MAS/320 Number Theory: Coursework 1

Franco VIVALDI

http://www.maths.qmw.ac.uk/~fv/teaching/nt/nt.html

This coursework will be assessed and count towards your final mark for the course

DEADLINE: Wednesday of week 3, at 1:00 pm.

CONTENT: Simple continued fractions.

MicroESSAY : Write an essay on rational numbers (approximately 100 words, and no mathematical symbols).

Problem 1. Expand into simple continued fractions (with the last term > 1):

$$a)\ \frac{118}{303} \qquad b)\ -\frac{21}{55} \qquad c)\ \frac{10001}{10101} \qquad d)\ \frac{12}{240005}.$$

Problem 2. Compute the convergents of the following continued fractions

$$a)\ [0;2,4,1,5] \qquad b)\ [-100;1,100].$$

Problem 3. Find the general solution of the following equation, and characterize the solutions for which x and y are positive.

$$75x - 131y = 19.$$

Problem 4. Let n be a positive integer. Expand into simple continued fractions:

$$a)\ \frac{n+1}{n} \qquad b)\ \frac{8n+5}{5n+3} \qquad c)\ \frac{n-1}{n^k}, \quad n > 1, \quad k \geq 1.$$

Problem 5. Show by induction that if $a_0, \ldots, a_n$ are positive integers, and $p_n/q_n = [a_0; \ldots, a_n]$, then

$$a)\ \frac{p_n}{p_{n-1}} = [a_n; a_{n-1}, \ldots, a_0] \qquad b)\ \frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_1].$$

Problem 6. Let p and q be coprime positive integers, with $p > q$, and let $p/q = [a_0; a_1, \ldots, a_n]$.

(a) Show that if the continued fraction is *symmetric,* that is, if $a_0 = a_n, a_1 = a_{n-1}, \ldots$, then p divides $q^2 + (-1)^{n+1}$. [*Hint:* use problem 5.]

(b) White the continued fraction $[3; 1, 1, 2, 2, 1, 1, 2, 1]$ in symmetric form, and verify the proposition in part (a).

(c) Conversely, prove that according as p divides $q^2 + 1$ or $q^2 - 1$, where $p > q > 0$ and p and q are coprime, then p/q develops into a symmetric continued fraction with an even or an odd number of partial quotients. [*Hint:* show that $q_n = p_{n-1}$, where $p_n = p$, $q_n = q$. Then use problem 5.]

(d) Verify the proposition in part (c) for $q = 13$ and p equal to the three largest divisors of $q^2 + 1$. Do the same for p equal to the three smallest divisors of $q^2 - 1$, which are greater than q.

Problem 7*. Let $[a_0; a_1, \ldots, a_k]$, with $a_k > 1$, be the continued fraction expansion of a rational number x. We define a function τ by letting $\tau(x) = k$, the number of convergents in the fractional part of x.

Let F_n be the set of rational numbers between 0 and 1, whose denominator is not greater than n. The set F_n contains a finite number of elements, e.g,

$$F_5 = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\}.$$

We let $\mathcal{T}(n)$ be the maximum value attained by $\tau(x)$ as x scans all the elements of the set F_n

$$\mathcal{T}(n) = \max_{x \in F_n} \tau(x).$$

Because $3/4 \in F_5$, and $3/4 = [0; 1, 3]$, we obtain the estimate $\mathcal{T}(5) \geq 2$ (in fact, $\mathcal{T}(5) = 3$).

Determine a lower bound for $\mathcal{T}(10000)$. (The greater the bound, the greater the mark.)