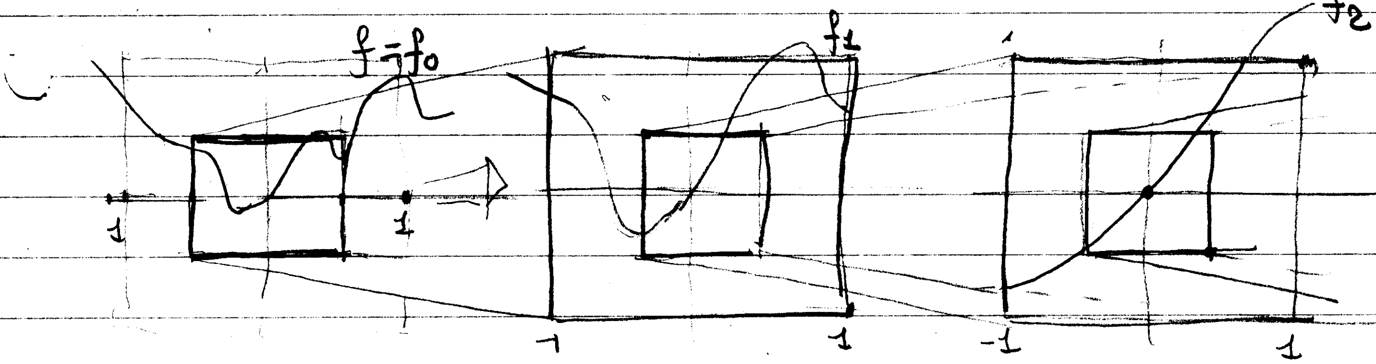
Introduction to renormalisation theory

Renormalisation = a mathematical microscope

Example. Let f be analytic a zero. Show that near zero the graph of f is a straight line.

Magnifying the graph of a function



$$\Sigma = \left\{ \text{functions } f : \mathbb{R} \circlearrowleft, \text{ analytic at zero.} \right\}$$

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \frac{1}{\rho} = \overline{\lim_{k \to \infty}} (c_k)^{1/k} < \infty \quad \text{or,} \quad \rho > 0.$$

Magnify by a factor $\alpha > 1$

$$R : \Sigma \circlearrowleft \qquad R f_t(x) = \alpha f_t(x/\alpha) = f_{t+1}(x)$$

$$= \sum_{k=0}^{\infty} \alpha c_k^{(t)} \left(\frac{x}{\alpha}\right)^k = \alpha c_0^{(t)} + c_1^{(t)} x + \frac{c_2^{(t)}}{\alpha} x^2 + \frac{c_3^{(t)}}{\alpha^2} x^3 + \dots + \frac{c_k^{(t)}}{\alpha^{k-1}} x^k$$

$$= c_0^{(t+1)} + c_1^{(t+1)} x + \dots$$

New coefficients: $\quad c_k^{(t+1)} = \dfrac{c_k^{(t)}}{\alpha^{k-1}} \qquad k = 0, 1, 2, \dots$

$t = 0, 1, 2, \dots$

R is a <u>linear operator</u> on Σ

$$\begin{pmatrix} c_0^{(t+1)} \\ c_1^{(t+1)} \\ \\ c_k^{(t+1)} \\ \\ \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots \\ 0 & 1 & & & \\ 0 & 0 & 1/\alpha & 0 & \\ 0 & 0 & 0 & 1/\alpha^2 & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} c_0^{(t)} \\ c_1^{(t)} \\ \\ c_k^{(t)} \\ \vdots \end{pmatrix}$$

"∞ matrix"

It is clear that under repeated applications of R,

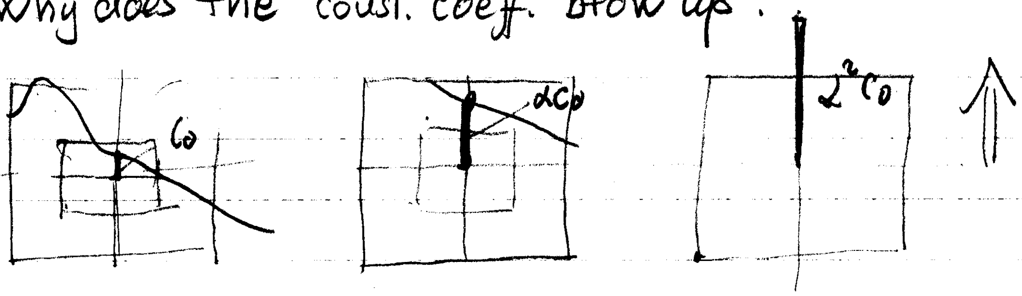
the constant coefficient $c_0^{(t)} = \alpha^t c_0 \to \infty$ if $c_0 \neq 0$.

the linear coefficient $c_1^{(t)} = c_1$ is invariant

all other coefficients $c_k^{(t)} = \left(\frac{1}{\alpha^{k-1}}\right)^t c_k \to 0$

Why does the const. coeff. blow up?

$c_0 \neq 0$



microscope is not aimed at the right point.

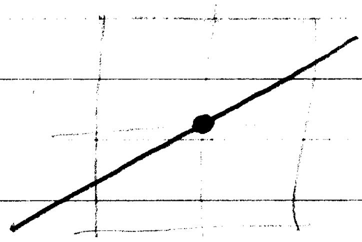
So must normalize in such a way that $c_0 = 0$.

Σ = set of $f : \mathbb{R} \circlearrowleft$ analytic at zero, and with $f(0) = 0$.

then for any $f_0 \in \Sigma$ let $f_{t+1} = R \cdot f_t$

If $f_0(x) = c_1 x + c_2 x^2 + \cdots$

then $\lim_{t \to \infty} f_t(x) = c_1 x$



Alternatively every function

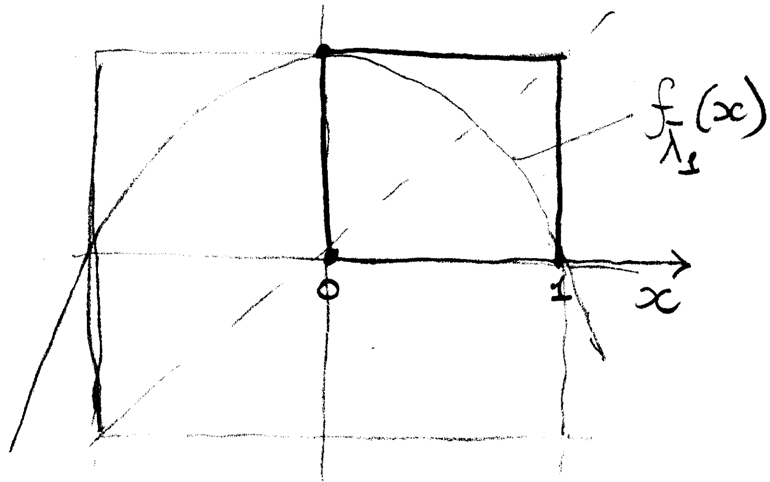
$$f^*(x) = c_1 x \qquad Rf^* = f^*$$

is a <u>fixed point</u> of the renormalisation operator.

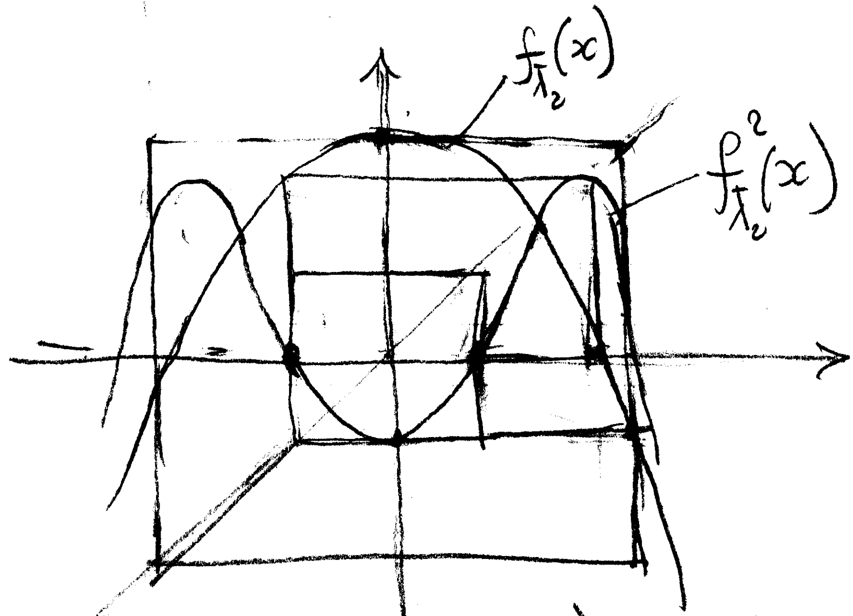
So R has <u>infinitely many</u> fixed points, forming a one-parameter family.

Renormalization for period-doubling

Superstable 2-cycle: $\lambda=\bar{\lambda}_1$



Superstable 4-cycle: $\lambda=\bar{\lambda}_2$



Key observation (Feigenbaum)

The map $f^2(x)$ at $\lambda=\bar{\lambda}_2$ resembles $f(x)$ at $\lambda=\bar{\lambda}_1$, provided that we scale, and change sign in both coordinate axes.

We verify this. Let $f(1) = 1/\alpha$.
At $\lambda = \bar{\lambda}_2$, we have

$$f^2(0) = f(1) = 1/\alpha$$

$$f^2(1/\alpha) = f^2(f(1)) = f^3(1) = 0.$$

Define $\quad (R(f))(x) = \bar{f}(x) = \alpha f\left(f\left(\frac{x}{\alpha}\right)\right).$

Then we have

$$\bar{f}(0) = \alpha f^2(0) = \alpha \cdot \frac{1}{\alpha} = 1$$

$$\bar{f}(1) = \alpha f^2\left(\frac{1}{\alpha}\right) = \alpha \cdot 0 = 0.$$

So $\bar{f}(x)$ has the 2-cycle $\{0, 1\}$.
Furthermore, 0 is a critical point for $\bar{f}$, since

$$\frac{d}{dx}\bar{f}(x) = \alpha \frac{d}{dx} f\left(f\left(\frac{x}{\alpha}\right)\right) = \alpha\, f'\left(\frac{x}{\alpha}\right) \cdot f'\left(f\left(\frac{x}{\alpha}\right)\right)$$

at $x=0$, $f'\left(\frac{x}{\alpha}\right) = f'(0) = 0$, so $\bar{f}'(0) = 0$, and the 2-cycle is superstable.

Likewise, at $\lambda = \bar{\lambda}_3$ we expect the superstable 8-cycle to resemble, after composition and scaling, to the 4-cycle at $\lambda = \bar{\lambda}_2$, and in general

$$(R(f_{\bar{\lambda}_K}))(x) \sim f_{\bar{\lambda}_{K-1}}(x)$$

on the interval $[-1, 1]$.

Look for a function f^* such that

$$(R(f^*))(x) = \alpha f^*(f^*(x/\alpha)) = f^*(x).$$

This is the famous Feigenbaum-Cvitanovic equation.

We are interested in solutions that satisfy (like the logistic map $f(x) = 1 - \lambda x^2$)

$$f^*(x) = \sum_{k=0}^{\infty} c_k x^{2k} \qquad \text{analytic \& even}$$

$$f^*(0) = 1. \qquad \text{normalised}$$

Hence $f^{*\prime}(0) = 0$. (since f^* is even & differentiable).

Numerical solution yields

$$\begin{aligned} f^*(x) \cong 1 &- 1.5276330 \cdot x^2 \\ &+ 0.1048152 \cdot x^4 \\ &+ 0.0267057 \cdot x^6 \\ &- 0.0035274 \cdot x^8 \\ &+ \ldots\ldots \end{aligned}$$

At $x = 0$, the F-C equation becomes

$$\alpha f^*(f^*(0)) = f^*(0)$$

$$\alpha f^*(1) = 1$$

or

$$\alpha = \frac{1}{f^*(1)} = -2.5029079 \ldots$$

So the constant α is a property of the <u>fixed</u>

Iterate the renormalization operator

$$R^2 f(x) = R \cdot R f(x) = R\, \alpha f^2(x/\alpha) = \alpha^2 f^4(x/\alpha^2)$$

and, in general

$$R^k f(x) = \alpha^k f^{2k}(x/\alpha^k).$$

For the fixed-point function $Rf^* = f^*$ so

$$\alpha^k f^{*2k}(x/\alpha^k) = f^*(x)$$

and so the iterated function is a rescaled version of the original one.

It can be shown that Feigenbaum's constant $\delta = 4.669...$ is an eigenvalue of the operator R.