COMPLEX ANALYTIC MAPS
A special case of 2-dimensional maps

$$z = x + iy \qquad x_{t+1} = u(x_t, y_t)$$
$$f(z) = u + iv \qquad y_{t+1} = v(x_t, y_t)$$

where u & v satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So the Jacobian is particularly simple

$$J(x,y) = J(z) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Now
$$|f'(z)|^2 = f'(z)\,\overline{f'(z)} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) =$$
$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \mathrm{Det}(J(z)).$$

while the eigenvalues of J are simply $f'(z)$ and $\overline{f'(z)}$.

So the stability of n-cycles is formally equivalent to the 1-D real case. If $f(z^*) = z^*$ and $f'(z^*) = \mu$, then

$\mu = 0$	z^* superstable
$\lvert\mu\rvert < 1$	stable
$\lvert\mu\rvert > 1$	unstable
$\lvert\mu\rvert = 1$	marginal.

The notions of multiplier & Lyapunov exponent generalize immediately.

Periodic orbits

The quadratic family: $f(z) = z^2 + c$.
We begin with $z = 0$.

$$f(z) = z^2 \qquad z = \rho e^{2\pi i\theta} \qquad z^2 = \rho^2 e^{2\pi i(2\theta)}$$

∞

$$\rho_{t+1} = \rho_t^2 \qquad \theta_{t+1} \equiv 2\theta_t \pmod 1$$

- Doubling map on the unit circle.
- Superstable fixed point at $z=0$.
- $z = \infty$? Conjugacy $h(z) = w = \frac{1}{z}$

$$g(w) = h \circ f \circ h^{-1}(w) = h \circ f\left(\frac{1}{z}\right) = h\left(\frac{1}{z^2}\right) = w^2$$

So g has a sstable fixed pt at $w=0$, whence f has a sstable f.p. at ∞.

<u>Def:</u> Let $f : \mathbb{C} \circlearrowleft$ be a rational function.
The <u>Julia set</u> $J(f)$ of f is the closure of the unstable cycles of f.

<u>Ex</u> Let $f(z) = z^2$. Then $J(f) = S^1$.

<u>Ex</u> Let $f(z) = \frac{z^2-1}{2z}$. Then $J(f) = \mathbb{R}$.

<u>Ex</u> Let $f(z) = 1 - 2z^2$. Then $J(f) = [-1, 1]$.

Theorem Let f be a rational map of degree greater than one, and let $J(f)$ be its Julia set. Then

i) $J(f)$ is non-empty, and contains uncountably many points.

ii) $J(f)$ is invariant under both f and f^{-1}.

iii) $J(f^n) = J(f)$

iv) $J(f)$ is the boundary of the basin of attraction of any attractive cycle of f.

v) If $J(f)$ has an interior point, it coincides with the Riemann sphere.

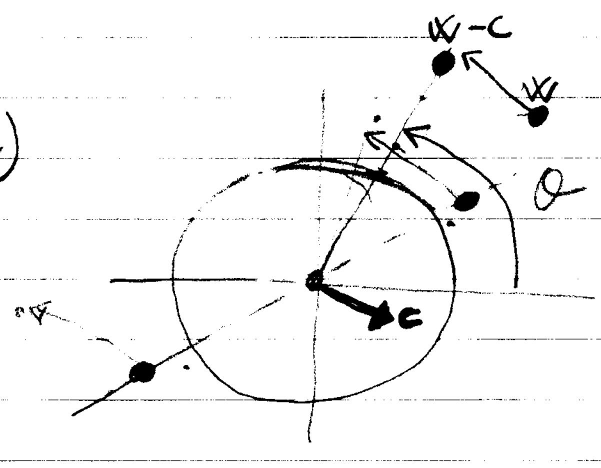
$J(f)$ is a repeller, so numerically difficult to compute by direct iteration. We turn it into an attractor by iterating backwards.

Ex $\quad w = f(z) = z^2 + c \qquad z = f^{-1}(w) = \mp\sqrt{w-c}$

Two pre-images. Write $\quad w - c = \rho e^{i\theta}$.

Then

$$z = \begin{cases} \sqrt{\rho}\, e^{i\theta/2} \\ \sqrt{\rho}\, e^{i(\theta/2+\pi)} \end{cases}$$



Def The <u>Mandelbrot set</u> $\mathcal{M} \subset \mathbb{C}$ is the set of all complex numbers c for which the orbit of the critical point of $f_c(z) = z^2 + c$ is bounded.

The critical point is $z = 0$. So the orbit is

$$0, \; c, \; c^2 + c, \; (c^2+c)^2 + c, \; \dots$$

One can prove that

i) $c \in \mathcal{M} \iff J(f_c)$ is connected

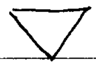
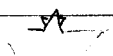
ii) $\mathcal{M}$ contains the disc $|c| \leq \frac{1}{2}$

iii) $\mathcal{M}$ is contained in the disc $|c| \leq 2$

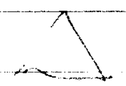
FRACTALS

Ex The Von Koch island (1904). It is the limit of a recursive sequence of polygons

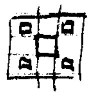
$$P_{t+1} = \Phi(P_t) \qquad (\Phi \text{ not yet specified})$$

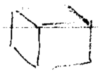
t	P_t	# sides	$\lvert\text{side}\rvert$	perimeter
0		3	$1/3$	1
1		$3\cdot 4$	$1/3^2$	$4/3$
2		$3\cdot 4^2$	$1/3^3$	$(4/3)^2$
k		$3\cdot 4^k$	$1/3^{k+1}$	$(4/3)^k \xrightarrow[k\to\infty]{} \infty$

Ex The Sierpinski triangle. The limit of a sequence of closed sets

t	P_t	# new holes	area hole	area P_t
0		0	0	1
1		1	$1/4$	$1-\frac{1}{4}$
2		3	$1/4^2$	$1-\left(\frac{1}{4}+3\cdot\frac{1}{4^2}\right)$
3		3^2	$1/4^3$	$1-\left(\frac{1}{4}-3\frac{1}{4^2}+3^2\frac{1}{4^3}\right)$
k		3^{k-1}	$1/4^k$	$1-\frac{1}{3}\left(\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots+\left(\frac{3}{4}\right)^k\right)$

Limit area: $1-\frac{1}{3}\sum_{t=1}^{\infty}\left(\frac{3}{4}\right)^t = 1-\frac{1}{3}\,\frac{3/4}{1-3/4} = 1-1=0.$

<u>Ex</u> Sierpinski carpet  $\subset \mathbb{R}^2$

Menger sponge  $\subset \mathbb{R}^3$

Ternary set  $\subset \mathbb{R}$

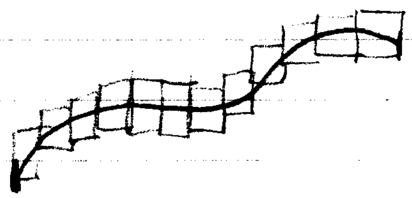
<u>Def</u> <u>Box dimension</u> (Kolmogorov capacity) of a set $A \subset \mathbb{R}^d$

Let $N(\varepsilon)$ = smallest # of cubes of volume ε^d necessary to cover A.

$$\mathcal{D}_0(A) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} \quad \text{(if the limit exists).}$$

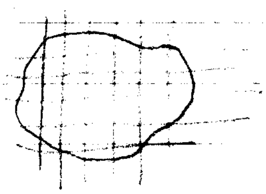
<u>Def</u> A is said to be a <u>FRACTAL</u> if $\mathcal{D}_0(A)$ is not an integer.

Why this definition?

1-dimensional sets in $\mathbb{R}^2$  length c

$$N(\varepsilon) \cong c \cdot \varepsilon^{-1}$$

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \lim \frac{\log(c) - \log(\varepsilon)}{-\log \varepsilon} = 1 - \lim_{\varepsilon \to 0} \frac{\log(c)}{\log \varepsilon} = 1 \checkmark$$

2-d sets in $\mathbb{R}^2$  Area c

$$N(\varepsilon) \cong c \cdot \varepsilon^{-2}$$

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \lim_{\varepsilon \to 0} \frac{\log(c) - \log \varepsilon^2}{-\log \varepsilon} = 2 - \lim_{\varepsilon \to 0} \frac{\log c}{\log \varepsilon} = 2 \checkmark$$

So, if $N(\varepsilon) \cong c \cdot \varepsilon^{-\mathcal{D}_0}$, then

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \mathcal{D}_0.$$

In all example considered we had two <u>scaling factors</u> appearing in each recursive step

Side length of boxes $\qquad \varepsilon_{t+1} = \alpha\, \varepsilon_t \qquad \varepsilon_0 = c$

Number of boxes $\qquad N_{t+1} = \beta N_t \qquad N_0 = 1$

Thus $\qquad \varepsilon_t = \varepsilon_0 \alpha^t, \qquad N_t = \beta^t,$
hence

$$D_0 = \lim_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \lim_{t\to\infty} \frac{\log N_t}{-\log \varepsilon_t} = \lim_{t\to\infty} \frac{t \log(\beta)}{-\log \varepsilon_0 - t\log(\alpha)}$$

$$= \frac{\log \beta}{\log(1/\alpha)}$$

Set	α	β	D_0
Koch's island	$1/3$	2^2	$2\frac{\log 2}{\log 3} = 1.2619\ldots$
Sierp. Triangle	$1/2$	3	$\frac{\log 3}{\log 2} = 1.5850\ldots$
Sierp. carpet	$1/3$	2^3	$3\frac{\log 2}{\log 3} = 1.8928\ldots$
Ternary set	$1/3$	2	$\frac{\log 2}{\log 3} = 0.6309\ldots$