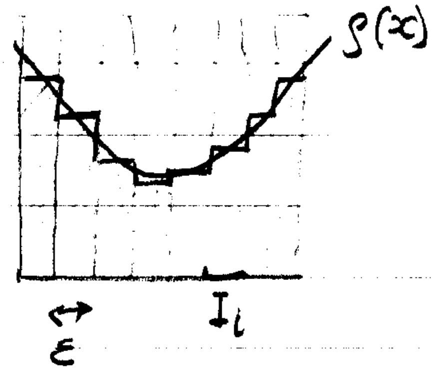
Probabilistic aspects of chaotic dynamics.

Unpredictability in the evolution of a chaotic orbit prompts question of probabilistic nature, such as

Given $f: \Sigma_1 \circlearrowleft$ with what probability we find the points of an orbit in a certain subregion $A \subset \Sigma_1$?

Make histogram: divide phase space into regions I_i of equal size ε, and count how many iterates visit I_i

$$\varphi_i = \frac{\#\text{ iterates in } I_i}{\text{total } \# \text{ of iterates}} = \frac{N_i}{N}$$



as $N \to \infty$ this gives a histogram.
As $\varepsilon \to 0$ we obtain (hopefully) a probability density $\rho(x)$.

The probability $\mu(A)$ of finding an iterate in some subset $A \subseteq \Sigma_1$ is

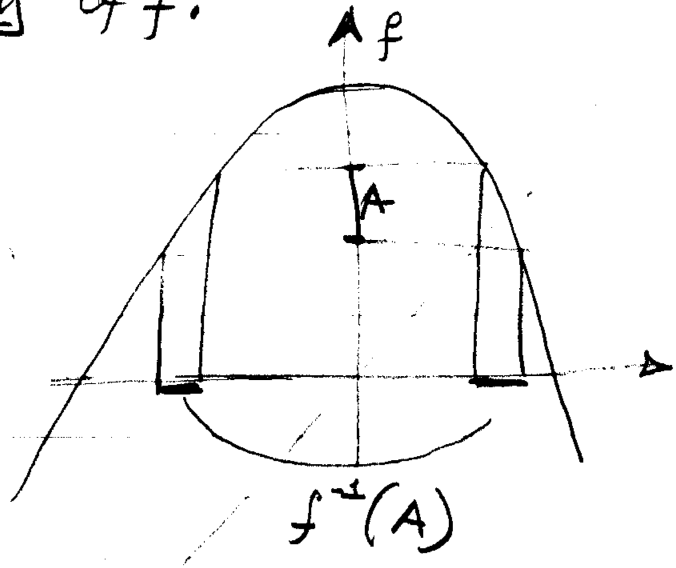
$$\mu(A) = \int_A \rho(x)\,dx.$$

This is the integral of the density function over A.

Def An *invariant measure* of a map $f: \Sigma \circlearrowleft$ is a probability measure μ satisfying

$$\mu(A) = \mu(f^{-1}(A)) \qquad \forall A \subset \Sigma,\ A \text{ measurable.}$$

The corresponding density is called an *invariant density* of f.

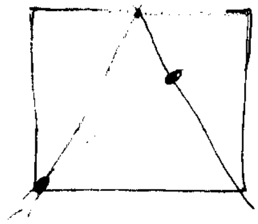


Remark We shall see that *almost all* orbits of a dynamical system generate the same invariant density.

Prob 3 -

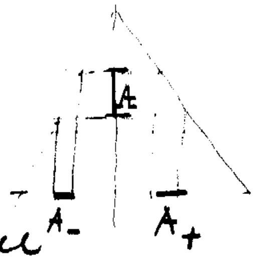
<u>Ex</u> The tent map $f:[0,1]\circlearrowleft$

$$f(x)=\begin{cases} 2x & 0\le x\le 1/2 \\ 2(1-x) & 1/2< x\le 1\end{cases}$$



f is chaotic : $\Lambda(x)=\log 2$.

Consider an interval A with $\mu(A)=\ell$.
$f^{-1}(A)$ consists of 2 equal intervals A_+ & A_- of length $\ell_+=\ell_-=\frac{1}{2}\ell$. If we choose μ to be the Lebesgue measure μ, then



$$\mu(f^{-1}(A))=\mu(A_+\cup A_-)=\mu(A_+)+\mu(A_2)=\mu(A)$$

A_+ & A_- are disjoint

If A is an elementary set (union of disjoint segments), then, repeating the argument above for all segments comprising A one finds that $\mu(f^{-1}(A))=\mu(A)$. Taking the limit of sequences of elementary sets, one establishes the invariance of the measure of any measurable set (we omit the details).

Thus the Lebesgue measure is an invariant measure of the tent map. A very similar argument shows that this measure is also invariant for the doubling map.

We wish to express the invariance of a measure

$$\mu(A) = \mu(f^{-1}(A))$$

in terms of the associated density. The above equation becomes

$$\int_A \rho(x)\,dx = \int_{f^{-1}(A)} \rho(x)\,dx = \sum_i \int_{f_i^{-1}(A)} \rho(x)\,dx$$

where i labels the various branches of the inverse function.

In the 1-D case, letting $y = f(x)$ we have $dy = f'(x)\,dx$, that is

$$dx = \frac{dy}{f'(f_i^{-1}(y))}.$$

We obtain

$$\int_A \rho(x)\,dx = \sum_i \left| \int_{f_i^{-1}(A)} \rho(f_i^{-1}(y)) \frac{dy}{f'(f_i^{-1}(y))} \right|.$$

If A is chosen so that the sign of $f'(f_i^{-1}(y))$ does not change in $f_i^{-1}(A)$ we obtain

$$\int_A \rho(y)\,dy = \sum_i \int_{f_i^{-1}(A)} \rho(f_i^{-1}(y)) \frac{dy}{|f'(f_i^{-1}(y))|}.$$

Equating the integrands, we obtain

$$\rho(y) = \sum_i \frac{\rho(f_i^{-1}(y))}{|f'(f_i^{-1}(y))|}.$$

we rewrite it as

$$\rho(y) = \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{|f'(x)|}$$

A solution ρ to this equation is an invariant density for f.

Given a differentiable map $f : \Sigma \to \Sigma$ ($\Sigma \subset \mathbb{R}$) we define the Perron-Frobenius operator of f $\mathcal{P}_f$ by

$$(\mathcal{P}_f\, \rho)(y) = \sum_{x = f^{-1}(y)} \frac{\rho(x)}{|f'(x)|} = \bar{\rho}(x).$$

P_f is made to act on the space of functions $\rho : \Sigma \to \mathbb{R}$, which assume non-negative real values, are Lebesgue integrable and satisfy the normalization condition

$$\int_X \rho(x)\, dx = 1.$$

Thus any fixed point ρ of the P-F operator $\mathcal{P}_f(\rho) = \rho$, is an invariant probability density for the map f.

Ex The tent map $y = f(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 < x \le 1 \end{cases}$

thus

$0 \le x \le 1/2 \qquad x = \frac{y}{2}$

$1/2 < x \le 1 \qquad x = 1 - \frac{y}{2}.$

P-F operator for the tent map

$$(\mathcal{P}\rho)(y) = \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{|f'(x)|} = \frac{\rho(y/2)}{|f'(y/2)|} + \frac{\rho(1-y/2)}{|f'(1-y/2)|}$$

$$= \frac{1}{2}\left(\rho(y/2) + \rho(1-y/2)\right).$$

Let $\rho(y) = c$, a constant. Then

$$(\mathcal{P}\rho)(y) = \frac{1}{2}(c+c) = c = \rho(y).$$

So such ρ is a fixed point of P-F. Normalization

$\int_0^1 \rho(y) = 1 \quad \Rightarrow \quad c = 1$ and $\rho(x)$ is the density function of the Lebesgue measure.

<u>Ex</u> The "Ulam point" ($\lambda = 2$) of the logistic map.

$$y = f(x) = 1 - 2x^2 \qquad \Sigma = [-1, 1].$$

Pre-images $\quad x = \mp\sqrt{\frac{1-y}{2}} = f^{-1}(y)$

$$|f'(x)| = 4|x| = 4\sqrt{\frac{1-y}{2}}$$

$$(\mathcal{P}_f \rho)(y) = \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{|f'(x)|} = \frac{1}{4\sqrt{\frac{1-y}{2}}}\left[\rho\left(\sqrt{\frac{1-y}{2}}\right) + \rho\left(-\sqrt{\frac{1-y}{2}}\right)\right].$$

We now verify that the function

$$\rho(y) = \frac{c}{\sqrt{1-y^2}}$$

is a fixed point of P-F.

$$\rho\left(\mp\sqrt{\frac{1-y}{2}}\right) = \frac{c}{\sqrt{1 - \frac{1-y}{2}}} = \frac{c\sqrt{2}}{\sqrt{1+y}}.$$

Thus

$$(\mathcal{P}\rho)(y) = \frac{1}{4\sqrt{\frac{1-y}{2}}} \frac{2 \cdot c \cdot \sqrt{2}}{\sqrt{1+y}} = \frac{c}{\sqrt{1-y}\sqrt{1+y}}$$

$$= \frac{c}{\sqrt{1-y^2}} = \rho(y).$$

The function ρ is non-negative on $[-1, 1]$.
Now we show it is integrable (by integrating it!).

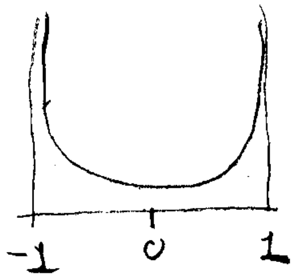
Normalization

$$c\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = 1$$

$$c\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = c \arcsin(x)\Big|_{-1}^{1} = c\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = c\pi$$

or $c = \frac{1}{\pi}$. So an invariant probability density for the logistic map at the Ulam point is

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$$



It turns out $\rho(x)$ is the unique invariant density of the Ulam map (this follows from ergodicity — see below).

ERGODICITY

Let $f : \Sigma \circlearrowleft$ be a map. A set $A \subset \Sigma$ is said to be <u>invariant</u> under f if $f(A) = A$. The map f is said to be <u>ergodic</u> (with respect to an invariant measure μ), if Σ cannot be decomposed into two invariant subsets of positive measure

Ergodicity is measure-dependent. However, if μ is the integral of a smooth density, one can check ergodicity using the Lebesgue measure (this will always be the case in this course).

<u>Thm</u> The doubling map is ergodic. $(\Sigma = [0,1)\ ;\ \mu = \text{Lebesgue m.})$

<u>Pf.</u> Let $A \subset \Sigma$ be an invariant set of measure $0 < \mu(A) < 1$. Then, from measure preservation, we have (neglecting, possibly, zero measure sets) $f^{-1}(A) = A$, and hence $f(\Sigma \setminus A) = \Sigma \setminus A$. Fix $\varepsilon > 0$, and find an open interval Δ of length 2^{-n}, for some n, such that

$$\mu(\Delta \setminus A) > (1-\varepsilon)\mu(\Delta) = \frac{1-\varepsilon}{2^n}.$$

Since $f'(x) = 2$, the map f doubles the measure of any set, as long as it remains injective on that set.

Thus $$\mu(f^n(\Delta \setminus A)) = 2^n \mu(\Delta \setminus A) > 1 - \varepsilon.$$

Since $f^n(\Delta \setminus A) \subset \Sigma \setminus A$ (apart from zero measure sets), we have $\mu(A) < \varepsilon$, and since ε was arbitrary, we have $\mu(A) = 0$, a contradiction.
So no such a set A exists. $\square$

An integrable function $\mathcal{X} : \Sigma \to \mathbb{R}$ will be called an <u>observable</u> (or <u>test function</u>, or <u>random variable</u>). We use $\mathcal{X}$ to perform measurements of a dynamical system $f : \Sigma \circlearrowleft$

<u>Def</u> The <u>time-average</u> of $\mathcal{X}$ along the orbit through x is given by

$$\overline{\mathcal{X}}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathcal{X}(f^t(x))$$

<u>Ex</u> let $\mathcal{X}(x) = \log|f'(x)|$. Then $\overline{\mathcal{X}}(x) = \Lambda(x)$ is the Lyapounov exponent.

<u>Def</u> The <u>phase-average</u> of $\mathcal{X}$ with respect to the invariant density ρ is given by

$$\langle \mathcal{X} \rangle = \int_\Sigma \mathcal{X}(x)\rho(x)\,dx$$

<u>Thm</u> If f is ergodic, then time and phase averages are the same almost everywhere, i.e,

$$\overline{\mathcal{X}(x)} = \langle \mathcal{X} \rangle \quad \text{for almost all } x.$$

- "ergodic" & "almost everywhere" are intended with respect to the same invariant measure.

Thus for an ergodic map, a time-average is the same for almost all initial conditions.

Ex Computing the average position.

let $X(x) = x$. Then, if f is ergodic

$$\bar{x} = \lim_{N \to N} \frac{1}{N} \sum_{t=0}^{N-1} x_t = \int_{\Sigma} \rho(x) \cdot x \cdot dx = \langle x \rangle$$

• For the doubling map: $\rho(x) = 1$, $\Sigma = [0, 1]$.

$$\bar{x} = \langle x \rangle = \int_0^1 x \, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

• For the Ulam map: $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$, $\Sigma = [-1, 1]$

$$\bar{x} = \langle x \rangle = \int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} \cdot x \cdot dx = 0,$$ since the integrand is odd.

Ex Computing the variance $\langle x^2 \rangle - \langle x \rangle^2$ for the doubling map.

$$\overline{x^2} = \frac{1}{N} \lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} x_t^2 = \langle x \rangle = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

whence

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{4-3}{12} = \frac{1}{12}$$

Ex Computing the Lyapounov exponent

Let $\chi(x) = \log|f'(x)|$.

- For the doubling map: $|f'(x)| = 2$.

$$\Lambda = \langle \chi \rangle = \int_0^1 \log 2 \, dx = \log 2 \int_0^1 dx = \log 2.$$

- For the Ulam map: $|f'(x)| = 4x$

$$\Lambda = \int_{-1}^{1} \frac{\log|4x|}{\pi\sqrt{1-x^2}} dx = 2\int_0^1 \frac{\log 4x}{\pi\sqrt{1-x^2}} dx =$$

$$= 2\log 4 \int_0^1 \frac{dx}{\pi\sqrt{1-x^2}} + \frac{2}{\pi}\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$$

$$= \cancel{4}^{2} \log 2 \cdot \frac{1}{\cancel{2}} + \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right)$$

$$= 2\log 2 - \log 2 = \log 2.$$

Same as the doubling map. This is no coincidence, as we shall see.

We can now justify the observation that the invariant density of an ergodic map is obtained from a histogram constructed from a single orbit.
Indeed let $\chi = \chi_A$ be the <u>characteristic function</u> of a set $A \subset \Sigma$,

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

If $(x_0, x_1, \ldots)$ is an orbits, then the # of iterates that belong to A is computed as

$$\sum_{t=0}^{N-1} \chi_A(x_t)$$

For an ergodic map, we get, a.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \chi_A(x_t) = \int_{\Sigma} \chi_A(x) \rho(x) dx$$

$$= \int_A \rho(x) dx = \mu(A).$$

The LHS is the limit frequency of points that land in A, i.e. the probability of finding a point of an orbit in A.
Thus the invariant probability measure gives that probability.