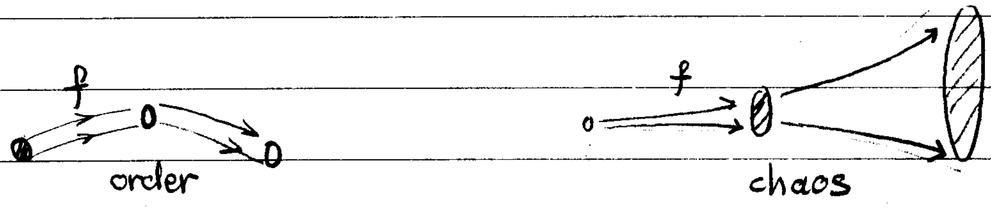
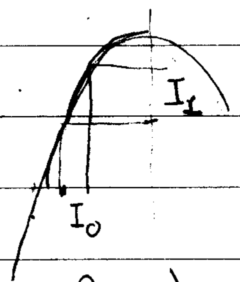
CHAOS

Chaotic behaviour means sensitive dependence on initial conditions



Orbits of nearby initial conditions separate very fast (exponentially) during time evolution. Any uncertainty in the determination of initial conditions results in loss of predictability.

Consider a 1-D map $f : [a,b] \circlearrowleft$ of class C^1, and a small interval I_0, of length Δ_0



Letting $I_1 = f(I_0)$ we find that

$$\Delta_1 = |f'(x_0)| \Delta_0$$

for a suitable $x_0 \in I_0$ (from MVT), resulting in expansion if $|f'(x_0)| > 1$ and contraction if $|f'(x_0)| < 1$.

We do the same with the iterated function f^N.
After N iterations, a small interval I_0 is transformed into an interval I_N of length

$$\Delta_N = \left| f^{N'}(x_0) \right| \Delta_0$$

where, from the chain rule

$$f^{N'}(x_0) = f'(x_0) \cdot f'(x_1) \cdots f'(x_{N-1}) = \prod_{t=0}^{N-1} f'(x_t).$$

The multiplier will be typically exponentially large or small, so we define the <u>local expansion rate</u> $\Lambda_N(x_0)$ by

$$\Delta_n = e^{N \cdot \Lambda_N(x_0)} \cdot \Delta_0$$

and we find

$$e^{N \cdot \Lambda_N(x_0)} = \frac{\Delta_N}{\Delta_0} = \left| \prod_{t=0}^{N-1} f'(x_t) \right| = \prod_{t=0}^{N-1} \left| f'(x_t) \right|$$

giving

$$N \cdot \Lambda_N(x_0) = \log \prod_{t=0}^{N-1} \left| f'(x_t) \right| = \sum_{t=0}^{N-1} \log(\left| f'(x_t) \right|)$$

hence

$$\Lambda_N(x_0) = \frac{1}{N} \sum_{t=0}^{N-1} \log(\left| f'(x_t) \right|).$$

Taking the limit $N \to \infty$ we get an important quantity.

<u>Def.</u> Let $f:[a,b]\circlearrowleft$ be a differentable map.
The <u>Lyapounov exponent</u> of f at $x\in[a,b]$ is given by

$$\Lambda(x)=\lim_{N\to\infty}\Lambda_N(x)=\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\log|f'(f^t(x))|$$

if the limit exists.

<u>Remarks</u>

- $\Lambda(x)$ is a time-average of the local expansion rate of the orbit through x
- In some important cases $\Lambda(x)$ is – in a sense to be made precise – almost independent on x, that is, it assumes the same value for almost all x. When this is the case, Λ becomes an intrinsic property of f, and we have

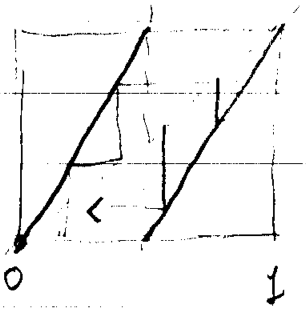
<u>Def</u>: A differentiable mapping $f:[a,b]\circlearrowleft$ is <u>chaotic</u> if it has a positive Lyapounov exponent.

Example The binary shift map (or doubling map).

We represent the circle as the unit interval $I=[0,1)$ with endpoints identified. Let $f: I \circlearrowleft$ be given by

$$f(x) \equiv 2x \pmod 1.$$

Thus $f\left(\frac{3}{5}\right)=\frac{1}{5}$, $f(0.99)=0.98$



We have $f'(x)=2$, $\forall x$, hence

$$\lambda = \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \log 2 = \lim_{N\to\infty} \frac{1}{\cancel{N}} \cdot \cancel{N} \log 2 = \log 2.$$

So the doubling map is chaotic. (We shall treat it in greater details later.)

Example Let $f: I \circlearrowleft$ be such that $0 < c_1 \le |f'(x)| \le c_2$. then

$$\Lambda(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{\infty} \log|f'(f^t(x))| \ge \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \log c_1 = \log c_1$$

and, likewise, $\Lambda(x) \le \log c_2$, and we get the bounds

$$\log c_1 \le \Lambda(x) \le \log c_2 .$$

<u>Theorem</u> If x is a point of a T-cycle with multiplier $\mu \neq 0$, then

$$\Lambda(x) = \frac{1}{T} \log|\mu|.$$

<u>Proof.</u> Let $\{x_0^*, \dots, x_{T-1}^*\}$ be the T-cycle.
We write: $N = kT + r$, $0 \le r < T-1$.
Then, letting $x_0 = x_0^*$, we have $x_N = x_r^*$.

$$\Lambda(x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \log|f'(x_t)| =$$
$$= \lim_{N\to\infty} \frac{1}{N} \left(\sum_{s=1}^{k} \sum_{t=0}^{T-1} \log|f'(x_t^*)| + \sum_{t=0}^{r} \log|f'(x_t^*)| \right)$$
$$= \lim_{N\to\infty} \frac{1}{N} \left(k \log\left|\prod_{t=0}^{T-1} f'(x_t^*)\right| + \sum_{t=0}^{r} \log|f'(x_t^*)| \right).$$

Now

$$\lim_{N\to\infty} \frac{k}{N} = \lim_{N\to\infty} \frac{k}{kT+r} = \frac{1}{T}; \qquad \prod_{t=0}^{T-1} f'(x_t^*) = \mu.$$

Moreover the periodic function

$$\sum_{t=0}^{r(N)} \log|f'(x_t^*)|$$

is <u>bounded</u>, since $f'(x_t^*) \neq 0$, by assumption, and so

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{r} \log|f'(x_t^*)| = 0.$$

Thus $\Lambda(x_0) = \frac{1}{T} \log|\mu|$ and since the choice of x_0 in the cycle was arbitrary, the result follows. $\square$.

Remarks

- If x belongs to a superstable cycle, we define $\Lambda(x) = -\infty$.
- $\Lambda(x)$ is positive for unstable orbits and negative for stable ones. So if there is chaos, we expect lots of unstable orbits, and there cannot be stable orbits, from the following result.

Thm Let $f \in C^1$ have a cycle with Lyap. exp Λ. If x belongs to its basin of attraction, and if the orbit of x does not contain critical pts, then $\Lambda(x) = \Lambda$.

Pf. We restrict the proof to 1-cycles (the general case is treated similarly). Let $x_t \to x^*$, with $f(x^*) = x^*$. Assume $f'(x^*) \neq 0$. From the continuity of f', given $\varepsilon > 0$, $\exists M$ such that, $\forall t \geq M$. We have

$$\log|f'(x^*)| - \frac{\varepsilon}{2} < \log|f'(x_t)| < \log|f'(x^*)| + \frac{\varepsilon}{2}.$$

Sum over t, to obtain

$$\frac{N-M}{N}\left(\log|f'(x^*)| < \frac{\varepsilon}{2}\right) < \frac{1}{N}\sum_{t=M}^{N-1} \log|f'(x_t)| < \frac{N-M}{N}\left(\log|f'(x^*)| + \frac{\varepsilon}{2}\right).$$

Letting $N \to \infty$, we find

$$\log|f'(x^*)| - \frac{\varepsilon}{2} < \lim_{N\to\infty} \frac{1}{N}\sum_{n=M}^{N-1} \log|f'(x_t)| < \log|f'(x^*)| + \frac{\varepsilon}{2}$$

but for N sufficiently large we also have, $(f'(x_t) \neq 0)$.

$$\frac{1}{N}\left|\sum_{n=0}^{M-1} \log|f'(x_t)|\right| < \frac{\varepsilon}{2}$$

and so

$$\log|f'(x^*)| - \varepsilon < \underbrace{\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \log|f'(x_t)|}_{\lambda(x_0)} < \log|f'(x^*)| + \varepsilon$$

and the result follows by letting $\varepsilon \to 0$.

If x^* is superstable, the argument must be modified as follows. Given $\varepsilon > 0$, $\exists M$ such that, $\forall t \geq M$ we have

$$\log|f'(x_t)| \leq -\frac{1}{\varepsilon},$$

then proceed as above. $\square$

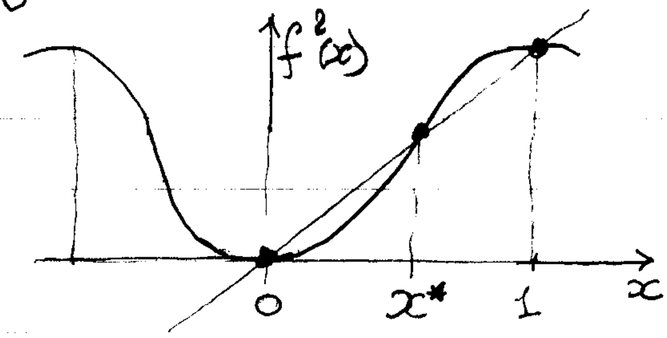
The above proof is easily adapted to the case of eventually periodic points: the Lyapounov exponent of the pre-images of a cycle is equal to the Lyapounov exponent of a cycle (excluding, as usual, critical points).

<u>Example</u> $f(x) = 1 - x^2$ in $[-1, 1]$.

We have a superstable 2-cycle $\{0, 1\}$ and an ustable 1-cycle at $x^* = \frac{\sqrt{5}-1}{2}$ with multiplier

$$\mu = -2x^* = 1 - \sqrt{5}.$$

So $\Lambda(0) = \Lambda(1) = -\infty$, while $\Lambda(x^*) = \log(\sqrt{5}-1) > 0$.
Looking at $f^2(x) = -x^4 + 2x^2$



it is clear that every $x \in [0, 1] \setminus \{x^*\}$ belongs to the basin of attraction of the 2-cycle.
Furthermore, $f([-1, 0]) = [0, 1]$ and so

$$\Lambda(x) = \begin{cases} \log \sqrt{5} - 1 & x = \mp \frac{\sqrt{5}-1}{2} \\ -\infty & \text{otherwise}. \end{cases}$$