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Dynamical Systems (LTCC)
Problem Sheet 1

Problem 1 ("Averaging")

Consider a differential equation which depends periodically on the independent variable ("a periodically driven system")

$$\dot{x}(t) = \varepsilon f(x(t), t), \qquad f(y,t) = f(y, t+T) \tag{1}$$

where T denotes the period and $0 < \varepsilon \ll T$ a small parameter. Since $\dot{x} \sim \varepsilon$ the solution $x(t)$ won't change considerably during one period, and it seems sensible to approximate equation(1) by the "time averaged" system

$$\dot{z}(t) = \varepsilon \bar{f}(z(t)), \qquad \bar{f}(y) = \frac{1}{T}\int_0^T f(y,t)dt\,.$$

a) Using Taylor series expansion determine a (T-periodic) coordinate transformation $z = x + \varepsilon h_1(x,t) + \mathcal{O}(\varepsilon^2)$, with $h_1(x,t) = h_1(x, t+T)$, such that equation (1) reads

$$\dot{z}(t) = \varepsilon \bar{f}(z(t)) + \mathcal{O}(\varepsilon^2).$$

(Hint: using $\dot{z} = \dot{x} + \varepsilon \frac{\partial h}{\partial x}\dot{x} + \varepsilon \frac{\partial h_1}{\partial t} + \mathcal{O}(\varepsilon^2)$ derive a condition for $\frac{\partial h_1}{\partial t}$ at first order in ε.)

b) The equation for the (weakly) driven van der Pol oscillator reads

$$\ddot{x}(t) + \varepsilon(x^2(t) - 1)\dot{x}(t) + \omega_0^2 x(t) = \varepsilon h \cos(\omega t) \tag{2}$$

where $\omega^2 - \omega_0^2 = \varepsilon\Delta$ denotes the small detuning. Use the linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\dfrac{\sin(\omega t)}{\omega} \\ \sin(\omega t) & \dfrac{\cos(\omega t)}{\omega} \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

to rewrite equation (2) as a system of the form (1). Apply first order averaging (i.e., take the average with respect to the explicit time-dependence) to obtain an autonomous system of differential equations for u and v.