Problem 4

a) Let $.\sigma_0\sigma_1\sigma_2\dots$ be a symbol sequence. Denote by $.\tau_0\tau_1\tau_2\dots$ the corresponding Milnor-Thurston sequence, and by $.\tilde{\tau}_1\tilde{\tau}_2\tilde{\tau}_3\dots$ the Milnor-Thurston sequence of the shifted symbol sequence $.\sigma_1\sigma_2\sigma_3\dots$ (which is certainly not necessarily a shift of the sequence $.\tau_0\tau_1\tau_2\dots$). Let

$$x = \sum_{k=0}^{\infty} \frac{\tau_k}{2^{k+1}}, \qquad y = \sum_{k=0}^{\infty} \frac{\tilde{\tau}_{k+1}}{2^{k+1}}.$$

Since $\tau_0 = \sigma_0$ and

$$0 \leq \sum_{k=1}^{\infty} \frac{\tau_k}{2^{k+1}} \leq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}$$

one has $x \in I_{\sigma_0}$ (and for a similar reason $y \in I_{\sigma_1}$).

Case I: suppose $\sigma_0 = 0$. Then the finite strings $\sigma_0\sigma_1\dots\sigma_{k-1}$ and $\sigma_1\sigma_2\dots\sigma_{k-1}$ have the same number of ones, i.e., $\tau_k = \tilde{\tau}_k$ (as both symbols are derived from σ_k !). Since $x \in I_0$ we have

$$T(x) = 2x = 2\sum_{k=1}^{\infty} \frac{\tau_k}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{\tau_{k+1}}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{\tilde{\tau}_{k+1}}{2^{k+1}} = y\,.$$

Case II: suppose $\sigma_0 = 1$. Then the number of ones in the finite string $\sigma_0\sigma_1\dots\sigma_{k-1}$ exceeds the number of ones in the finite string $\sigma_1\sigma_2\dots\sigma_{k-1}$ by one, i.e., $\tilde{\tau}_k = 1 - \tau_k$ (!). Since $x \in I_1$ we have

$$T(x) = 2(1-x) = 2\left(\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} - \sum_{k=0}^{\infty} \frac{\tau_k}{2^{k+1}}\right) = 2\sum_{k=1} \frac{1-\tau_k}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\tilde{\tau}_k}{2^k} = y\,.$$

b) Let $x \in [0,1]$ and let $.\tau_0\tau_1\tau_2\dots$ denote a binary representation of x. Then the recursive construction

$$\begin{aligned}
\sigma_0 &= \tau_0 \\
\sigma_k &= \begin{cases} \tau_k & \text{if the string } \sigma_0\sigma_1\dots\sigma_{k-1} \text{ contains an even number of ones} \\ 1-\tau_k & \text{if the string } \sigma_0\sigma_1\dots\sigma_{k-1} \text{ contains an odd number of ones} \end{cases}
\end{aligned}$$

gives the unique symbol sequence $.\sigma_0\sigma_1\sigma_2\dots$ such that the corresponding Milnor-Thurston sequence coincides with $.\tau_0\tau_1\tau_2\dots$, i.e., $h(.\sigma_0\sigma_1\sigma_2\dots) = x$.

c) The only way for a binary representation to be not unique is an infinite tail of zeros or ones, i.e., for $N \geq 0$

$$.\tau_0\tau_1\dots\tau_{N-1}10000\dots \quad \text{and} \quad .\tau_0\tau_1\dots\tau_{N-1}01111\dots$$

give the same real number. The two symbol sequences corresponding to these Milnor-Thurston sequences are (see part b)

$$.\sigma_0\sigma_1\dots\sigma_{N-1}110000\dots \quad \text{and} \quad .\sigma_0\sigma_1\dots\sigma_{N-1}010000\dots \tag{1}$$

with suitable symbol string $\sigma_0\sigma_1\dots\sigma_{N-1}$ (if $\sigma_0\sigma_1\dots\sigma_{N-1}$ contains an even number of ones then $.\sigma_0\sigma_1\dots\sigma_{N-1}110000\dots$ has Milnor-Thurston sequence $.\tau_0\tau_1\dots\tau_{N-1}10000\dots$ and $.\sigma_0\sigma_1\dots\sigma_{N-1}010000\dots$ has Milnor-Thurston sequence $.\tau_0\tau_1\dots\tau_{N-1}01111\dots$, if $\sigma_0\sigma_1\dots\sigma_{N-1}$ contains an odd number of ones then $\sigma_0\sigma_1\dots\sigma_{N-1}110000\dots$ has Milnor-Thurston sequence $.\tau_0\tau_1\dots\tau_{N-1}01111\dots$ and $\sigma_0\sigma_1\dots\sigma_{N-1}010000\dots$ has Milnor-Thurston sequence $.\tau_0\tau_1\dots\tau_{N-1}10000\dots$).

The two sequences in equation (1) are mapped after N symbol shifts to $.11000\dots$ and $.01000\dots$ respectively, i.e., the point $x \in [0,1]$ which has symbol sequences given in equation (1) is mapped to 1/2 after N applications of the tent map (it is easy to check that 1/2 is the unique value with symbolic coding $.11000\ldots$ and $.01000\ldots$). Thus

$$J = \bigcup_{N \geq 0} T^{-N}(1/2) = \{m/2^n : 1 \leq m \leq 2^n - 1 \text{ and } n \geq 1\}.$$

The symbolic dynamics fails to be one to one at the boundaries of the cylinder sets, but the set J is countable (i.e. "small" with respect to the Lebesgue measure).