also reveals an interesting relationship between covering sets and two well-known solution concepts, which we introduce next.

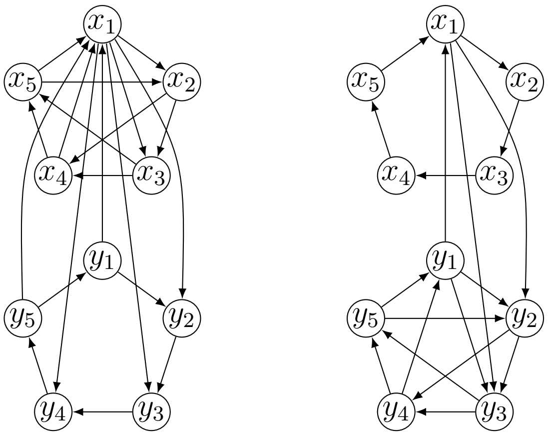

Fig. 3. Not all minimal upward or downward covering sets need to intersect with the minimal bidirectional covering set. In order to improve readability, edges between alternatives x_i and y_i have been drawn only for x_1. The remaining edges follow by rotational symmetry.

A set S of alternatives is called *stable* if no element inside the set can be removed on the grounds of being dominated by some other element in the set, and no element outside the set can be included in the set because some element inside the set dominates it (von Neumann and Morgenstern, 1944).

Definition 5 (stable set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then $S \subseteq A$ is a* (von Neumann-Morgenstern) stable set *if*

(i) $a \succ b$ for no $a, b \in S$ and
(ii) for all $a \notin S$ there is some $b \in S$ with $b \succ a$.

Stable sets are neither guaranteed to exist nor to be unique. Elementary counterexamples are cycles consisting of three or four alternatives, respectively.

The *Banks set* consists of those elements that are the maximal element of $\succ$ for some subset of the alternatives on which $\succ$ is complete and transitive and which is itself maximal with respect to set inclusion (Banks, 1985).[5]

Definition 6 (Banks set) *Let A be a set of alternatives, $\succ$ a dominance relation on A. Then, an alternative a is in the* Banks set *of A, denoted $a \in BA(A)$, if there exists $X \subseteq A$ such that $\succ$ is complete and transitive on X with maximal element a and there is no $b \in A$ such that $b \succ x$ for all $x \in X$.*

Returning to the dominance graph of Figure 1, it is easily verified that there exists a unique stable set $S = \{a, c, e\}$, and that $BA(A) = \{b, d, f\}$. It turns

[5] There are various possible generalizations of the Banks set, which was originally defined for tournaments, to general dominance graphs. This one is referred to as B_1 by Banks and Bordes (1988).

out that the apparent relationship between upward covering and stable sets on the one hand and downward covering and the Banks set on the other is no mere coincidence.

Theorem 2 *Every stable set is a minimal upward covering set and thus contained in the upward uncovered set.*

Proof: To see that every stable set is an upward covering set, consider a particular stable set S and an arbitrary alternative $x \notin S$. By external stability of S, there must be some $y \in S$ such that $y \succ x$. On the other hand, we have $x \not\succ y$ by asymmetry of $\succ$, and $z \not\succ y$ for all $z \in S$ by internal stability of S. Hence, y upward covers x in $S \cup \{x\}$. Minimality of S as a covering set follows directly from its internal stability, because any element $s \in S$ is uncovered in $S' \cup \{s\}$ for every $S' \subseteq S$. $\square$

The previous result is interesting insofar as stable sets are not guaranteed to exist while there always is at least one minimal upward covering set. It further is worth noting that there can be additional minimal upward covering sets, which may have an empty intersection with all stable sets.

Banks and Bordes (1988) have shown that the Banks set is contained in the downward uncovered set. We extend this result by proving that every downward covering set has a non-empty intersection with the Banks set.

Theorem 3 *The Banks set intersects with every downward covering set.*

Proof: It is well-known that the Banks set and the minimal covering set are not contained in each other in tournaments and thus also in general dominance graphs (Laffond et al., 1995). Now consider a downward covering set $X \subseteq A$ and assume for contradiction that $X \cap BA(A) = \emptyset$. We will show by induction on $|X|$ that a necessary condition for X to be a downward covering set is that $\succ$ is complete and transitive on X with maximal element $x \in X$. Then, since $x \notin BA(A)$, there must be some $y \in A$ such that $y \succ x'$ for all $x' \in X$, meaning that X is *not* a covering set. For the basis, $X = \{x\}$ for some $x \in A$ can only be a covering set if x is a Condorcet winner and thus $x \in BA(A)$. Now assume that $|X| = n$ for some $n > 1$. By the induction hypothesis, there must be a set $X' \subseteq X$, $|X'| = n - 1$, such that $\succ$ is complete and transitive on X' with maximal element $x \in X'$. Then, since $x \notin BA(A)$, there must be some $y \in BA(A)$ such that $y \succ x'$ for all $x' \in X'$. In turn, since $y \notin X$, there has to be some $z \in X$ such that $z \succ y$ and $y \succ x'$ implies $z \succ x'$ for all $x' \in A$. In particular, $z \succ x'$ for all $x' \in X$, $x' \neq z$. $\square$

5 Computing Choice Sets

As pointed out in Section 1, computational intractability is a crucial deficiency of a solution concept. To this end, it has been shown that computing the Banks set, the Slater set, and stable sets is NP-hard (Woeginger, 2003; Bartholdi, III et al., 1989; Alon, 2006; Conitzer, 2006; Charbit et al., 2007; Brandt et al., 2007), which is considered very strong evidence that efficient algorithms for computing these sets do not exist. In the following we investigate whether the solution concepts defined in Section 3 are computationally tractable.

5.1 (Iterated) Uncovered Set

We start by showing that all variants of the uncovered set are very easy to compute.

Theorem 4 *Deciding whether an alternative is contained in the upward, downward, or bidirectional uncovered set is in P.*

Proof: We actually show membership, under constant depth reducibility, in the class $AC^0 \subset P$ of problems that can be decided using a Boolean circuit with constant depth and unbounded fan-in (see, *e.g.*, Papadimitriou, 1994). Besides membership in P this also implies that the problem is amenable to parallel computation. For this, we observe that it can be decided in AC^0 whether for a particular pair of alternatives $x, y \in A$, x covers y under any of the three covering relations by checking whether the (relevant part of the) dominance graph for the pairwise majority relation satisfies the conditions of Definition 1. The complete covering relation can be computed by checking all pairs of alternatives in parallel. Finally, a particular alternative is in one of the uncovered sets if and only if it has indegree zero in the graph of the respective covering relation. $\Box$

This result in fact holds for all other covering relations considered by Bordes (1983) and also implies that the iterated uncovered set for each of these covering relations can be computed in polynomial time.

5.2 Essential Set

We continue with the finest solution concept studied in this paper, the essential set. By Definition 4, the essential set can be computed by finding those actions of the adjacency game that are played with positive probability in some Nash equilibrium. Since the set of equilibria in zero-sum games is convex, there is a

Algorithm 1 Essential set

procedure $ES(A, \succ)$

$(m_{ij})_{i,j\in A} \leftarrow M_{A,\succ}$

maximize ε

subject to $\sum_{j\in A} s_j \cdot m_{ij} \leq 0 \quad \forall i \in A$

$\sum_{j\in A} s_j = 1$

$s_j \geq 0 \quad \forall j \in A$

$s_i - \sum_{j\in A} s_j \cdot m_{ij} - \varepsilon \geq 0 \; \forall i \in A$

$B \leftarrow \{\, a \in A \mid s_a > 0 \,\}$

return B

unique support profile of maximum size that corresponds to an equilibrium. This equilibrium further has to be *quasi-strict* in the sense of Harsanyi (1973), which means that equilibrium actions yield *strictly* more expected payoff than non-equilibrium actions, and *symmetric* due to the symmetry of the adjacency game (see Brandt and Fischer, 2008, for more details). The well-known fact that the value of any *symmetric* zero-sum game is zero allows us to construct a straightforward linear feasibility program for finding some equilibrium of the adjacency game. The additional requirement of quasi-strictness can be translated to strict inequalities which require that the probability each action is played with exceeds the payoff of this action. Finally, strict inequalities are converted into weak inequalities by introducing a variable ε to be maximized (see Algorithm 1). It turns out that computing the essential set is not only contained in the class P of efficiently solvable problems, but also belongs to the hardest problems in this class.

Theorem 5 *Deciding whether an alternative is contained in the essential set is P-complete.*

Proof: For membership we need to show that Algorithm 1 is correct and can be executed in polynomial time. Correctness follows from the equivalence of the essential set and the unique support of a quasi-strict equilibrium in the adjacency game (Brandt and Fischer, 2008). The running time is determined by a linear programming problem, which can be solved in polynomial time (Khachiyan, 1979).

For hardness, we take advantage of the well-known reduction from linear programs to symmetric two-player zero-sum games due to Dantzig (1951) (see also Luce and Raiffa, 1957, pp. 419–423). More precisely, this reduction shows that deciding whether a particular action in a zero-sum game is played with positive probability is at least as hard as deciding feasibility of a linear program. The latter problem is known to be P-complete, even when coefficients are restricted to $\{-1, 0, 1\}$, which in turn is ensured by Dantzig's reduction (Greenlaw et al., 1995). □

Algorithm 2 Minimal bidirectional covering set

procedure $MC(A, \succ)$
  $B \leftarrow ES(A, \succ)$
  loop
    $A' \leftarrow \bigcup_{a \in A \setminus B} (UC(B \cup \{a\}) \cap \{a\})$
    if $A' = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup ES(A', \succ)$
  end loop

5.3 Minimal Bidirectional Covering Set

Since its introduction in 1988, there has been doubt whether the minimal bidirectional covering set can be computed efficiently (Dutta, 1988; Laslier, 1997). Interestingly, in contrast to all other solution concepts considered in this paper, there is no obvious reason why the corresponding decision problem should be in NP, *i.e.*, even *verifying* whether a given set is indeed a minimal covering set is a non-trivial task. While it can easily be checked whether a set is a covering set, verification of minimality is problematic. On the other hand, the problem of verifying a minimal covering set and the more general problem of deciding whether a given alternative a is contained in MC are both in coNP. This is due to the fact that MC is contained in all covering sets. In other words, $a \notin MC$ if and only if there is a (not necessarily minimal) covering set $B \subseteq A$ with $a \notin B$.

By inclusion of ES in MC, Algorithm 1 provides a way to efficiently compute a *non-empty subset* of MC. While in general the existence of an efficiently computable subset cannot be exploited to efficiently compute the set itself,[6] it is of great benefit in our context.

The algorithm we propose for computing MC (Algorithm 2) indeed starts with the essential set and then iteratively adds specific elements outside the current set that are still uncovered. The crux of the matter is to only add elements that may not be covered in a later iteration, and it is not obvious which elements these should be. We show that elements in the minimal bidirectional covering set of the *restriction* of the dominance relation to the uncovered alternatives can be safely added to the current set. Since a non-empty subset of any minimal covering set, *viz.*, the essential set, can be found efficiently, this completes the algorithm. The backbone of Algorithm 2 is the following insight.

Lemma 1 *Let $(A, \succ)$ be a dominance graph, $B \subseteq MC(A)$, and $A' =$*

[6] For example, Hudry (2004) has pointed out that single elements of the Banks set—and thus a subset—can be found efficiently, whereas computing the entire set is NP-hard.

$\bigcup_{a\in A\setminus B}(UC(B\cup\{a\})\cap\{a\})$. *Then,* $MC(A')\subseteq MC(A)$.

Proof: Partition A', the set of alternatives not covered by B, into two sets C and C' of elements contained in $MC(A)$ and elements not contained in $MC(A)$, *i.e.*, $C = A'\cap MC(A)$ and $C' = A'\setminus MC(A)$. We will show that C is externally stable for A'. Since $MC(A')$ must lie in the intersection of all sets that are externally stable for A', this means that $MC(A')\subseteq MC(A)$.

In the following, we will use an easy consequence of the definition of the bidirectional covering relation: for two sets X, X' with $X\subseteq X'\subseteq A$ and two alternatives $x, y\in X$, if y covers x in X', then y also covers x in X. We will refer to this property as "covering in subsets".

Let $x\in C'$. Since $x\notin MC(A)$, there has to be some $y\in MC(A)$ that covers x in $MC(A)\cup\{x\}$. It is easy to see that $y\notin B$. Otherwise, since $B\subseteq MC(A)$ and by covering in subsets, y would cover x in $B\cup\{x\}$, contradicting the assumption that $x\in A'$. On the other hand, assume that $y\in MC(A)\setminus(B\cup C)$. Since y covers x in $MC(A)\cup\{x\}$, $B\subseteq MC(A)$, and by covering in subsets, we have that $y\succ x$, $\overline{D}_B(y)\subseteq\overline{D}_B(x)$, and $D_B(x)\subseteq D_B(y)$. Furthermore, since $y\notin A'$, there has to be some $z\in B$ such that $z\succ y$, $\overline{D}_B(z)\subseteq\overline{D}_B(y)$, and $D_B(y)\subseteq D_B(z)$. Combining the two, we get $z\succ x$, $\overline{D}_B(z)\subseteq\overline{D}_B(x)$, and $D_B(x)\subseteq D_B(z)$, *i.e.*, z covers x in $B\cup\{x\}$. This again contradicts the assumption that $x\in A'$. It thus has to be the case that $y\in C$. Since $C\subseteq MC(A)$, it follows from covering in subsets that y also covers x in $C\cup\{x\}$. We have shown that for every $x\in C'$, there exists $y\in C$ such that y covers x in $C\cup\{x\}$, *i.e.*, C is externally stable for A'. $\square$

Since B and A' in the statement of Lemma 1 are always disjoint, we obtain a valuable tool: For every proper subset of the minimal bidirectional covering set, the lemma tells us how to find another disjoint and non-empty subset. This insight can be used to iteratively compute MC.[7]

Theorem 6 *The minimal* bidirectional *covering set can be computed in time polynomial in the number of alternatives.*

Proof: We prove that Algorithm 2 computes the minimal bidirectional covering set and runs in time polynomial in the number of alternatives. In each iteration of the algorithm, at least one element is added to set B, so the algorithm is guaranteed to terminate after a linear number of iterations. During each iteration the algorithm determines A', which is feasible in polynomial time according to Theorem 4, and computes ES for a subset of the alternatives, which by Theorem 5 requires only polynomial time.

[7] Lemma 1 can also be used to construct a *recursive* algorithm for computing MC. However, such an algorithm has exponential worst-case running time.

As for correctness, a simple induction on the number of iterations shows that $B \subseteq MC(A)$ holds at any time. When the algorithm terminates, B is a covering set for A, so we must actually have $B = MC(A)$. The base case follows directly from the fact that $ES(A) \subseteq MC(A)$. Lemma 1 implies the induction step. $\square$

5.4 Minimal Unidirectional Covering Sets

A potential problem of bidirectional covering is that it is not very discriminatory. One might thus try to obtain smaller choice sets by considering covering in one direction only. It turns out that this renders the computational problems hard.

Theorem 7 *Deciding whether an alternative is contained in some minimal* upward *covering set is NP-hard.*

Proof: We give a reduction from the NP-complete Boolean satisfiability problem (SAT, see, *e.g.*, Papadimitriou, 1994). For a particular Boolean formula $B = \bigwedge_{i=1}^{k} c_i$ over a set V of $|V| = n$ variables, we construct a dominance relation $\succ$ over a set A of alternatives such that a particular alternative $d \in A$ is in some minimal upward covering set for A if and only if B is satisfiable.

For each variable $v_i \in V$, $1 \leq i \leq n$, we introduce four alternatives x_i, $\bar{x}_i$, x'_i, and $\bar{x}'_i$ such that $x_i \succ \bar{x}_i \succ x'_i \succ \bar{x}'_i \succ x_i$. This corresponds to the 4-cycle on the left of Figure 2, which we have observed to have exactly two minimal upward covering sets. For each clause c_j, $1 \leq j \leq k$, we add an alternative y_j such that $x_i \succ y_j$ if variable $v_i \in V$ appears in clause c_j as a positive literal, and $\bar{x}_i \succ y_j$ if variable v_i appears in clause c_j as a negative literal. We finally add an alternative d for which we want to decide membership in a minimal upward covering sets and let $y_j \succ d$ for $1 \leq j \leq k$.

First of all, we observe that every minimal upward covering set for A and $\succ$ must contain either x_i and x'_i or $\bar{x}_i$ and $\bar{x}'_i$, but not both, for all i, $1 \leq i \leq n$. Obviously, at least one of the pairs is needed to cover the other pair. On the other hand, consider a covering set that contains all four alternatives for some i. Then, neither x_i nor $\bar{x}_i$ can cover any of the alternatives y_j because $\bar{x}'_i \succ x_i$ but $\bar{x}'_i \not\succ y_i$ and we can assume w.l.o.g. that $x_i \succ y_j$ and $\bar{x}_i \succ y_j$ do not hold at the same time. We could thus obtain a smaller covering set by removing either pair of alternatives, meaning that the original covering set was not minimal.

Now assume that there exists a satisfying assignment $\phi : V \to \{true, false\}$ for B, and consider the set $A' \subseteq A$ of alternatives containing d along with x_i and x'_i if $\phi(v_i) = true$ and $\bar{x}_i$ and $\bar{x}'_i$ if $\phi(v_i) = false$, for all i, $1 \leq i \leq n$. Since ϕ

is a satisfying assignment, there exists, for each j, $1 \leq j \leq k$, some $a_j \in A'$ with $a_j \succ y_j$ and $b \succ a_j$ for no $b \in A'$, such that a_j covers c_j. Furthermore, alternatives x_i, $\bar{x}_i$, x'_i, and $\bar{x}'_i$, $1 \leq i \leq n$, are either in A' or covered by some element of A'. It is easily verified that removing any set of elements from A' leaves some element uncovered. Thus, A' is a minimal upward covering set for A and $\succ$.

On the other hand, consider a minimal upward covering set A' for A and $\succ$ such that $d \in A'$. Again, A' will contain exactly one pair of alternatives x_i and x'_i or $\bar{x}_i$ and $\bar{x}'_i$ for each i, $1 \leq i \leq n$. Furthermore, A' may not contain y_j for some j, $1 \leq j \leq k$, as any $a \in A'$ with $a \succ y_j$ would cover y_j, while y_j with $a \succ y_j$ for no $a \in A$ would itself cover d. It is easily verified that for all i, $1 \leq i \leq n$, A' must contain exactly one of x_i or $\bar{x}_i$, and letting $\phi(v_i) = \mathit{true}$ if $x_i \in A'$ and $\phi(v_i) = \mathit{false}$ otherwise yields a satisfying assignment for B. $\square$

Theorem 8 *Deciding whether an alternative is contained in some minimal* downward *covering set is NP-hard, even if a downward covering set is guaranteed to exist.*

Proof: We again give a reduction from SAT. For a particular Boolean formula $B = \bigwedge_{i=1}^{k} c_i$ over a set V of $|V| = n$ variables, we construct a dominance relation $\succ$ over a set A of alternatives such that a particular alternative $d \in A$ is in some minimal downward covering set for A if and only if B is satisfiable.

For each variable $v_i \in V$, $1 \leq i \leq n$ we introduce six alternatives x_i, $\bar{x}_i$, x'_i, $\bar{x}'_i$, x''_i, and $\bar{x}''_i$ such that $x_i \succ \bar{x}_i \succ x'_i \succ \bar{x}'_i \succ x''_i \succ \bar{x}''_i \succ x_i$, $x_i \succ x'_i \succ x''_i \succ x_i$, and $\bar{x}_i \succ \bar{x}'_i \succ \bar{x}''_i \succ \bar{x}_i$. This corresponds to the dominance graph shown on the right of Figure 2, which we have observed to have exactly two minimal downward covering sets. For each clause c_j, $1 \leq j \leq k$, we add an alternative y_j such that $y_j \succ x_i$ if variable v_i appears in clause c_j as a positive literal, and $y_i \succ \bar{x}_i$ if v_i appears in c_j as a negative literal. We finally add an alternative d for which we want to decide membership in a minimal downward covering sets and let $d \succ y_j$ for all j, $1 \leq j \leq k$, as well as k additional alternatives z_i, $1 \leq i \leq k$, such that $z_i \succ d$, and $z_i \succ y_j$ if $j \neq i$.

We observe the following properties of downward covering sets for A and $\succ$:

- Every downward covering set must contain the undominated alternatives z_j, $1 \leq j \leq k$.
- Every *minimal* downward covering set must contain either x_i, x'_i, and x''_i or $\bar{x}_i$, $\bar{x}'_i$, and $\bar{x}'_i$, but not both, for all i, $1 \leq i \leq n$. For one, a set not containing at least one of the two three-cycles cannot be externally stable. Assume for contradiction that X is such a set. Then, an alternative $x \in \{x_i, \bar{x}_i | 1 \leq i \leq n\}$ would have to be covered by c_j for some j, which in turn can only be the case if X does not contain the two alternatives dominated by x. In this case, one of the latter two alternatives is not dominated by any

element of X, contradicting external stability. On the other hand, consider a covering set that contains all six alternatives for some i. Since none of these alternatives dominates any alternative outside the six-cycle, we can obtain a smaller covering set by removing either one of the three-cycles, meaning that the original covering set was not minimal.

- For each j, $1 \leq j \leq k$, y_j can be in a covering set only if this set also contains some alternative $a \in A$ with $y_j \succ a$, *i.e.*, an alternative corresponding to a literal that occurs in clause c_i. Otherwise, y_j is covered by some z_i, $i \neq j$.
- d is in a covering set if and only if this set also contains all y_j, $1 \leq j \leq k$. In a covering set not containing y_j, d is covered by z_j.

Now assume that there exists a satisfying assignment $\phi : V \to \{\mathit{true}, \mathit{false}\}$ for B, and consider the set $A' \subseteq A$ of alternatives containing d, y_j and z_j for all j, $1 \leq j \leq k$, along with x_i, x_i', and x_i'' if $\phi(v_i) = \mathit{true}$ and $\bar{x}_i$, $\bar{x}_i'$, and $\bar{x}_i''$ if $\phi(v_i) = \mathit{false}$. Since ϕ is a satisfying assignment, there exists, for each j, $1 \leq j \leq k$, some $a_j \in A'$ with $y_j \succ a_j$, and $b \succ y_j$ and $b \succ a_j$ for no $b \in A'$, such that y_j is uncovered in A'. Furthermore, d is uncovered in A' because $y_j \in A'$ for all j, $1 \leq j \leq k$. It is easily verified that no proper subset of A' is a covering set, as this would require removal of x_i or $\bar{x}_i$ for some i, $1 \leq i \leq n$.

On the other hand, consider a minimal downward covering set A' for $\succ$ such that $d \in A'$. Again, A' will contain exactly one of the three-cycles x_i, x_i', and x_i'' or $\bar{x}_i$, $\bar{x}_i'$, and $\bar{x}_i''$. Furthermore, for d to be uncovered, A' must contain y_j for all j, $1 \leq j \leq k$. This means, however, that for every j, $1 \leq j \leq k$, A' must contain $a_j = x_i$ or $a_j = \bar{x}_i$ for some i, $1 \leq i \leq n$, such that $y_j \succ a_j$. Letting $\phi(v_i) = \mathit{true}$ if $x_i \in A'$ and $\phi(v_i) = \mathit{false}$ otherwise yields a satisfying assignment for B. $\Box$

Even deciding whether a minimal downward covering set exists is computationally intractable unless P equals NP.

Theorem 9 *Deciding whether there exists a minimal downward covering set is NP-complete.*

Proof: Membership in NP is obvious. We can simply guess a subset of the alternatives and verify that it is downward covering. Either this set is a minimal downward covering set itself or it contains a minimal downward covering set.

For *hardness*, recall that in the construction used in the proof of Theorem 8, a minimal downward covering set containing alternative d exists if and only if the Boolean formula B is satisfiable. We add three additional alternatives e_1, e_2, and e_3 to this construction such that $e_1 \succ e_2 \succ e_3 \succ d$, and observe the following:

- If there exists a satisfying assignment for B, we can construct a set A'

of alternatives that is internally and externally stable with respect to the original set of alternatives. It is easily verified that $A' \cup \{e_1, e_2, e_3\}$ is a covering set for the new set.
- On the other hand, assume there exists a set A' of alternatives that does *not* contain d and is both internally and externally stable. Due to external stability, A' must contain the undominated element e_1, and it must therefore contain e_3 if it contains e_2. Because of internal stability, on the other hand, A' cannot contain e_3, and thus cannot contain e_2 by the previous argument. This leaves e_3 uncovered in $A' \cup \{e_3\}$, contradicting external stability.

We have provided a reduction from SAT to the problem of deciding whether there exists a minimal downward covering set, showing that the latter is NP-hard. $\square$

Since minimal unidirectional covering sets are not unique, another computational problem of interest is whether an alternative is contained in *all* minimal covering sets. It can be inferred from our previous constructions that this problem is coNP-complete.

Theorem 10 *Deciding whether an alternative is contained in all minimal upward covering sets or all minimal downward covering sets is coNP-complete.*

Proof: Membership in coNP follows from the argument given in Section 5.3. Hardness can be established by a reduction from the coNP-complete problem VALIDITY, which asks whether a given Boolean formula is valid, *i.e.*, satisfied for all truth assignments (see, *e.g.*, Papadimitriou, 1994). In both constructions used in the proofs of Theorem 7 and 8, d is contained in *all* minimal covering sets if and only if ϕ is valid. $\square$

6 Conclusions

We have investigated solution concepts for dominance graphs that are based on the notion of covering, and analyzed their computational complexity. It turned out that polynomial-time algorithms exist for computing (iterated) uncovered sets, the essential set, and the minimal bidirectional covering set. In contrast, we showed that deciding whether an alternative is in some minimal upward or downward covering set is NP-hard. This is particularly interesting, because we further showed that these sets are related to von Neumann-Morgenstern stable sets and to the Banks set, respectively, which are also known to be computationally intractable unless P equals NP (Brandt et al., 2007; Woeginger, 2003). Table 1 summarizes our main results.

Our algorithm for computing the minimal bidirectional covering set MC underlines the significance of MC as a practical solution concept. MC was orig-

	existence	uniqueness	complexity
UC_b, UC_u, UC_d	✓	✓	in P
UC_b^∞, UC_u^∞, UC_d^∞	✓	✓	in P
MC	✓	✓	in P
minimal upward covering	✓	—	NP-hard
minimal downward covering	—	—	NP-hard
ES	✓	✓	P-complete

Table 1
Existence, uniqueness, and complexity of the solution concepts studied in this paper

inally introduced as a refinement of the uncovered set that is superior to the Slater set because the latter fails to satisfy a number of rather weak expansion-consistency conditions (Dutta, 1988).[8] Now it has turned out that *MC* can further be computed in polynomial time, whereas computing the Slater set is NP-hard and all known algorithms have exponential worst-case complexity (see Conitzer, 2006; Charon and Hudry, 2007). Due to the equivalence pointed out by Duggan and Le Breton (1996), our algorithm for computing *MC* can also be applied for finding the unique weak saddle in a subclass of symmetric two-player zero-sum games.

Typically, more sophisticated solution concepts are also harder to compute than simple ones. What makes *MC* very appealing is the fact that its computation is non-trivial, yet barely manageable in polynomial time.

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[8] The same criticism also holds for the Kemeny set.

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