Queen Mary & Westfield College
UNIVERSITY OF LONDON

MAS/320 NUMBER THEORY

Thu June 1 2000, 14:30

Duration: 3 hrs.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are permitted, but any programming, plotting or algebraic facility may not be used.

[**1**] *Continued fractions* [4+8]

(a) Define a *best rational approximant* of a real number, explaining the connection with continued fractions. Give an example of a best approximant.

(b) Let

$$\alpha = \sqrt{\log 2} = [0, 1, 4, 1, 34, 1, 5, \ldots].$$

i) Show that no convergent of α has two decimal digits in the denominator.

ii) State the *approximation theorem* for continued fractions. Use it to prove that there exists a rational number with a single decimal digit at denominator whose distance from α is less than 10^{-3}.

[**2**] *Quadratic surds* [4+4+12+14]

(a) Let n be a positive integer. Expand

$$\sqrt{\frac{n+1}{n}}$$

into a continued fraction.

(b) Define a *reduced* quadratic surd. Give an example. Characterize reduced surds in terms of continued fractions.

(c) Let D be a fixed positive integer, not a square.

i) Describe the structure of the continued fraction expansion of $\sqrt{D}$, and characterize the values of D for which the periodic part has unit period.

ii) Express 269 as a sum of two squares, using continued fractions.

iii) Let $r(D)$ be the number of reduced surds of the form $(\sqrt{D}+P)/Q$, with P and Q integers such that Q divides $D-P^2$. Prove that the period of the continued fraction expansion of $\sqrt{D}$ cannot be greater than $r(D)$.

(d)

i) Write an essay on *Pell's equation.* The essay should be approximately 200 words long. The use of mathematical symbols should be kept to a minimum.

ii) Find one solution of the equation $x^2 - 269\,y^2 = -1$.

[**3**] *Quadratic forms* [4+8+8+14]

(a) List all principal forms of discriminant D, for $-100 < D < -90$.

(b) Consider the following forms

$$Q_1(x, y) = 5x^2 - 4xy + 2y^2; \qquad Q_2(x, y) = 11x^2 + 30xy + 21y^2.$$

i) Show that Q_1 and Q_2 are equivalent.

ii) Represent 5 with Q_1 (by inspection), whence exploit the above equivalence to represent 5 by Q_2.

(c) Let $Q(x, y) = -2\,x^2 + xy + 2y^2$.

i) Show that Q is reduced, whence determine the number of reduced quadratic forms which are equivalent to Q.

ii) Explain what is meant by a *right neighbour* of a quadratic form, whence determine the right neighbour of Q which is reduced.

(d)

i) Write an essay on the *class number.* The essay should be approximately 200 words long. The use of mathematical symbols should be kept to a minimum.

ii) Compute the class number for the discriminant $D = -52$.

[**4**] *Modular arithmetic* [4+4+12]

(a) Let ϕ be the Euler's ϕ-function. Prove that $\phi(n)$ is even if and only if $n > 2$.

(b) Define a *primitive root.* Show that there are 32 primitive roots modulo 97. Given that 5 is a primitive root modulo 97, find another primitive root and an element of order 24.

(c) Let (a/p) be the Legendre symbol.

i) Compute $(127/179)$.

ii) Prove that if p is an odd prime, and $p \neq 7$, we have

$$\left(\frac{-7}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 2, 4 \pmod 7 \\ -1 & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

iii) Prove that a prime p is represented by the quadratic form $x^2 + xy + 2y^2$ if and only if $p = 7$ or $p \equiv 1, 2, 4 \pmod 7$.

End of examination paper