# MAS111 Convergence and Continuity

## Key Objectives

**At the end of the course, students should know the following topics and be able to apply the basic principles and theorems therein to solving various problems concerning convergence of sequences and continuity of functions.**

**Real numbers:** definition of algebraic and transcendental numbers, proving basic inequalities, finding supremum and infimum of a set of real numbers, stating the completeness axiom.

**Sequences:** definition of limit, proving results concerning limits of sequences, finding the limit of a bounded monotone sequence, proof and application of the sandwich theorem, proof and application of the Bolzano-Weierstrass Theorem, calculation of limits.

**Series:** definition of convergence, application of the comparison test, root test and ratio test for convergence, geometric and harmonic series, alternating series and absolute convergence, power series, finding radius and domain of convergence, stating the power series expansion of $\sin x$, $\cos x$ and $\exp x$, calculation of sums of simple series.

**Real functions:** definition of the limit of a function, definition of one-sided limits, use of the sandwich theorem, calculation of limits of functions.

**Continuous functions:** definition of continuity, derivation of basic properties of continuous functions on closed intervals, statement and application of the Intermediate Value Theorem, proving results concerning the roots of polynomials.

# 1 Real Numbers

$\mathbb{R}$ is a *complete ordered field.*

**Axioms for addition:**

**A1.** $\forall x, y \in \mathbb{R},\ \ x + y = y + x.$

**A2.** $\forall x, y, z \in \mathbb{R},\ \ (x + y) + z = x + (y + z).$

**A3.** $\exists 0 \in \mathbb{R}$, *called zero, such that* $x + 0 = x$ *for all* $x \in \mathbb{R}$.

**A4.** $\forall x,\ \ \exists(-x) \in \mathbb{R}$ *such that* $x + (-x) = 0.$

**Axioms for multiplication:**

**M1.** $\forall x, y \in \mathbb{R},\ \ xy = yx.$

**M2.** $\forall x, y, z \in \mathbb{R},\ \ (xy)z = x(yz).$

**M3.** $\exists 1 \in \mathbb{R}$, *called one, such that* $x1 = x.$

**M4.** it $1 \neq 0$ and $\forall x \in \mathbb{R}\backslash\{0\},\ \ \exists x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1.$

**Distributive law:**

$\forall x, y, z \in \mathbb{R},\ \ x(y + z) = xy + xz.$

**Axioms for order:**

**O1.** $\forall x, y \in \mathbb{R}$, *exactly one of the following holds*:

$$x < y, \quad x = y, \quad y < x.$$

**02.** $\forall x, y, z \in \mathbb{R}$,

$$x < y \ \ \textit{and} \ \ y < z \Longrightarrow x < z.$$

**03.** $\forall x, y, z \in \mathbb{R}$,

$$x < y \Longrightarrow x + z < y + z.$$

**04.** $\forall x, y \in \mathbb{R}$,

$$0 < x \ \ \textit{and} \ \ 0 < y \Longrightarrow 0 < xy.$$

**Completeness axiom:**

*If* $A$ *is a non-empty subset of* $\mathbb{R}$ *and has an upper bound,*
*then it has a least upper bound.*

**Theorem 1.1.** (Archimedes Principle ) *Let* $x, y \in \mathbb{R}$ *and let* $x > 0$. *Then there exists* $n \in \mathbb{N}$ *such that* $nx > y$.

**Corollary 1.2.** *Let* $x, y \in \mathbb{R}$ *be such that* $x < y$. *Then there exists a rational number* $r \in \mathbb{R}$ *such that* $x < r < y$.

**Corollary 1.3.** *There exists a unique number* $\ell \in \mathbb{R}$ *such that* $\ell > 0$ *and* $\ell^2 = 2$.

**Proposition 1.4.** *Let* $A$ *be a non-empty subset of* $\mathbb{R}$ *with an upper bound. Let* $\ell \in \mathbb{R}$. *The following conditions are equivalent:*

*(i)* $\ell = \sup A$;

*(ii)* $\ell$ *is an upper bound of* $A$ *such that* $\forall \varepsilon > 0$, $\exists a \in A$ *with* $\ell - \varepsilon < a$.

**Proposition 1.5.** *Let* $A$ *be a non-empty subset of* $\mathbb{R}$ *with a lower bound. Let* $\ell \in \mathbb{R}$. *The following conditions are equivalent:*

*(i)* $\ell = \inf A$;

*(ii)* $\ell$ *is a lower bound of* $A$ *such that* $\forall \varepsilon > 0$, $\exists a \in A$ *with* $a < \ell + \varepsilon$.

**Proposition 1.6.** *Let* $p/q$ *be a rational root of*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o = 0$$

*where* $n \geq 1$ *and* $a_0, \cdots, a_n \in \mathbb{Z}$ *with* $a_0 a_n \neq 0$. *If* $p$ *and* $q$ *are coprime, then* $p | a_0$ *and* $q | a_n$.

**Question 1.7.** What is an algebraic number? a transcendental number?

**Question 1.8.** What are the sup and inf of

$$\left\{ \frac{x}{1 + x^2} : x \in \mathbb{R} \right\} ?$$

**Question 1.9.** Show that

(i) $|x - a| < b \Longleftrightarrow a - b < x < a + b$.

(ii) $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$ and $x \geq -1$.

**Theorem 1.10.** *Let* $A$ *and* $B$ *be non-empty bounded subsets of* $\mathbb{R}$. *Then*

*(i)* $\sup(A + B) = \sup A + \sup B$ ;

*(ii)* $\inf(A + B) = \inf A + \inf B$.

**Theorem 1.11.** (Arithmetic-Geometric mean inequality) *For any non-negative real numbers* $a_1, a_2, \cdots, a_n$, *we have*

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

## 2 Sequences

A (real) sequence is a function $x : \mathbb{N} \longrightarrow \mathbb{R}$. We study the behaviour of $x(n)$ for *'large'* $n$. We often write $x_n$ for $x(n)$ and denote a sequence $x : \mathbb{N} \longrightarrow \mathbb{R}$ by listing its image:

$$x_1,\ x_2,\ x_3,\ \cdots,\ x_n,\ \cdots$$

or by

$$(x_n)_{n=1}^{\infty}, \quad (x_n)_{n\in\mathbb{N}}$$

or simply, $(x_n)$.

**Definition 2.1.** Let $(x_n)$ be a sequence. We say that $(x_n)$ *converges to the limit* $\ell$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Longrightarrow |x_n - \ell| < \varepsilon$$

in which case, we say that the sequence $(x_n)$ *is convergent* or, *converges.* We say that $(x_n)$ is *divergent* or *diverges* if it is not convergent.

**Theorem 2.2.** *A sequence can converge to at most one limit.*

**Notation** We denote *the* limit of $(x_n)$, if it exists, by

$$\lim_{n\to\infty} x_n.$$

We also write

$$x_n \to \ell \ \text{ as } \ n \to \infty \qquad \text{or simply,} \quad x_n \to \ell$$

to mean that $(x_n)$ converges to the limit $\ell$.

**Theorem 2.3.** $\lim_{n\to\infty} \frac{1}{n} = 0.$

*Proof.* By Archimedes Principle. □

A sequence $(x_n)$ is said to be *bounded above* if there exists some constant $K \in \mathbb{R}$ such that $x_n \leq K$ for all $n$.

A sequence $(x_n)$ is said to be *bounded below* if there exists some constant $K \in \mathbb{R}$ such that $x_n \geq K$ for all $n$.

A sequence $(x_n)$ is said to be *bounded* if it is both bounded above and below which is equivalent to saying that there exists some constant $K \in \mathbb{R}$ such that $|x_n| \leq K$ for all $n$.

**Theorem 2.4.** *Every convergent sequence is bounded.*

**Example 2.5.**

(i) Theorem 2.4 does **not** say that a bounded sequence converges, indeed, the following sequence is bounded and divergent:

$$1,\ 0,\ 1,\ 0,\ \cdots .$$

(ii) The sequence $1,\ 2,\ 3,\ \cdots,\ n,\ \cdots$ diverges because it is unbounded.

**Theorem 2.6.** *Let $(a_n) \to a$ and $(b_n) \to b$. Then we have*

*(i)* $a_n + b_n \to a + b$;

*(ii)* $a_n - b_n \to a - b$;

*(iii)* $a_n b_n \to ab$;

*(iv)* $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ *if* $b_n, b \neq 0$.

**Theorem 2.7.** *(Sandwich Theorem) Given that $a_n \leq x_n \leq b_n$ and that both sequences $(a_n)$ and $(b_n)$ converge to $\ell$, then the sequence $(x_n)$ also converges to $\ell$*

**Example 2.8.** *Let $0 < a < 1$. Then $a^n \to 0$.*

**Definition 2.9.** A sequence $(a_n)$ is called *increasing* if $n \geq m \implies a_n \geq a_m$; it is called *decreasing* if $n \geq m \implies a_n \leq a_m$. Further, $(a_n)$ is called *monotone* if it is either increasing or decreasing.

**Example 2.10.** Given $a > 0$, the sequence $\sqrt[n]{a}$ is a monotone sequence. What is the limit $\lim\limits_{n\to\infty} \sqrt[n]{a}$ ?

**Theorem 2.11.** *Let the sequence $(a_n)$ be increasing and bounded above. Then it converges, moreover, we have*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

**Theorem 2.12.** *Let the sequence $(a_n)$ be decreasing and bounded below. Then it converges, moreover, we have*

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

**Example 2.13.** The sequence

$$\sqrt{3}, \quad \sqrt{3+\sqrt{3}}, \quad \sqrt{3+\sqrt{3+\sqrt{3}}}, \quad \ldots$$

converges. What is the limit?

**Example 2.14.** *The sequence*

$$\left(\left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}$$

*is increasing and bounded by* 3*. Therefore it converges and the limit is denoted by* $e$*.*

**Theorem 2.15.** *Given* $a_n \to a$ *as* $n \to \infty$*, we have*

$$\frac{a_1 + \cdots + a_n}{n} \to a.$$

**Definition 2.16.** Let $(x_n)$ be a sequence. A *subsequence* of $(x_n)$ is any sequence of the form

$$x_{n_1}\,,\ x_{n_2}\,,\ x_{n_3}\,,\ \cdots\,,\ x_{n_k}\,,\ \cdots$$

where $n_k \in \mathbb{N}$ and

$$n_1 \,<\, n_2 \,<\, n_3 \,<\, \cdots \,<\, n_k \,<\, \cdots\,.$$

**Example 2.17.**

(i) $(x_n)$ is a subsequence of itself.

(ii) $(x_{2n})$ and $(x_{2n+1})$ are subsequences of $(x_n)$.

(iii) $x_2\,,\ x_5\,,\ x_6\,,\ x_{23}\,,\ x_{31}\,,\ x_{31}\,,\ x_{31}\,,\ x_{31}\,,\ x_{31}\,,\ \cdots$ is **not** a subsequence of $(x_n)$.

(iv) $1\,,3\,,5\,,5\,,5\,,6\,,7\,,8\,,\cdots$ *is* a subsequence of
$1\,,2\,,3\,,4\,,5\,,5\,,5\,,5\,,5\,,6\,,7\,,8\,,\cdots$ .

**Proposition 2.18.** *Let* $(x_n)$ *be a sequence. Given that both subsequences* $(x_{2n})$ *and* $(x_{2n+1})$ *converge to the same limit* $\ell$*, we also have* $x_n \to \ell$ *as* $n \to \infty$*.*

**Theorem 2.19.** *Every sequence has a monotone subsequence.*

**Theorem 2.20.** (Bolzano-Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

**Definition 2.21.** A sequence $(x_n)$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0\,, \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon.$$

Notation: $\lim\limits_{n,m\to\infty} |x_n - x_m| = 0$.

**Theorem 2.22.** (Cauchy Criterion for Convergence) *A real sequence* $(x_n)$ *is convergent if, and only if, it is a Cauchy sequence.*

## 3 series

The symbols

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots ,$$

called a (real) *series*, denote the (real) sequence $(s_n)$ of partial sums, where

$$s_n = a_1 + a_2 + \cdots + a_n$$

is called the *$n$-th partial sum* of the series. We say that a series $\sum_{n=1}^{\infty} a_n$ *converges* or, *is convergent*, if the sequence $(s_n)$ of partial sums converges, in which case, the limit of $(s_n)$ is called the *sum* of the series and we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) .$$

A series $\sum_{n=1}^{\infty} a_n$ is said to be *divergent* or, *diverge*, if it is not convergent.

**CAUTION.** Do not confuse the *sequence*

$$a_1 , \ a_2 , \ \cdots \ a_n , \ \cdots$$

with the *series*

$$a_1 + a_2 + \cdots + a_n + \cdots ,$$

the latter *is* the following sequence :

$$a_1 , \quad a_1 + a_2 , \quad a_1 + a_2 + a_3 , \quad \cdots , \quad a_1 + a_2 + \cdots + a_n , \quad \cdots \ !!!$$

**Example 3.1.** The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$$

converges and the sum is 1. We compute the $n$-th partial sum:

$$s_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = 1 - \frac{1}{2^n}$$

which converges to 1 as $n \to \infty$, that is, $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

**Example 3.2.** The series

$$\sum_{n=1}^{\infty} r^n$$

converges for $|r| < 1$. What is the sum?

**Example 3.3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2}$.

**Example 3.4.** The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. Indeed, its sequence $(s_n)$ of partial sums is unbounded and hence diverges. For any $K > 0$, pick $m > 2K$, then for any $n > 2^m$, we have

$$\begin{aligned} s_n &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) \\ &\quad + \cdots + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right) + \cdots + \frac{1}{n} \\ &> \frac{1}{2} + (2)\left(\frac{1}{4}\right) + (4)\left(\frac{1}{8}\right) + \cdots + (2^{m-1})\left(\frac{1}{2^m}\right) \\ &= \frac{m}{2} > K \end{aligned}$$

which shows that $(s_n)$ is unbounded.

**Theorem 3.5.** *If a series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$ *as* $n \to \infty$.

**CAUTION.** If $a_n \to 0$, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ NEED NOT converge! See Example 3.4.

**Theorem 3.6.** *Let* $a_n \geq 0$ *for all* $n \in \mathbb{N}$. *Then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges if it has* **bounded** *partial sums* $s_n$.

**Example 3.7.** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent because $s_n < 2$ for all $n$.

**Theorem 3.8.** (General Principle of convergence) *A series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges if, and only if, for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$m > n > N \Longrightarrow |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon.$$

The following are three very useful tests for convergence of series.

**Theorem 3.9.** (Comparison Test) *Let* $0 \le a_n \le b_n$ *for all* $n$ (*or, from some* $n$ *onwards* ). *If the series* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges. Equivalently, if* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

**Theorem 3.10.** (Root Test) *If* $a_n > 0$ *and* $\sqrt[n]{a_n} \le r < 1$ *for some fixed* $r$, *and for all* $n$ (*or, from some* $n$ *onwards*), *then* $\sum_{n=1}^{\infty} a_n$ *converges. If* $\sqrt[n]{a_n} \ge 1$ *from some* $n$ *onwards, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

**Theorem 3.11.** (Ration Test) *If* $a_n > 0$ *and*

$$\frac{a_{n+1}}{a_n} \le r < 1$$

*for some fixed* $r$, *and for all* $n$ ( *or, from some* $n$ *onwards* ), *then* $\sum_{n=1}^{\infty} a_n$ *converges. If*

$$\frac{a_{n+1}}{a_n} \ge 1$$

*from some* $n$ *onwards, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

**Example 3.12.** What can you say about the convergence or divergence of the following series;

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \ ; \quad \sum_{n=1}^{\infty} \frac{1}{\log n} \ ; \quad \sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n \ ; \quad \sum_{n=1}^{\infty} \sin\frac{1}{n} \ .$$

A series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with $a_n \ge 0$ is called an *alternating series.*

**Theorem 3.13.** (Leibniz) *An alternating series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *is convergent if the sequence* $(a_n)$ *decreases to* $0$, *that is,* $(a_n)$ *is decreasing and* $\lim_{n\to\infty} a_n = 0$.

**Example 3.14.** The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges.

**Definition 3.15.** A series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* or, *converge absolutely*, if the series $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

**Proposition 3.16.** *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges; in other words, every absolutely convergent series is convergent.*

**Cauchy Product**

**Theorem 3.17.** *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be absolutely convergent series. Define*

$$c_n = \sum_{p+q=n} a_p b_q = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

*Then the series* $\sum_{n=0}^{\infty} c_n$ *converges absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left( \sum_{n=0}^{\infty} a_n \right) \left( \sum_{n=0}^{\infty} b_n \right).$$

*Proof.* Write $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$. Let

$$s_n = a_0 + a_1 + \cdots + a_n \,,$$

$$t_n = b_0 + b_1 + \cdots + b_n \,.$$

Then we have

$$w_n := s_n t_n \longrightarrow AB \quad \text{as} \quad n \to \infty.$$

Since

$$0 \le |c_0| + \cdots + |c_n| \le (|a_0| + \cdots + |a_n|)(|b_0| + \cdots + |b_n|) \le \left( \sum_{n=0}^{\infty} |a_n| \right) \left( \sum_{n=0}^{\infty} |b_n| \right),$$

the series $\sum_{n=0}^{\infty} |c_n|$ converges because it has bounded partial sums.

Next, we show $\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$. First, assume that $a_n, b_n \geq 0$ for all $n$. Then the inequalities

$$w_{[\frac{n}{2}]} \leq c_0 + \cdots + c_n \leq w_n$$

and the fact that both sequences $w_n$ and $w_{[\frac{n}{2}]}$ converge to $AB$ yield

$$\sum_{n=0}^{\infty} c_n = \lim_{n\to\infty} (c_0 + \cdots + c_n) = AB.$$

Finally, make no assumption on $a_n$ and $b_n$. For any $x \in \mathbb{R}$, we can write

$$x = x^+ - x^-$$

with $x^+, x^- \geq 0$. Indeed, we can let $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. Now we have

$$\begin{aligned} c_n &= \sum_{p+q=n} a_p b_q = \sum_{p+q=n} (a_p^+ - a_p^-)(b_q^+ - b_q^-) \\ &= \sum_{p+q=n} a_p^+ b_q^+ - \sum_{p+q=n} a_p^+ b_q^- - \sum_{p+q=n} a_p^- b_q^+ + \sum_{p+q=n} a_p^- b_q^- \\ &:= x_n - y_n - u_n + v_n. \end{aligned}$$

By the above arguments, we have

$$\begin{aligned} \sum_{n=0}^{\infty} x_n &= \left(\sum_{n=0}^{\infty} a_n^+\right)\left(\sum_{n=0}^{\infty} b_n^+\right), \quad \sum_{n=0}^{\infty} y_n = \left(\sum_{n=0}^{\infty} a_n^+\right)\left(\sum_{n=0}^{\infty} b_n^-\right), \\ \sum_{n=0}^{\infty} u_n &= \left(\sum_{n=0}^{\infty} a_n^-\right)\left(\sum_{n=0}^{\infty} b_n^+\right), \quad \sum_{n=0}^{\infty} v_n = \left(\sum_{n=0}^{\infty} a_n^-\right)\left(\sum_{n=0}^{\infty} b_n^-\right). \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{n=0}^{\infty} c_n &= \sum_{n=0}^{\infty} x_n - \sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} v_n \\ &= \left(\sum_{n=0}^{\infty} a_n^+ - \sum_{n=0}^{\infty} a_n^-\right)\left(\sum_{n=0}^{\infty} b_n^+ - \sum_{n=0}^{\infty} b_n^-\right) \\ &= AB. \end{aligned}$$

□

# 4 Power series

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

is called a *power series* (in $x$) *centred at* $a$ with coefficients $a_n$. We often consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

centred at 0.

**Theorem 4.1.** *Given any power series* $\sum_{n=0}^{\infty} a_n x^n$, *one of the following three conditions holds:*

*(i)* $\sum_{n=0}^{\infty} a_n x^n$ *converges only at* $x = 0$*;*

*(ii)* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for all* $x \in \mathbb{R}$*;*

*(iii) there exists* $R > 0$ *such that* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $|x| < R$ *and diverges for* $|x| > R$.

**Definition 4.2.** When condition (iii) above occurs, the number $R > 0$ is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n x^n$. For conditions (i) and (ii) above, we define the *radius of convergence* $R$ to be 0 and $\infty$, respectively. Thus, every power series has a radius of convergence $R$.

**Definition 4.3.** The *domain of convergence* of a power series $\sum_{n=0}^{\infty} a_n x^n$ is the set

$$\left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

If a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R$, then its domain of convergence is an interval with endpoints $-R$ and $R$, for example, $(-R, R)$ or $[-R, R)$, or some other form. Note that the domain of convergence always contains the open interval $(-R, R)$.

Given a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

we can compute its radius of convergence $R$ by the following formulae:

$$\begin{aligned} R &= \frac{1}{\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|} \\ &= \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} \end{aligned}$$

provided the above limits exist.

**Example 4.4.** The power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ has radius of convergence

$$R = \frac{1}{\lim\limits_{n\to\infty} \left| \dfrac{1/(n+1)}{1/n} \right|} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The series converges for $x \in (-1, 1)$. At $x = 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. At $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. So the domain of convergence is $[-1, 1)$.

**Example 4.5.** Find the radius of convergence of each of the following series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \quad \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}\,, \quad \sum_{n=1}^{\infty} n^n x^n\,, \quad \sum_{n=0}^{\infty} (3 + (-1)^n)^n x^n\,.$$

# 5 Limits of functions

**Definition 5.1.** Let $a \in S \subset \mathbb{R}$. Let

$$f : S \backslash \{a\} \longrightarrow \mathbb{R}$$

be a function. We say that $f(x)$ *tends to* $\ell$ *as* $x$ *tends to* $a$ (in $S$) if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in S \text{ and } 0 < |x - a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Usually, $S$ is an open interval, we often omit the words "in $S$" if it is understood and also say that $f$ has a limit at $a$ in the above situation.

**Lemma 5.2.** (Uniqueness of limit) *If $f(x)$ tends to $\ell$ and $\ell'$ as $x$ tends to $a$, then $\ell = \ell'$.*

**Definition 5.3.** *If $f(x)$ tends to $\ell$ as $x$ tends to $a$, then we call $\ell$ the limit of $f(x)$ as $x$ tends to $a$ and write*

$$\lim_{x \to a} f(x) = \ell.$$

**We note that $f$ need not be defined at $a$.**

**Example 5.4.** The function $f(x) = x \sin \frac{1}{x}$ is defined on $\mathbb{R} \backslash \{0\}$ and

$$\lim_{x \to 0} \; x \sin \frac{1}{x} = 0.$$

Indeed, let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then

$$\begin{aligned} 0 < |x - 0| < \delta &\implies |x| < \delta \\ &\implies \left| x \sin \frac{1}{x} - 0 \right| = \left| x \sin \frac{1}{x} \right| \leq |x| < \delta = \varepsilon. \end{aligned}$$

The following two propositions are very useful for computing limits of functions.

**Proposition 5.5.** *Given $\lim_{x \to a} f(x) = \ell$ and $\lim_{x \to a} g(x) = \rho$, we have*

$$\lim_{x \to a} (f + g)(x) = \ell + \rho;$$

$$\lim_{x \to a} (fg)(x) = \ell \rho;$$

$$\lim_{x \to a} \left( \frac{1}{f} \right)(x) = \frac{1}{\ell} \text{ if } \ell \neq 0.$$

*Further, if $f(x) \geq g(x)$ for all $x$, then $\ell \geq \rho$.*

**Proposition 5.6.** *If $h(x) \leq f(x) \leq g(x)$ and $\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = \ell$, then*

$$\lim_{x \to a} f(x) = \ell.$$

**Example 5.7.** $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \frac{1}{2}$; $\quad \lim_{x \to 0} \frac{\sin x}{x} = 1.$

## One-sided limits

**Definition 5.8.** Let $a \in S \subset \mathbb{R}$ and let $f : S\backslash\{a\} \longrightarrow \mathbb{R}$ be a function. We say that $f(x)$ *tends to the limit* $\ell$ *as* $x$ *tends to* $a$ *from the left* ( or increases to $a$) (in $S$) if

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that } x \in S \text{ and } a - \delta < x < a \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Usually, $S$ is an interval with an end point $a$, we often omit the words "in $S$" if it is understood and denote the limit by

$$\lim_{x \to a^-} f(x) = \ell\,, \quad \text{or} \quad \lim_{x \uparrow a} f(x) = \ell$$

which is called the left-hand limit of $f$ at $a$.

We define the right-hand limit of a function likewise.

**Definition 5.9.** Let $a \in S \subset \mathbb{R}$ and let $f : S\backslash\{a\} \longrightarrow \mathbb{R}$ be a function. We say that $f(x)$ *tends to the limit* $\ell$ *as* $x$ *tends to* $a$ *from the right* ( or decreases to $a$) (in $S$) if

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that } x \in S \text{ and } a < x < a + \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Usually, $S$ is an interval with an end point $a$, we often omit the words "in $S$" if it is understood and denote the limit by

$$\lim_{x \to a^+} f(x) = \ell\,, \quad \text{or} \quad \lim_{x \downarrow a} f(x) = \ell$$

which is called the right-hand limit of $f$ at $a$.

**Example 5.10.** $\lim_{x \to 1^-} [x] = 0\,, \quad \lim_{x \to 1^+} [x] = 1\,;$

$$\lim_{x \to 0^-} \exp\left(\frac{1}{x}\right) = 0\,, \quad \lim_{x \to 0^+} \exp\left(\frac{1}{x}\right) \text{ does not exist.}$$

**Theorem 5.11.** *A function has a limit at a if, and only if, it has equal one-sided limits at a, in other words, the following two conditions are equal:*

*(i)* $\lim_{x \to a} f(x) = \ell\,;$

*(ii)* $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \ell\,.$

### Application of limits : Derivatives

Let $f : (a, b) \longrightarrow \mathbb{R}$ be a function and let $c \in (a, b)$. We say that $f$ is *differentiable* at $c$ if the following limit exists

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

in which case, the limit is called the *derivative* of $f$ at $c$, and is denoted by $f'(c)$.

**Example 5.12.** Let $f(x) = x^n$. Then $f'(c) = nc^{n-1}$ for all $c \in \mathbb{R}$.

More generally, we have:

**Theorem 5.13.** *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R$*. Then* $f$ *is differentiable at every point* $x \in (-R, R)$ *with derivative*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

**Example 5.14.** $\exp{}'(x) = \left(1 + x + \frac{x^2}{2!} + \cdots\right)' = 1 + x + \frac{x^2}{2!} + \cdots = \exp(x)$ ;

$\sin{}'(x) = \left(x - \frac{x^3}{3!} + \cdots\right)' = 1 - \frac{x^2}{2!} + \cdots = \cos(x)$ .

**Proposition 5.15.** *The exponential function* $\exp : \mathbb{R} \longrightarrow (0, \infty)$ *is a bijection.*

*Proof.* See Appendix. □

**Definition 5.16.** The inverse $\exp^{-1} : (0, \infty) \longrightarrow \mathbb{R}$ of the exponential function exp is called the *natural logarithmic function* and is denoted by

$$\log : (0, \infty) \longrightarrow \mathbb{R} .$$

Therefore we have

$$\log \exp(x) = \exp \log(x) = x$$

for $x$ in appropriate domains.

# 6 Continuous functions

Let $(a, b)$ be an open interval in $\mathbb{R}$. We include the case of $(a, b) = (-\infty, \infty) = \mathbb{R}$.

**Definition 6.1.** Let $f : (a, b) \longrightarrow \mathbb{R}$. We say that $f$ is *continuous at a point* $c \in (a, b)$ if

$$\lim_{x \to c} f(x) = f(c),$$

in other words,

$$\forall \varepsilon > 0\,,\ \exists \delta > 0 \ \text{ such that } \ |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon.$$

**Example 6.2.** Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function. Then $f$ is continuous.
**CAUTION.** The converse is false!
The exponential function exp is differentiable, hence it is continuous.

**Proposition 6.3.** *Let $f, g : (a, b) \longrightarrow \mathbb{R}$ be continuous functions. Then $f + g$, $f - g$ and $fg$ are continuous functions on $(a, b)$. Further, if $g(x) \neq 0$ for every $x \in (a, b)$, then the quotient $\dfrac{f}{g}$ is continuous on $(a, b)$.*

*Proof.* This follows from the arithmetics of limits in Proposition 5.5. □

**Proposition 6.4.** *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Then their composite $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ is also a continuous function.*

**Example 6.5.** *The Gaussian function* $\exp(-x^2)$ *is continuous.*

**Proposition 6.6.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous at a point $c$ and $f(c) > 0$. Then there exists $\delta > 0$ such that*

$$f(x) > 0$$

*for all $x \in (c - \delta, c + \delta)$.*

**Remark.** We have similar result to the above for $f(c) < 0$.

A useful criterion of continuity is that a function is continuous if, and only if, it preserves convergence of sequences.

**Theorem 6.7.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and let $c \in \mathbb{R}$. The following conditions are equivalent:*

*(i) $f$ is continuous at $c$;*

*(ii) if $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) = f(c)$.*

**Example 6.8.** We have $\lim_{n\to\infty} \sqrt[n]{n} = 1$ because $0 \le \sqrt[n]{n} - 1 < \frac{2}{\sqrt{n}}$. The function log is continuous at the point 1, therefore $\lim_{n\to\infty} \log(\sqrt[n]{n}) = \log 1 = 0$, or

$$\lim_{n\to\infty} \frac{\log n}{n} = 0\,.$$

Finally, continuous functions have the following important properties.

**Theorem 6.9.** (Intermediate Value Theorem) *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function such that $f(a)f(b) < 0$. Then there is a point $t \in (a, b)$ satisfying $f(t) = 0$.*

**Example 6.10.** *The following equation has no rational root, but has a negative root:*

$$3x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

**Example 6.11.** Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Then there is a point $t \in [0, 1]$ such that $f(t) = t$.

**Definition 6.12.** A function $f : S \longrightarrow \mathbb{R}$ is said to be *bounded* if there is a constant $M$ such that

$$|f(x)| \le M$$

for all $x \in S$.

**Theorem 6.13.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then $f$ is bounded.*

**Example 6.14.** In the above theorem, it is important that $f$ is defined on a *closed* interval. The theorem is false for continuous functions defined on *open* intervals, for instance, the function $f(x) = \frac{1}{x}$ is continuous on the open interval on $(0, 1)$, but is unbounded!

The following theorem says that a continuous function on a closed interval achieves its supremum and infimum.

**Theorem 6.15.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then there exist $s, t \in [a, b]$ such that*

$$f(s) = \sup\{f(x) : x \in [a, b]\}\,,$$
$$f(t) = \inf\{f(x) : x \in [a, b]\}\,.$$

# Appendix

**Theorem.** *The exponential function* $\exp : \mathbb{R} \longrightarrow (0,\infty)$ *is a continuous bijection.*

*Proof.* We first show that

$$\exp(x+y) = \exp(x)\ \exp(y) \qquad (x, y \in \mathbb{R}).$$

This follows from the product formula in Theorem 3.17:

$$\begin{aligned}\exp(x)\ \exp(y) &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!} \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{n!}\right)\left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k y^{n-k}\right) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\ &= \exp(x+y).\end{aligned}$$

In particular, we have

$$\exp(x)\ \exp(-x) = \exp(0) = 1 \tag{1}$$

and $\exp(x) \neq 0$ for all $x \in \mathbb{R}$. Since $\exp(x) > 0$ for $x > 0$, we have $\exp(x) > 0$ for all $x \in \mathbb{R}$.

By the definition of exp, we have $\exp(x) > \exp(y) > \exp(0)$ for $x > y > 0$. It follows from (1) that $\exp(-y) > \exp(-x)$. Hence exp is strictly increasing and injective.

To show that exp is surjective, we apply the Intermediate Value Theorem since exp is continuous. Let $r \in (0,\infty)$. If $r = 1$, then $r = \exp(0)$. If $r > 1$, then $\exp(r) > r > \exp(0)$ implies that $r = \exp(x)$ for some $x \in (0, r)$ by Intermediate Value Theorem. If $r < 1$, then $\dfrac{1}{r} > 1$ implies that $\dfrac{1}{r} = \exp(y)$ for some $y \in \mathbb{R}$ and therefore $r = \dfrac{1}{\exp(y)} = \exp(-y)$. This proves surjectivity of exp. □