MAS/320 Number Theory: Coursework 3

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http://www.maths.qmw.ac.uk/~fv/teaching/nt/nt.html

This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 7, at 1:00 pm.

CONTENT: Periodic continued fractions and applications.

M★croESSAY : Write an essay on periodic continued fractions. (Approximately 100 words, and no mathematical symbols.)

Problem 1. For the following values of α, find a quadratic polynomial with integer coefficients having α as a root. Such coefficients must have no common factor.

$$a)\quad \frac{(\sqrt{n})^3-\sqrt{n}}{(\sqrt{n})^5+(\sqrt{n})^6} \qquad b)\quad (1+\sqrt{3})^5 \qquad c)\quad 2\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{5}}\right)^k$$

Problem 2. In the following cases

$$a)\quad D=2; \qquad b)\quad D=5; \qquad c)\quad D=11$$

list all pairs (P,Q) for which the quadratic expression $(\sqrt{D}+P)/Q$ is reduced, identifying those pairs for which $(D-P^2)/Q$ is an integer.

Problem 3. Write the following integers as a sum of two squares by using the Legendre construction.

$$a)\quad 197; \qquad b)\quad 181; \qquad c)\quad 10733.$$

Problem 4. In the following cases

$$a)\quad D=23; \qquad b)\quad D=13; \qquad c)\quad D=97$$

find the fundamental solution of Pell's equation, and determine if the equation $x^2-Dy^2=-1$ is also solvable. If so, find the fundamental solution.

Problem 5. Let $x=p$, $y=q$ be the fundamental solution to Pell's equation $x^2-Dy^2=1$, and let $x=x_i, y=y_i$, for $i=m-1,m,m+1$ be three consecutive solutions.

(a) Prove that $x_{m+1}=px_m+qDy_m$ and $y_{m+1}=q\,x_m+p\,y_m$.

(b) Prove that $x_{m+1}=2\,p\,x_m-x_{m-1}$ and $y_{m+1}=2\,p\,y_m-y_{m-1}$.

(c) Consider Pell's equation $x^2 - 2y^2 = 1$. Find the fundamental solution, and hence use one of the above recursion formulae to find a solution for which both x and y are greater than 10^6.

Problem 6. We define the *Fibonacci numbers* F_n recursively as follows

$$F_0 = 0 \qquad F_1 = 1 \qquad\qquad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1. \tag{1}$$

The first terms in the Fibonacci sequence are: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

(a) Show that the convergents of

$$\gamma = \frac{1+\sqrt{5}}{2}$$

satisfy the relation

$$\frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}} \qquad n \geq 0. \tag{2}$$

(b) Prove the following formula

$$F_n = \frac{1}{\sqrt{5}}\left[\gamma^n - (\gamma')^n\right] \qquad n \geq 0 \tag{3}$$

where γ' is the algebraic conjugate of γ.

[*Hint:* verify that the right hand side of the above equation satisfies (1).]

(c) Prove the *Cassini identity:*

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \qquad n \geq 1. \tag{4}$$

(d) Prove (by induction on k) the following formula

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \qquad\qquad n \geq 0, \quad k \geq 1. \tag{5}$$

(e) Use (5) to prove by induction that for any integer $s \geq 0$, F_{sn} is a multiple of F_n, for all $n \geq 0$.

(f)* The *generating function* G of the sequence F_k is the following power series

$$G(x) = \sum_{k=0}^{\infty} F_k\, x^k = x + x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots. \tag{6}$$

Prove that

$$G(x) = \frac{x}{1 - x - x^2},$$

and hence, or otherwise, determine the radius of convergence of the power series (6).

[*Hint:* compute $G(x) - xG(x) - x^2G(x)$.]