MTH5117 Mathematical writing: Coursework 8

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DEADLINE: Sunday of week 11, at 23.55.

ASSESSED PROBLEMS [with allocated marks].
Problem 1: 2, 3 [35]. Problem 2: [25]. Problem 3: (b) [40].

Problem 1. Each of the following statements is (equivalent to) an implication, which may be true or false. For each implication

i) state whether it is true or false;
if it is false, produce a counterexample;
ii) state the contrapositive, and whether it is true or false;
iii) state the converse, and whether it is true or false;
if it is false, produce a counterexample;
iv) state the negation, and whether it is true or false.

[*Make sure that the truth or falsehood of i), ii), iv) are consistent.*]

1. *Two integers that are coprime are also prime.*

2. *Two complex numbers whose sum is a real number must be complex conjugates of each other.*

3. *Any function with an increasing derivative is increasing.*[1]

4. *Every integer which is the product of two primes has four distinct divisors.*[2]

[1] Increasing here is intended in the strict sense (see notes).
[2] You may assume that all relevant integers are positive.

Problem 2.

The following definition has several flaws. (a) List them explicitly; (b) write a correct, clearer definition.

> *Let X be a subset of $\mathbb{R}$, and let $f(X)$ be the number of integers in X, namely*
>
> $$f : \mathbb{R} \to \mathbb{N} \qquad\qquad X \to \#((x \in X) \cap (x \in \mathbb{Z})).$$

Problem 3.

Read carefully the text displayed on the next two pages. Then

(a) Ask three questions about this document.
[*Avoid vague, general questions, or questions concerning specific examples, such as: "What is a rational number?" (too general), or "Are 16 and 27 co-prime?" (too specific).*]

(b) Write a report on this document, comprising

i) a short title [$\notin$];

ii) two or three concise key points [$\notin$];

iii) a summary of the document [$\notin$, 150].

The minimal distance between two distinct integers is 1; by contrast, there is no minimal distance between two distinct rational numbers[3]. Indeed if x is a rational number, by letting $y = x \pm 1/n$, where n is a sufficiently large integer, we can make the distance between x and y as small as we please.

We analyse the properties of two rational numbers that are very close to each other. Let $x = a/b$ and $y = c/d$ (with $x \neq y$), and assume that the distance between x and y is small (in particular, less than 1). Then we can find a positive integer n such that

$$|x - y| \; = \; \frac{|ad - cb|}{|bd|} \; \leq \frac{1}{n} \tag{1}$$

and we shall choose n to be as large as possible, namely

$$\frac{1}{n+1} < |x - y| \leq \frac{1}{n}. \tag{2}$$

Because $x \neq y$, the integer $ad - cb$ is non-zero, and hence $|ad - cb| \geq 1$. From (1), we then obtain

$$|bd| \; \geq \; |ad - cb| \, n \; \geq \; n \tag{3}$$

which says that the product bd of the denominators of x and y is at least as large as n, in absolute value. From (2) we see that as x approaches y, the integer n must approach infinity, and so must the product bd of the denominators of our rationals, from (3). So the closer two rationals are to each other, the larger is the product of their denominators.

Let us consider a specific example. Let $x = 16/27$. A rational very close to x is $y = x + 1/10^8$, since we have $|x - y| = 10^{-8}$. We compute y explicitly

$$y = \frac{16}{27} + \frac{1}{10^8} = \frac{1600000027}{2700000000}. \tag{4}$$

While it was easy to find a rational very close to x, we paid a high price for it: the denominator of y has 10 digits. Is it possible to do the same with fewer digits? Inspecting (1) we see that, for a given distance, the product of the denominators of two rationals is minimal when $|ad - bc| = 1$. The quantity

$$|ad - bc| = |bd||x - y| = |\text{product of denominators}| \times \text{ distance}$$

[3] The distance between two numbers x and y, real or complex, is defined to be $|x - y|$.

may be regarded as a measure of the effectiveness of approximation process. In our example this quantity is equal to $27^2 = 729$, far from its optimal value 1. To find an optimal set of values of $y = c/d$, we must therefore solve the equation

$$|ad - bc| = |16d - 27c| = 1.$$

Now, the integers 16 and 27 are relatively prime, and hence this equation has lots of solutions —infinitely many of them. To find them, we first use Euclid's algorithm, to get the basic solution

$$16 \cdot (-5) - 27 \cdot (-3) = 1 \qquad\qquad (d = -5,\ c = -3).$$

Then, adding and subtracting $16 \cdot 27\,t$ from the left hand side, we find that, for all integers t

$$16 \cdot (-5 + 27t) - 27 \cdot (-3 + 16t) = 1.$$

The expressions in parentheses are the desired integers d and c, giving one pair of solutions for each value of t. Now, to do better than in equation (4), we require

$$27d = 27(-5 + 27t) > 10^8,$$

and the smallest solution is $t = 137175$. This value of t gives the desired nearby rational:

$$\frac{c}{d} = \frac{2194797}{3703720}, \qquad\qquad \left|\frac{16}{27} - \frac{2194797}{3703720}\right| = \frac{1}{100000440} < \frac{1}{10^8}.$$

We have done better than in (4) with only seven digits at denominator!

It should be clear that the above construction works with any rational a/b, not just $16/27$. The only requirement is that a and b must be coprime.