Tangent bifurcations

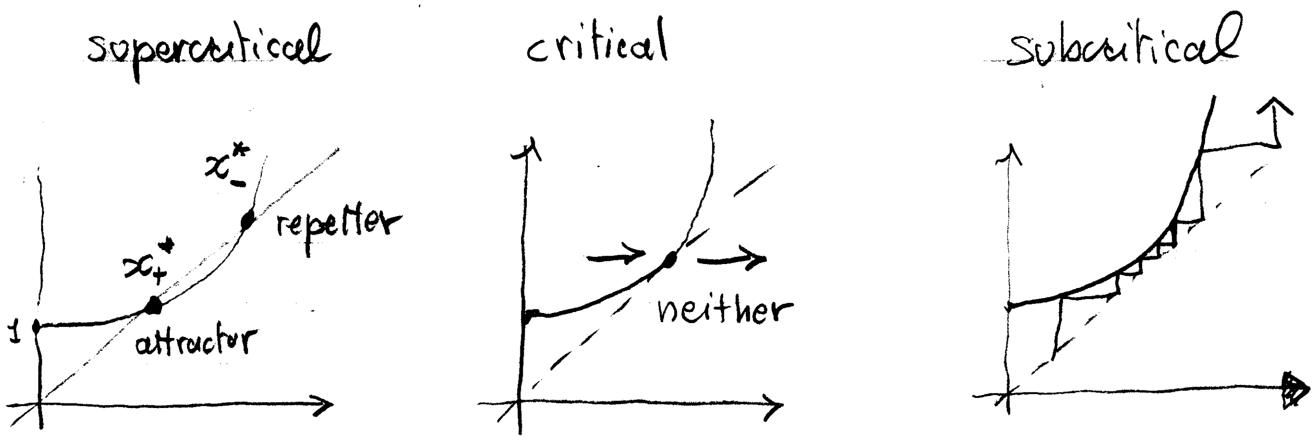
We consider again the 1-cycles of the logistic map. They are root of the polynomial

$$\Phi_1(x) = \lambda x^2 + x - 1$$

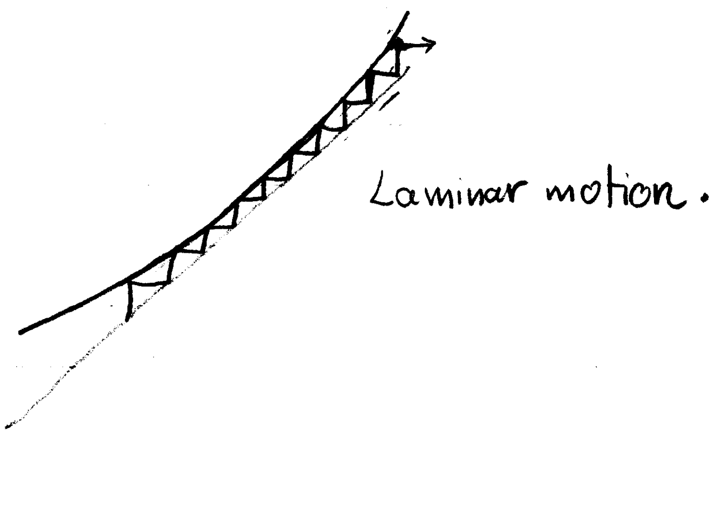
giving

$$x^*_{\mp} = \frac{-1 \mp \sqrt{1+4\lambda}}{2\lambda}$$

Such cycles are real for $\lambda \geq -\frac{1}{4}$. What happens near $\lambda = -1/4$?



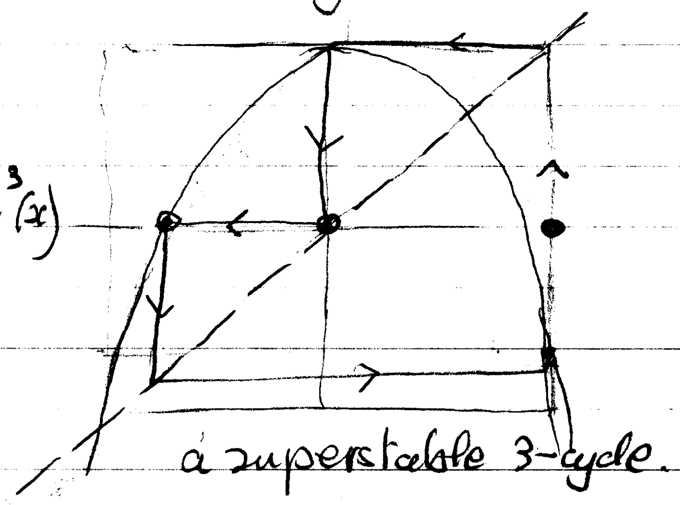
This is called a <u>tangent bifurcation</u>.
The behaviour of interest to us is the subcritical one.



Tangent bifurcations and periodic windows

We have seen that at some $1 < \lambda_T < 2$ the logistic map supports a superstable 3-cycle.
How is this 3-cycle born?

We must study the graph of $f^3(x)$



a superstable 3-cycle.

Look at fixed points of $f^3(x)$.

$$f(x) = 1 - \lambda x^2$$

$$f^2(x) = 1 - \lambda(1 - \lambda x^2)^2 = 1 - \lambda + 2\lambda^2 x^2 - \lambda^3 x^4$$

$$f^3(x) = 1 - \lambda(1 - \lambda + 2\lambda^2 x^2 - \lambda^3 x^4)^2$$

Thus

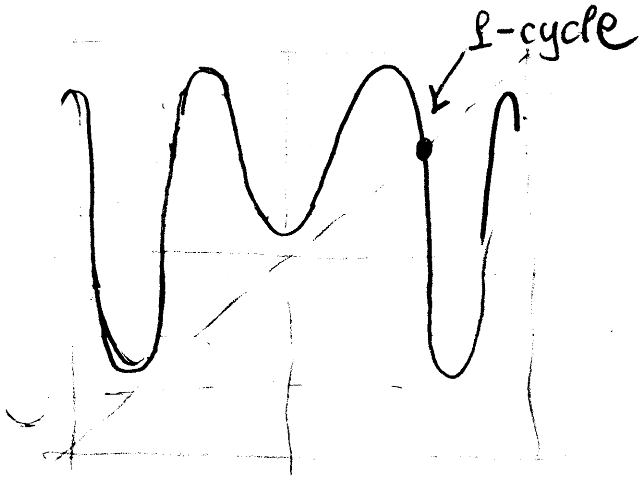
$$\Phi_3(x) = \frac{f^3(x) - x}{f(x) - x} = \frac{\lambda^7 x^8 - 4\lambda^6 x^6 + 4\lambda^3(\lambda^2 - \lambda + 1)x^4 + 4\lambda^3(1-\lambda)x^2 + x + (1-\lambda)^2 \cdot \lambda - 1}{\lambda x^2 + x - 1}$$

$$= \lambda^6 x^6 - \lambda^5 x^5 - \lambda^4(3\lambda - 1)x^4 + \lambda^3(2\lambda - 1)x^3 + \lambda^2(3\lambda^2 - 3\lambda + 1)x^2 - \lambda(\lambda - 1)^2 x + 1 - \lambda + 2\lambda^2 - \lambda^3$$

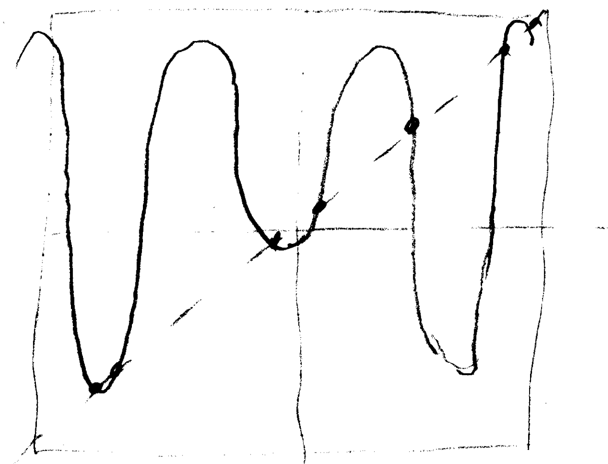
degree 6, so <u>two</u> 3-cycles.

At some critical parameter value, two 3-cycles are born, via a tangent bifurcation

subcritical



supercritical



How do we determine the critical value?

Let $\Phi(x)$ be a polynomial (over any field) of degree n.
Let $\alpha_1, \dots, \alpha_n$ be its roots (not necessarily distinct)

The <u>discriminant</u> of $\Phi(x)$ is the following quantity

$$\Delta\phi = a^n \prod_{i<j} (\alpha_i - \alpha_j)^2 \qquad \text{where } a \text{ is the leading coeff.}$$

So $\Delta\phi = 0 \iff$ two roots of ϕ are the same

Let $\phi(x) = ax^2 + bx + c \qquad n = 2$

$$\alpha_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \alpha_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta\phi = a^2(\alpha_1 - \alpha_2)^2 = \cancel{a^2}\,\frac{b^2 - 4ac}{\cancel{a^2}} = b^2 - 4ac$$

Let $\phi(x) = ax^3 + bx^2 + cx + d \qquad n = 3$

Then $\Delta\phi = -27a^2d^2 + b^2c^2 - 4(b^3d + ac^3) + 18abcd$

The discriminant of $\Phi_3(x)$ is computed as

$$\Delta\phi_3 = \lambda^{30}(16\lambda^2-4\lambda+7)^2(4\lambda-7)^2.$$

The discriminant of $16\lambda^2-4\lambda+7$ (w.r.to λ) is -432, so no real roots. So apart from the trivial solution $\lambda=0$, we find

$$\lambda_T = \frac{7}{4}$$

at this value, a stable-unstable pair of 3-cycles is born from a tangent bifurcation.

At $\lambda = 7/4$ we find

$$\Phi_3(x) = \frac{1}{2^{12}}(343x^3-98x^2-252x+8)^2$$

that is, the two 3-cycles coincide.

The product of the multipliers of the two 3-cycles is given by

this is $f_\lambda^3(0)$!

$$2^6\Phi_3(0) = 2^6(1-\lambda+2\lambda^2-\lambda^3) = \mu_1\cdot\mu_2.$$

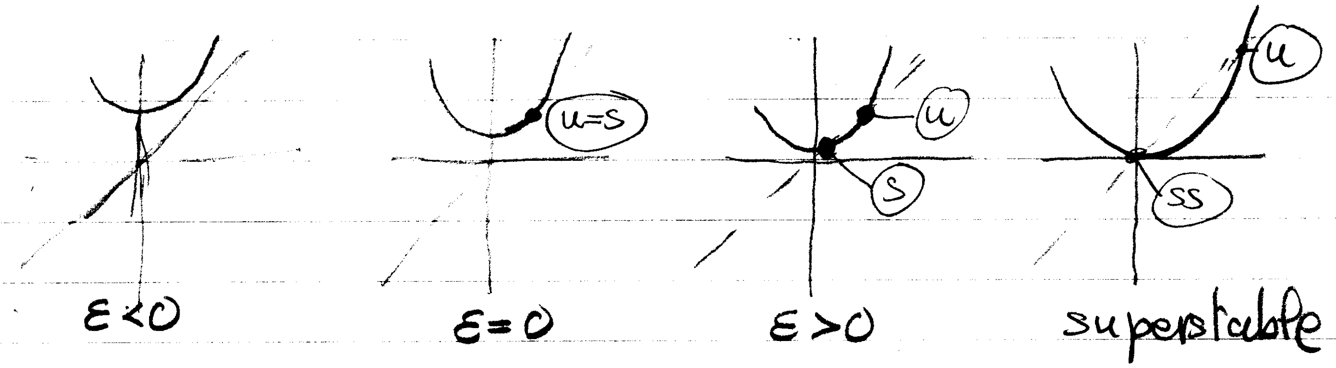
substituting $\lambda = 7/4+\varepsilon$ we find

$$\mu_1\cdot\mu_2 = 1-204\varepsilon+O(\varepsilon^2). \qquad (*)$$

So for $\varepsilon>0$ at least one 3-cycle is stable since $\mu_1\mu_2<1$ means that one of the 2 terms is <1.

A qualitative analysis of the behaviour of $f^3(x)$ near zero, shows that the other cycle is unstable

Magnifying $f^3_{7/4+\varepsilon}(x)$ near zero



The parameter value corresponding to superstability may be estimated from (*), as follows

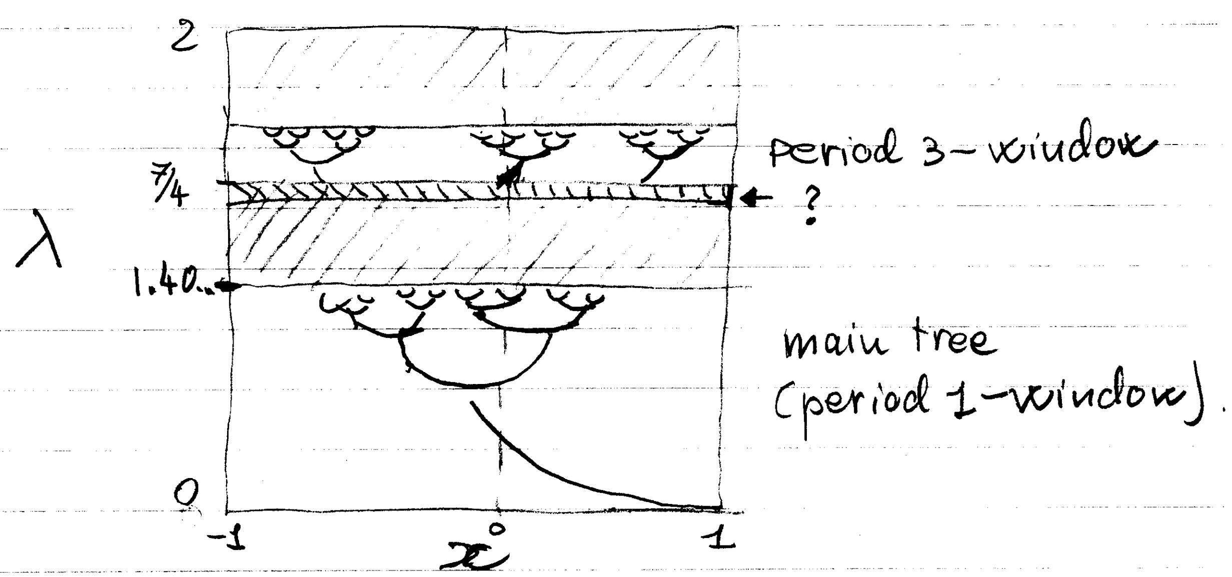
$$\mu_1 \cdot \mu_2 = 0 \Rightarrow \varepsilon = \frac{1}{204} \Rightarrow \lambda = \frac{7}{4} + \frac{1}{204} \cong 1.75490\ldots$$

in excellent agreement with the exact value 1.75487...

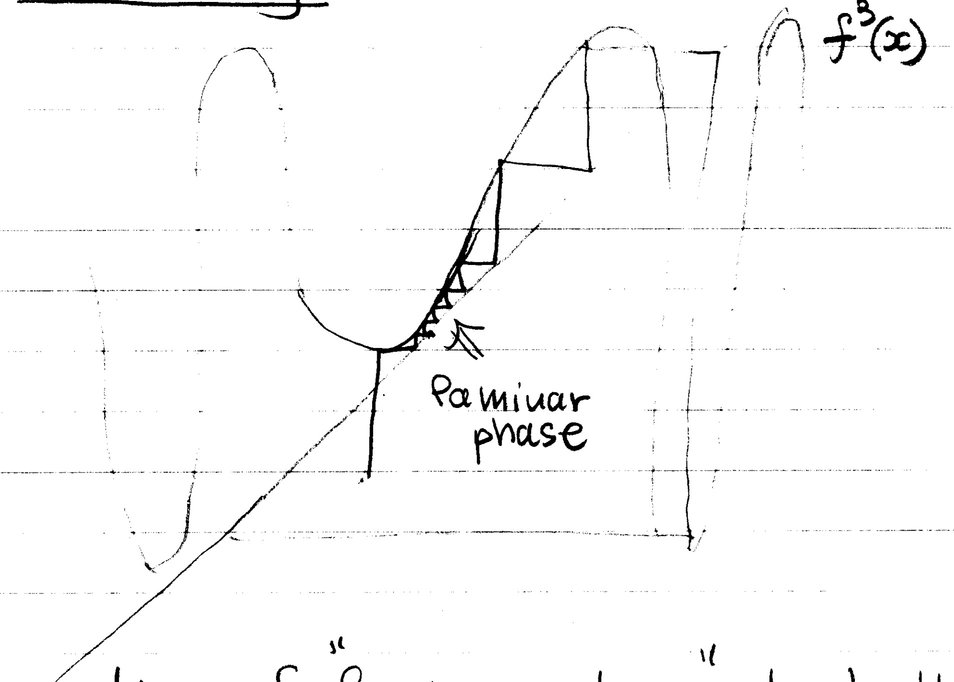
Further increase of λ will cause the 3-cycle to lose stability and generate a period-doubling sequence with periods

$$3, \quad 3\cdot 2, \quad 3\cdot 2^2, \; 3\cdot 2^3, \ldots, 3\cdot 2^j, \ldots \; 3\cdot 2^\infty$$

much like for the 1-cycle. This generates a "periodic window"



Just below the critical value $\varepsilon = 0$ ($\lambda = 7/4$), one observes <u>intermittency</u>



an alternating of "laminar motions" close to the 3-cycle (which is still complex), and "chaotic outbursts".

<u>It can be shown that</u> the average length $\langle L \rangle$ of the laminar phase goes as

$$\langle L \rangle \sim \frac{1}{\sqrt{|\varepsilon|}}$$

thus $\quad \langle L \rangle \to \infty \quad$ as $\quad \varepsilon \to 0^-$

So far we have found two routes to chaos

- period-doubling cascade (pitchfork bifs.)
- intermittency (tangent bif.)

These phenomena are observed in the physical world.

Sharkovsky's Theorem

Def The Sharkovsky ordering of the natural integers is defined by

$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd \cdots \lhd 2^m \lhd \cdots$$
$$\cdots \lhd 2^k(2n-1) \lhd \cdots \lhd 2^k \cdot 7 \lhd 2^k \cdot 5 \lhd 2^k \cdot 3 \lhd \cdots$$
$$\cdots \lhd 2(2n-1) \lhd \cdots \lhd 2\cdot 7 \lhd 2 \cdot 5 \lhd 2 \cdot 3 \lhd \cdots$$
$$\cdots \lhd 2n-1 \lhd \cdots \lhd 7 \lhd 5 \lhd 3 .$$

Theorem Let $f : [a,b] \to \mathbb{R}$ be a continuous map. If f has a periodic orbit of minimal period n, then it has a periodic orbit of minimal period m for all $m \lhd n$.

(Ukrainian Math. Z. 16 61 (1964)).

Ex Suppose f has a 4-cycle. Then f has a 2-cycle, and a 1-cycle

Ex Suppose f has a 10-cycle. Then f has n-cycles for all even $n > 6$.
(The cases $n=6$ and n odd are undecided).

Ex If f has a 3-cycle, then it has an n-cycle for every n.

(The above corollary of S's thm, was discovered -independently- by Li & Yorke in 1975)
'Period 3 implies chaos', Amer. Math. Monthly 82 985 (1975)).

<u>Ex</u> Logistic map $f_\lambda(x) = 1-\lambda x^2$.

We found a pair of 3-cycles being born at $\lambda = 7/4$. Furthermore, we found that the discriminant of $\Phi_3(x)$

$$\Delta\Phi_3 = \lambda^{30}(16\lambda^2-4\lambda+7)^2(4\lambda-7)^2 > 0 \quad \lambda > 7/4.$$

and so these 3-cycles exist for <u>all</u> $\lambda \geq 7/4$ (including $\lambda > 2$!!). Sharkovsky thm then implies that for $\lambda \geq 7/4$ $f_\lambda(x)$ has n-cycles for <u>all</u> values of n. These are usually unstable, except in the windows, where one is stable for a short λ-range

(The proof of Sharkovsky thm will be omitted.)