Problem 3

a) By substitution if follows that $(u_*, v_*) = (\sqrt{10}/4, \sqrt{2}/4)$ is fixed point at $\gamma_* = 3\sqrt{2}/8$, $\sigma_* = \sqrt{5}/4$. The Jacobian matrix reads

$$Df(x_*) = \begin{pmatrix} 1-3u_*^2 - v_*^2 & -\sigma_* - 2u_*v_* \\ \sigma_* - 2u_*v_* & 1-3v_*^2 - u_*^2 \end{pmatrix} = \begin{pmatrix} -1 & -\sqrt{5}/2 \\ 0 & 0 \end{pmatrix} \tag{1}$$

Obviously, eigenvalues are given by $\lambda^2 = -1$ and $\lambda^c = 0$ with the centre eigenvector given by

$$\underline{e}^c = \begin{pmatrix} \sqrt{5}/2 \\ -1 \end{pmatrix} \tag{2}$$

b) Using e.g. second order Taylor series expansion for the equations of motion

$$\begin{aligned} \dot{u}(t) &= u(t) - \sigma v(t) - u(t)(u^2(t) + v^2(t)) \\ \dot{v}(t) &= \sigma u(t) + v(t) - v(t)(u^2(t) + v^2(t)) - \gamma \\ \dot{\gamma} &= 0 \end{aligned}$$

one obtains

$$\begin{aligned} \begin{pmatrix} \delta\dot{u} \\ \delta\dot{v} \\ \delta\dot{\gamma} \end{pmatrix} &= \begin{pmatrix} 1-3u_*^2 - v_*^2 & -\sigma_* - 2u_*v_* & 0 \\ \sigma_* - 2u_*v_* & 1-3v_*^2 - u_*^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \\ \delta\gamma \end{pmatrix} \\ &+ \begin{pmatrix} -3u_*(\delta u)^2 - u_*(\delta v)^2 - 2v_*\delta u\delta v \\ -3v_*(\delta v)^2 - v_*(\delta u)^2 - 2u_*\delta u\delta v - a(\delta\gamma)^2 \\ 0 \end{pmatrix} + \ldots \end{aligned} \tag{3}$$

where ... denotes the contributions of higher (i.e. third) order. In fact, if we employ equation (1) the expression simplifies considerably

$$\begin{aligned} \delta\dot{u} &= -\delta u - \frac{\sqrt{5}}{2}\delta v - 3u_*(\delta u)^2 - u_*(\delta v)^2 - 2v_*\delta u\delta v + \ldots \\ \delta\dot{v} &= -3v_*(\delta v)^2 - v_*(\delta u)^2 - 2u_*\delta u\delta v - a(\delta\gamma)^2 + \ldots \\ \delta\dot{\gamma} &= 0\,. \end{aligned} \tag{4}$$

c) The linear part in equation (3) has now a doubly degenerate eigenvalue zero. Normally that would not guarantee the existence of two linearly independent eigenvectors. But since we have introduced $\delta\gamma$ in a slightly weird way, we have avoided any additional contribution at first order. Therefore, the two eigenvectors are given by (see equation (2) as well)

$$\underline{e}_1^c = \begin{pmatrix} \sqrt{5}/2 \\ -1 \\ 0 \end{pmatrix}, \qquad \underline{e}_2^c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The two dimensional centre manifold which is tangential to the linear space spanned by $\underline{e}_1^c$ and $\underline{e}_2^c$ thus reads

$$\delta u = h(\delta v, \delta\gamma) = -\frac{\sqrt{5}}{2}\delta v + \ldots \tag{5}$$

where ... denotes contributions of second and higher order.

d) Since the derivative of δv according to equation (4) is of second order, the expansion (5) is sufficient to obtain the equation of motion on the centre manifold to second order

$$\begin{aligned}
\delta\dot{v} &= -3v_*(\delta v)^2 - v_*\left(-\frac{\sqrt{5}}{2}\delta v\right)^2 - 2u_*\left(-\frac{\sqrt{5}}{2}\delta v\right)\delta v - a(\delta\gamma)^2 + \ldots \\
&= \frac{3\sqrt{2}}{4}(\delta v)^2 - a(\delta\gamma)^2 + \ldots
\end{aligned}$$

Using the linear scaling $z = (-3\sqrt{2}/4)\delta v$ the expression reduces to the normal form

$$\dot{z} = \mu - z^2$$

where

$$\mu = \frac{3\sqrt{2}}{4}a(\delta\gamma)^2 = \frac{3\sqrt{2}}{4}(\gamma - \gamma_*)\,.$$

The pair of fixed points is generated for $\mu > 0$, i.e. $\gamma > \gamma_*$. The result is consistent with the analysis in the lecture notes.