Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNIT 2012

MTH5117 Mathematical Writing

Duration: 2 hours
Date and time: May 30 2013, at 10.00am

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions; marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorised material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

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Exam papers must not be removed from the examination room.

Examiner: Franco Vivaldi

 TURN OVER

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation [$\not\in$, n] indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals. The integer n —when present— prescribes the *approximate* length (in words). In the absence of this notation, mathematical symbols may be used freely.

Question 1. [Marks: $(5, 5, 5, 5, 5), (4, 5, 5, 5)$]

(a) For each of the following mathematical objects, provide two levels of description: 1) a coarse description, which only identifies the class to which the object belongs (set, function, etc.); 2) a finer description, which characterises the object in question as accurately as possible. [$\notin$]

i) $f : \mathbb{R}^3 \to \mathbb{Z}$

ii) $\{z \in \mathbb{C} \,:\, |z - 1| < 1\}$

iii) $\prod_{k=1}^{\infty} f(k)$

iv) $\cos(2z) = 2\cos(z)^2 - 1$

v) $\{(1, 3), (3, 5), (5, 7), \ldots\}$

(b) Express each of the following statements with symbols, **using at least one quantifier**.

i) *The set A is a subset of the odd integers.*

ii) *The function f is not increasing.*

iii) *The set A contains a largest element.*

iv) *Only finitely many terms of the sequence (a_k) are zero.*

Question 2. [Marks: 8, 8, 8] Explain the following concepts, as clearly as you can. Combine words and symbols, as appropriate, and provide an illustrative example of each concept.

(a) The **image** and **inverse image** of a set under a function.

(b) **Relational operators** and **relational expressions**.

(c) **Periodic functions**.

Question 3. [Marks: 7,8] Each of the following definitions has faults. *i*) Explain what they are; *ii*) write out an appropriate revision.

(*a*) Let f be the following real function

$$f:\mathbb{R}\to\mathbb{R} \qquad f(x_1,x_2)=\frac{\sqrt{x_1+x_2}}{(x_1+1)(x_2+1)}.$$

(*b*) Let X be a subset of $\mathbb{R}$, and let $f(X)$ be the number of integers in X. Denoting by $\#A$ the number of elements of any set A, we have

$$f:\mathbb{R}\to\mathbb{N} \qquad f(X)=\#(x\in X\cap x\in\mathbb{Z}).$$

Question 4. [Marks: 2,4,11]

Read the text displayed on the next two pages. Then write a report on it, comprising

i) a short title [$\notin$];

ii) two concise key points [$\notin$];

iii) a summary of the document [$\notin$, 150].

End of paper. An appendix of 2 pages follows.

THIS PAGE AND THE NEXT PAGE CONTAIN MATERIAL FOR QUESTION 4.

The Pythagorean theorem says that the sum of the squares of the sides of a right triangle equals the square of the hypotenuse. In symbols

$$a^2 + b^2 = c^2. \tag{1}$$

There are many right triangles all of whose sides are natural numbers, e.g.,

$$3^2 + 4^2 = 5^2, \qquad 5^2 + 12^2 = 13^2, \qquad 8^2 + 15^2 = 17^2.$$

A triple (a, b, c) of natural numbers satisfying (1) is called a Pythagorean triple. Given a Pythagorean triple (a, b, c) and any natural number d, we obtain at once a new Pythagorean triple (ad, bd, cd), because

$$(ad)^2 + (bd)^2 = (a^2 + b^2)d^2 = c^2d^2 = (cd)^2.$$

So clearly there are infinitely many triples, but they are uninteresting being scaled versions of the same triple. This prompts the following definition.

> A *Primitive Pythagorean triple* (PPT) is a Pythagorean triple whose components have no common factor.

It's easy to show that in a PPT one of a and b must be odd, and the other even. Clearly a and b cannot both be even, otherwise c is also even, and a, b, c are not co-prime. If $a = 2x + 1$ and $b = 2y + 1$ are both odd, then

$$a^2 + b^2 = 4x^2 + 4y^2 + 4x + 4y + 2$$

is even but not divisible by 4. However, c^2 is even (being the sum of two odd numbers), and being a square it's divisible by 4, which is impossible.

By interchanging a and b, if necessary, we may assume that a is odd and b is even. We find

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

Suppose that d is a common factor of $c - b$ and $c + b$. Then d also divides their sum and difference:

$$(c + b) + (c - b) = 2c \qquad (c + b) - (c - b) = 2b$$

Now, (a, b, c) is a PPT, and hence b and c have no common factor. Hence d must divide 2, that is, $d = 1$ or $d = 2$. However, d also divides $(c-b)(c+b) = a^2$ which is odd, so $d = 1$.

We have shown that $c - b$ and $c + b$ have no common factor and that their product is a square. The only way this can happens is if $c - b$ and $c + b$ are themselves squares. (To see this, factor $c-b$ and $c+b$ into primes. Then the primes appearing in the factorization of $c-b$ will be distinct from the primes in the factorisation of $c + b$, from the fundamental theorem of arithmetic.) We write

$$c + b = s^2 \qquad \text{and} \qquad c - b = t^2$$

where $s > t \geqslant 1$ are odd integers (because b and c have opposite parity) with no common factors. Solving these equations for b and c yields

$$c = \frac{s^2 + t^2}{2} \qquad \text{and} \qquad b = \frac{s^2 - t^2}{2} \tag{2}$$

and then

$$a = \sqrt{(c - b)(c + b)} = st. \tag{3}$$

So we have proved:

Theorem. *Every primitive Pythagorean triple (a, b, c) with a odd and b even is given by the formulae (2) and (3) where $s > t \geqslant 1$ are any odd integers with no common factors.*

In particular, this result shows that there are infinitely many primitive triples. Dividing equation (1) by c^2 we obtain

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

so every PPT gives a rational point $(x, y) = (a/c, b/c)$ on the unit circle, lying in the first quadrant. Using (2) and (3) we obtain

$$(x, y) = \left(\frac{2st}{s^2 + t^2}, \frac{s^2 - t^2}{s^2 + t^2}\right)$$

expressing our rational point in terms of the parameters t and s. This formula gives us infinitely many rational points on the unit circle.