# Higher-dimensional mappings

General form for $F: \mathbb{R}^d \to \mathbb{R}^d$

$$X_{t+1} = F(X_t)$$

$$X_t = (x_t^{(1)}, x_t^{(2)}, \ldots, x_t^{(d)}) \in \mathbb{R}^d$$

$$F(X) = (f^{(1)}(X), f^{(2)}(X), \ldots, f^{(d)}(X)), \quad f^{(k)}: \mathbb{R}^d \to \mathbb{R}.$$

Examples $(d=2)$ $\quad X = (x, y)$

- The Henón mapping $(\Sigma = \mathbb{R}^2)$

$$x_{t+1} = -\lambda x_t^2 + y_t + 1$$
$$y_{t+1} = \mu x_t$$

- The standard mapping $(\Sigma = S \times \mathbb{R})$

$$x_{t+1} = x_t + y_{t+1} \pmod{2\pi}$$
$$y_{t+1} = y_t + \varepsilon \sin(x_t) \qquad \varepsilon \geq 0$$

- The Kaplan-Yorke mapping $(\Sigma = \mathbb{R}^2)$

$$x_{t+1} = 1 - 2x_t^2$$
$$y_{t+1} = \lambda y_t + x_t$$

## STABILITY OF PERIODIC ORBITS

Let $F(X^*) = X^*$ with $X = X + \delta$ $\quad \delta = (\delta^{(1)}, \ldots, \delta^{(d)})$
Expanding in Taylor series:

$$X^{*(1)} + \delta_{t+1}^{(1)} = f^{(1)}(X^*) + \frac{\partial f^{(1)}}{\partial x^{(1)}} \delta_t^{(1)} + \cdots + \frac{\partial f^{(1)}}{\partial x^{(d)}} \delta_t^{(d)} +$$

$$\vdots$$

$$X^{*(d)} + \delta_{t+1}^{(d)} = f^{(d)}(X^*) + \frac{\partial f^{(d)}}{\partial x^{(1)}} \delta_t^{(1)} + \cdots + \frac{\partial f^{(d)}}{\partial x^{(d)}} \delta_t^{(d)} + \cdots$$

But $f^{(j)}(X^*) = X^{*(j)}$ so $\quad \delta_{t+1} = \mathbf{J}\,\delta_t + \cdots$
where

$$J = J(X^*) = \begin{pmatrix} \frac{\partial f^{(1)}}{\partial x^{(1)}} & \cdots & \frac{\partial f^{(1)}}{\partial x^{(d)}} \\ \vdots & & \vdots \\ \frac{\partial f^{(d)}}{\partial x^{(1)}} & \cdots & \frac{\partial f^{(d)}}{\partial x^{(d)}} \end{pmatrix}.$$

$J$ is called the Jacobian matrix of $F$. $(J = DF)$.

We have $\delta_t = J^t \delta_0$, so $\|\delta_t\|$ will decrease if all the eigenvalue of $J$ are inside the unit circle in the complex plane. If at least one eigenvalue is outside the unit circle, any $\delta_t$ lying in the associated eigenspace will grow in size. So we have

**Def** Let $\mu_1, \ldots, \mu_d$ be the eigenvalues of $J(X^*)$. Then $X^*$ is said to be

| | | |
|---|---|---|
| STABLE | if $\lvert\mu_j\rvert < 1$ | $j = 1, \ldots, d$ |
| UNSTABLE | if $\exists j$ s.t. $\lvert\mu_j\rvert > 1$ | |
| NEUTRAL | if $\lvert\mu_j\rvert = 1$ | $j = 1, \ldots, d$. |

Remark. In 1-D $\quad J(x) = f'(x) .= \lambda(x)$.

Example Kaplan-Yorke map.

Fixed pts: $\begin{cases} x = 1 - 2x^2 \\ y = \lambda y + x \end{cases}$

$$2x^2 + x - 1 = 0 \quad \Rightarrow \quad x = \frac{-1 \mp \sqrt{9}}{4} = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$$

$$y(1-\lambda) = x \quad \Rightarrow \quad y = \frac{x}{1-\lambda}.$$

Two fixed points:

$$X_1^* = \left(-1, \frac{1}{\lambda - 1}\right) \qquad X_2^* \left(\frac{1}{2}, \frac{1}{2(1-\lambda)}\right).$$

Jacobian

$$J(x,y) = \begin{pmatrix} -4x & 0 \\ 1 & \lambda \end{pmatrix}.$$

Char. Equation

$$z^2 - (\lambda - 4x)z - 4\lambda x = (z - \lambda)(z + 4x) = 0$$

so the eigenvalues are $\mu_1 = \lambda$ and $\mu_2 = -4x$.

So at $X_1^*$: $\quad \mu_1 = \lambda \quad \mu_2 = 4 \quad$ unstable.

$X_2^*$: $\quad \mu_1 = \lambda \quad \mu_2 = -2 \quad$ unstable.

**Example** Standard map

F.P.: $x \equiv x + y \pmod{2\pi}$
$y = y + \varepsilon \sin(x)$ $\qquad \varepsilon > 0$

$y = 2\pi k$, $k \in \mathbb{Z}$ $\qquad x = 0, \pi$, so two families

$X_1^* = (0, 2\pi k) \qquad X_2^* = (\pi, 2\pi k)$

Jacobian (note: $x_{t+1} = x_t + \varepsilon \sin(x_t)$)

$$J(x,y) = \begin{pmatrix} 1 + \varepsilon\cos(x) & 1 \\ \varepsilon\cos(x) & 1 \end{pmatrix}$$

Since $\operatorname{Det} J = 1$ we have the char poly $z^2 - T z + 1 = 0$ where $T = \operatorname{Trace}(J)$. The eigenvalues are $\mu$ and $1/\mu$, where

$$\mu = \frac{T + \sqrt{T^2 - 4}}{2}$$

So one of the eigenvalues is outside the unit circle if $|T| > 2$, while for $|T| \leq 2$ they are complex conjugates on the circle.

We have

$X = X_1^* \qquad Tr(J) = 2 + \varepsilon \qquad$ unstable for $\varepsilon > 0$

$X = X_2^* \qquad Tr(J) = 2 - \varepsilon \qquad$ ustable for $\varepsilon > 4$
neutral for $0 \leq \varepsilon \leq 4$.

The standard map commutes with translations by an integer multiple of $2\pi$ in the $y$-direction.

Indeed, the standard map $F : \Sigma \circlearrowleft$ $(\Sigma = \mathbb{R} \times S^1)$ is given by

$$(x, y) \mapsto \left(x + y + \varepsilon \sin(x) \, (\mathrm{mod}\ 2\pi),\ y + \varepsilon \sin(x)\right)$$

and we let

$$H : \Sigma \circlearrowleft \qquad (x, y) \mapsto (x, y + 2\pi).$$

We verify that $F \circ H = H \circ F$, namely that $F$ is conjugate to itself via $H$

$$\begin{aligned}(H \circ F)(x, y) &= \left(x + y + 2\pi + \varepsilon \sin(x) (\mathrm{mod}\ 2\pi),\ y + 2\pi + \varepsilon \sin(x)\right) \\ &= \left(x + y + \varepsilon \sin(x) (\mathrm{mod}\ 2\pi),\ y + 2\pi + \varepsilon \sin(x)\right). \\ &= (F \circ H)(x, y).\end{aligned}$$

So the orbit structure is <u>periodic</u>, in the $y$-direction