The O-notation

Let f be a real or complex-valued function. Write

$$f(x) = O(g(x)) \qquad x \in X$$

to mean that there exists a constant A such that, for all $x \in X$ we have

$$|f(x)| \leq A|g(x)|.$$

The set X is usually an interval in $\mathbb{R}$ (possibly infinite) and it is often specified more informally as

$$f(x) = O(g(x)) \qquad \begin{cases} x \to \infty \\ |x| \to \infty \\ x \to 0, \quad \text{etc.} \end{cases}$$

So $x \to \infty$ denotes the interval $(a, +\infty)$ for a suitable large a. Likewise $x \to 0$ denotes an interval $(-a, a)$ for suitably small a, while $x \to 0^+$ denotes $(0, a)$.
(If X is clear from the context, it may not be specified at all.)

<u>Example</u> as $x \to \infty$ we have

$$3x + 1 = O(x) \qquad x = O(x^2)$$

$$\frac{x^2+1}{x^3} = O\left(\frac{1}{x}\right) \qquad \arctan(x) = O(1)$$

$$x \log(x) = O(x^2) \qquad \sin(x) = O(1)$$

$$\log(x)^6 = O(x^{1/2}) \qquad x + \sin(x) = O(x)$$

Note that $O(1)$ just means "bounded".

<u>Example</u> as $x \to 0$ we have

$$\sin(x) = O(x) \qquad \cos(x) - 1 = O(x^2)$$

$$x^2 = O(x) \qquad \cos(x) + 1 = O(1)$$

$$\frac{e^x - 1}{x} = 1 + O(x) \qquad \frac{1}{1-x} = 1 + x + O(x^2).$$

More formally, $f(x) = O(g(x))$ should be written as $f(x) \in O(g(x))$, where $O(g(x))$ is the <u>set</u> of all functions f such that $|f(x)| < c|g(x)|$ for some constant c. With this in mind, one can prove the following:

1) $f(x) = O(f(x))$
2) $cf(x) = O(f(x))$
3) $O(f(x))O(g(x)) = O(f(x)g(x))$
4) $O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|)$
5) $O(f(x)g(x)) = f(x)\, O(g(x))$
6) $O(f(x)^2) = O(f(x))^2$.

Example

$$O(x) = \{ x, x^2, \sin(x), \log(1+x), \tan(x), \dots \} \qquad x \to 0$$

Example Taylor series

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k + O(x^{N+1}) \qquad x \to 0$$

Example

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \qquad |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \dots \qquad \text{any } x$$

$$(1+x)^y = e^{y \log(1+x)} = 1 + y \log(1+x) + O(y^2 \log(1+x)^2)$$

$$= 1 + yx + O(yx^2) + O(y^2 x^2) =$$

$$= 1 + yx + O(yx^2) \qquad \begin{matrix} x \to 0 \\ y \to 0 \end{matrix}$$

Thus

$$(1+x)^y = \begin{cases} 1 + yx + O(x^2) & y \text{ fixed} \\ 1 + O(y) & x \text{ fixed} \end{cases}$$

<u>Example</u> Assume that $f(x) \prec 1$ (meaning: $\lim_{x \to 0} f(x) = 0$)
Then, for sufficiently small $|x|$

$$\log(1+f(x)) = f(x)\left(1 - \frac{1}{2}f(x) + \frac{1}{3}f(x)^2 + \cdots\right)$$
$$= f(x)\,O(1)$$

Thus

$$\log(1+O(f(x))) = O(f(x))\,O(1) = O(f(x))$$

<u>Example</u> Assume that $f(x) = O(1)$. Then

$$e^{O(f(x))} = 1 + O(f(x))$$

<u>Example</u> Assume that $f(x) \prec 1$ and $f(x)g(x) = O(1)$
Then

$$(1 + O(f(x)))^{O(g(x))} = 1 + O(f(x)g(x))$$

<u>Example</u> Assume that $f(x) \prec 1$.

$$\frac{1}{a + O(f(x))} = \frac{1}{a}(1 + O(f(x)))^{-1} = \frac{1}{a}(1 + O(f(x))$$
$$= \frac{1}{a} + O(f(x))$$

Exercises

Prove or disprove

- $\frac{1}{1+x^2} = 1 + O(x) \qquad x \to 0$
- $\cos(x)\sin(x) = \begin{cases} O(x^2) & x \to \infty \\ O(x^2) & x \to 0 \end{cases}$
- $\cos(O(x)) = 1 + O(x^2) \qquad \text{all } x$
- $O(x+y)^2 = O(x^2) + O(y^2)$
- $e^{(1+O(1/n))^2} = e + O(1/n) \qquad n \to \infty$
- $n^{\log(n)} = O(\log(n)^n) \qquad n \to \infty$

Prove that

$$1 + 2x + O(x^2) = (1+2x)(1+O(x^2)) \qquad x \to 0$$