MAS/202 Algorithmic Mathematics: Coursework 9

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DEADLINE: Wednesday of week 11, at 12:00 pm.

CONTENT: vectors

M$\star$croESSAY : Write an essay on vectors (in the context of this course). [$\not c$, 100]

Problem 1. Let $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or $\mathbb{Z}/(p)$, p a prime (or indeed let F be any field), and let $v_1, \dots, v_k \in F^n$.

(a) Let $0 \neq \alpha \in F$. Prove that

$$\langle v_1, \dots, v_{k-1}, \alpha v_k \rangle = \langle v_1, \dots, v_k \rangle.$$

[*Hint:* show that the left-hand side is contained in the rigth-hand side, and vice-versa.]

(b) Prove that if the sequence $(v_1, \dots, v_k)$ is in echelon form, then $v_1, \dots, v_k$ are linearly independent.
[*Hint:* this is linear algebra.]

(c) Let $F = \mathbb{Z}/(p)$, p a prime. Using `Echelonize`, prove that if $\mathcal{W}$ is the subspace of F^n generated by the vectors $W = (w_1, \dots, w_m)$, then $\#\mathcal{W} = p^s$, for some s, with $0 \leq s \leq n$. Explain what is s.
[*Hint:* use the result of the previous problem.]

Problem 2. Let $F = \mathbb{Z}/(2)$, and let $v_1 = (0, 1, 0, 1)$, $v_2 = (0, 0, 0, 0)$ $v_3 = (1, 1, 0, 1)$, $v_4 = (1, 0, 1, 0)$, $v_5 = (0, 1, 1, 1)$ be in F^4. Use the algorithm `Echelonize` to determine a sequence U, in echelon form, of vectors in F^4, such that $\langle U \rangle = \langle v_1, \dots, v_5 \rangle$.

Problem 3. Let $F = \mathbb{Z}/(5)$, and let

$$v_1 = (4,3,1,3) \qquad v_2 = (1,3,4,1) \qquad v_3 = (2,3,3,2) \in F^4.$$

(a) Use the algorithm `Echelonize` to determine a sequence U, in echelon form, of vectors in F^4, such that $\langle U \rangle = \langle v_1, v_2, v_3 \rangle$.

(b) Let $\mathcal{V} = \langle v_1, v_2, v_3 \rangle$. Which of the following vectors belong to $\mathcal{V}$?

$$(2,1,3,1) \qquad (3,1,4,2) \qquad (2,2,4,3).$$

In each case, explain why.

Problem 4. Write an algorithm to the following specifications

`Algorithm Echelon`

`INPUT`: W, a finite sequence of n-dimensional vectors, over the same field.

`OUTPUT`: TRUE, if W is in echelon form, FALSE otherwise.

Explain what you are doing. Use the notation $W = (W_1, W_2, \ldots)$, and $W_k = (W_{k,1}, W_{k,2}, \ldots)$, and the operator # to access input data. Assume that the algorithm `ldindx` and `ldterm` are available. Decide how `ldterm` behaves for the zero vector, and design the algorithm accordingly.