

A friendly introduction to
EXPERIMENTAL MATHEMATICS

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Introduction

With the aid of computers, mathematical experimentation is undergoing an unprecedented development, which is defining new ways of doing mathematical research. This course is intended to give pre-university students a taste of this exciting new trend. Using the state of the art computer algebra software Maple, the students will be invited to explore some mathematical phenomena, to make a synthesis of the results, and to formulate precise conjectures. No previous computing experience is necessary.

The material is organized into three worksheets:

Worksheet 1: A Guided tour to Maple. A quick introduction to Maple's immense capabilities.

Worksheet 2: Prime numbers. Prime numbers provide an inexhaustible supply of interesting and surprising mathematical problems. Maple has several built-in facilities for studying them.

Worksheet 3: Sequences. Constructing sequences and studying them numerically and graphically marks the beginning of advanced computing.

This book and supplementary material
can be found on the Course's web page:

<http://www.maths.qmw.ac.uk/~fv/ss/>

A guided tour of Maple

This worksheet is intended to give you a taste of Maple's exciting capabilities, while illustrating some basic computer usage.

FOLLOW THESE INSTRUCTIONS CLOSELY:

- Choose a workstation and *log in*, entering your user name.
- Select the standard Windows environment, and then enter your password.
- Once you are into Windows, select from the bottom toolbar:

Public → Maths & Stats → Maple 15 → Classic Worksheet

- The *worksheet window* will appear, which occupies a portion of the screen. To enlarge it to full screen, click on the middle button placed on the right of the top frame of the window. You must now type the Maple commands that are listed below in boldface, after the Maple prompt '>'. I will also display Maple's output, for your convenience. What you type will appear where the cursor bar '|' is. Remember to press the (ENTER) key after typing the semicolon at the end of each command. Ready?

Let n be equal to $50!$ — the product of the first 50 positive integers:

$$50! = 1 \cdot 2 \cdot 3 \cdots 49 \cdot 50.$$

> **n:=50!;**

$n := 30414093201713378043612608166064768844377641568960512000000000000$

Decompose n into prime factors.

> **ifactor(n);**

$(2)^{47} (3)^{22} (5)^{12} (7)^8 (11)^4 (13)^3 (17)^2 (19)^2 (23)^2 (29) (31) (37) (41) (43) (47)$

(Does this factorization make sense to you? Can you understand from it why the last 12 decimal digits of n are zero?)

Compute a famous prime number: $2^{1279} - 1$

```
> 2^1279-1;
```

```
10407932194664399081925240327364085538615262247266704805319112350\  
4036080596733602980122394417323241848424216139542810077913835\  
6624832346490813990660567732076292412950938922034577318334966\  
1583550472959420547689811211693677147548478866962501384438260\  
2917323488853111608285384165850282556046662248318909188018470\  
6822220314052102669843548873295802887805086973618690071472071\  
0555703168729087
```

The integer $2^{1279} - 1$ has 386 decimal digits (you can check this using the Maple function `length`). As the result does not fit on a single line, Maple has automatically appended the continuation symbol ‘\’ which indicates that the output continues on the following line. In principle we can compute and display extremely large integers, and indeed integers with up to $2^{19} - 9 = 524279$ decimal digits (this magic number derives from Maple’s internal representations of integers). In practice however, to compute and display very large integers, the computer may take longer than the maximum time we are prepared to wait (100 years, for instance).

Display the first 100 digits of π , but make a deliberate typing error (forget to close the parenthesis). Note that `Pi` has upper case `P` and lower case `i`.

```
> evalf(Pi,100;
```

syntax error:

```
evalf(Pi,100;  
      ^
```

Retype it

```
> evalf(Pi,100);
```

```
3.14159265358979323846264338327950288419716939937510582097494\  
4592307816406286208998628034825342117068
```

Compute the derivative of $\sin(x^2 \ln(1 - x^3))$.

```
> diff(sin(x^2*ln(1-x^3)),x);
```

$$\cos(x^2 \ln(1 - x^3)) \left(2x \ln(1 - x^3) - 3 \frac{x^4}{1 - x^3} \right)$$

Compute the following indefinite integral

$$\int x^2 \sqrt{x^2 - a^2} dx$$

```
> int(x^2*sqrt(x^2-a^2), x);
```

$$\frac{1}{4}x(x^2 - a^2)^{3/2} + \frac{1}{8}a^2x\sqrt{x^2 - a^2} - \frac{1}{8}a^4\ln(x + \sqrt{x^2 - a^2})$$

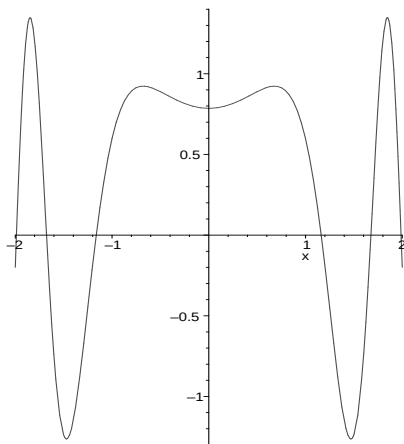
Expand the binomial $(x + \sqrt{y})^{10}$

```
> expand((x+sqrt(y))^10);
```

$$x^{10} + 10x^9\sqrt{y} + 45x^8y + 120x^7y^{3/2} + 210x^6y^2 + 252x^5y^{5/2} + 210x^4y^3 \\ + 120x^3y^{7/2} + 45x^2y^4 + 10xy^{9/2} + y^5$$

Plot the graph of the function $\cos(x^3) \arctan(1 + x^2)$ for x in the range $-2 \leq x \leq 2$

```
> plot(cos(x^3)*arctan(1+x^2), x=-2..2);
```



- Edit your worksheet. First remove the Maple command that had a typing error, including the error message. Then go to the top of the worksheet and insert a new region above the first Maple command, by choosing the **Execution Group** option from the **Insert** menu. Turn this region into a text region by clicking on the button ‘T’ on the top margin of the window. You may wish to type your name at the top of the worksheet, and also a suitable heading for the worksheet (such as **WORKSHEET 1**).
- Save the worksheet, by choosing the **Save** option from the **File** menu. When the dialog box appears, you will see the string ***.mws** in a small window. Replace the character ***** with a suitable file name (such as **work1**, etc.). The full file name will then become **work1.mws**. The suffix **mws** informs the computer that your file contains a Maple worksheet. A copy of the file **work1.mws** is now in your private file space. You will be able to retrieve this worksheet whenever you like.
- Print your worksheet (**File** menu), and collect it at the printer.
- Exit Maple and log out

Prime numbers

Given two integers a and b , we say that a divides b if there exists an integer q such that $b = aq$. Thus 3 divides 21 because $21 = 3 \cdot 7$, while 12 does not divide 21 because the equation $21 = 12 \cdot q$ has no integer solution q .

A positive integer n greater than 1 is said to be *prime* if it has precisely two divisors: 1 and n . If this is not the case we say that n is *composite*. Thus 7 is prime, while $10 = 2 \cdot 5$ is composite, being divisible by 1, 2, 5 and 10. Note that 1 is not considered prime, for convenience. There are 25 primes less than 100:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 \quad (2.1)$$

Let us check some values in the above list, with the Maple function `isprime`

```
> isprime(97);
```

true

Confirmed. The integer 97 is prime.

```
> isprime(93);
```

false

So 93 is not prime. What are its factors?

```
> ifactor(93);
```

(3) (31)

So we have $93 = 3 \cdot 31$.

What is the next term of the sequence (2.1)?

```
> nextprime(97);
```

101

We can also go back to the previous prime

```
> prevprime(101);
```

97

The arithmetic of integers is based upon the *fundamental theorem of arithmetic*:

Theorem. Any positive integer greater than one can be expressed in one and only one way as a product of primes (ignoring different arrangements of the same primes).

Thus $84 = 2^2 \cdot 3 \cdot 7$ is such unique factorization. The function `ifactor` shows the fundamental theorem in action

```
> ifactor(84);
```

$$(2)^2 (3) (7)$$

Information about Maple functions is available through the on-line help system. For instance, typing `?ifactor` you get on-line help on the function `ifactor`.

Problem 1. A LARGE PRIME.

- (a) By means of the function `prevprime` determine the largest prime with 8 decimal digits.

SOLUTIONS: We look for the largest prime smaller than 100 millions (10^8).

```
> prevprime(10^8);
```

99999989

So the largest prime with 8 digits is 99999989.

- (b) Verify the result of part (a) directly, by testing for primality the integers less than 10^8 , starting from $10^8 - 1$ and working your way backward. You must use the `isprime` function, but you should avoid testing those numbers that are divisible by 2, 3 or 5 (for these you do not need Maple!).

SOLUTIONS: Because 3 divides 99999999 and 2 divides 99999998, the first prime to test is 99999997

```
> isprime(99999997);
```

false

So 99999997 is composite. We skip the next five integers, which are clearly composite (2 divides 99999996, 99999994 and 99999992, 3 divides 99999993 and 5 divides 99999995), and we test 99999991

```
> isprime(99999991);
```

false

Skip 99999990, which is even. Finally

```
> isprime(99999989);
```

true

Problem 2. *COUNTING PRIMES.*

There are 25 primes not exceeding 100 (cf. sequence (2.1)). Indeed, the 25th prime is 97, and the 26th prime is 101.

- (a) Verify the above statement with the function `ithprime`.

SOLUTIONS:

```
> ithprime(25);
```

97

```
> nextprime(%);
```

101

The character `%` means “the output of the previous command”, that is, `nextprime(%)` is the same as `nextprime(97)`. You can save a lot of typing by using it.

Let us solve the same problem more elegantly: what we really need from Maple is the answer to the following question: “Let $x = 100$ and $n = 15$. Is it true that the n th prime is less than or equal to x , and the $(n+1)$ th prime is greater than x ?”

```
> x:=100:n:=25:
```

```
> ithprime(n) <= x and ithprime(n+1) > x;
```

true

(Note that the colon at the end of a command hides the output from view, and that `<=` means $\leq$.) Maple confirms that there are 25 primes less than 100, without displaying any digit! Double check:

```
> n:=24:
```

```
> ithprime(n) < x and ithprime(n+1) > x;
```

false

- (b) Let $\pi(x)$ be the number of primes less than or equal to x . For instance, $\pi(1) = 0$, $\pi(11) = 5$, $\pi(100) = 25$. The function $\pi(x)$ is of great interest in number theory, it has been studied for hundreds of years.

Use the function `ithprime` to determine $\pi(200)$. This calculation is likely to require a certain amount of trial and error. (The function `nextprime` may also be useful.) You will find the solution of this problem at the end of this worksheet. Can you do it without looking at it?

- (c) Compute $\pi(x)$ for $x = 400, 600, 800$ and 1000 . Tabulate the values of $\pi(x)$ (in a worksheet text region) as follows:

x	$\pi(x)$
200	?
400	?
600	?
800	?
1000	?

Look also at the *difference* between successive values of $\pi(x)$ in this table. Can you spot any trend in the behaviour of $\pi(x)$?

The largest known prime

What is the largest prime known to date? This question brings us to consider numbers of the form

$$M_p = 2^p - 1 \quad p \text{ prime}$$

which are called *Mersenne numbers*, from the name of the 17th century priest, father Marin Mersenne, who first studied their properties. He originally thought that they were all primes, but this turned out not to be the case. For instance, for the primes $p = 2, 3$ and 5 , we find that M_p is prime

$$2^2 - 1 = 3, \quad 2^3 - 1 = 7, \quad 2^5 - 1 = 31.$$

However, for $p = 11$ we have the first disappointment

```
> isprime(2^11-1);
```

false

Maple shows that $M_{11} = 2047$ is not prime. Indeed, as of July 2005, M_p is known to be prime only for the following values of p

$$\begin{aligned} &2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, \\ &9689, 9941, 11213, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 859433 \\ &1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951 \end{aligned} \quad (2.2)$$

It is not known whether there are infinitely many Mersenne primes. The importance of Mersenne numbers derives from the discovery — made by Lucas in 1878 — of a very efficient algorithm for testing their primality. It was Lucas himself who found in this way the largest prime to be known before the computer age, namely

```
> 2^127-1;
```

170141183460469231731687303715884105727

The current world record corresponds to the largest prime in (2.2), that is, $2^{25964951} - 1$ with 7,816,230 digits (check the sites www.mersenne.org and <http://www.utm.edu/research/primes/> for interesting info about primes numbers and their records). If you try to compute this prime with Maple, you will find that your computer is not quite fast enough for the job. (Besides, you would not know what to do with seven million digits scrolling on you screen! In Maple it is possible to interrupt a calculation at any time, but the procedure is system-dependent: with a window interface, one typically clicks on some ‘interrupt’ button.)

The primality of small Mersenne numbers can be tested with the function `isprime`. Thus, with reference to the sequences (2.1) and (2.2)

```
> isprime(2^89-1),isprime(2^97-1);
```

true, false

Even though the algorithm used by `isprime` is quite sophisticated, this function does not implement Lucas’ test, which works only for Mersenne primes. Maple has specialized software for Mersenne numbers — type `?mersenne` for detail.

Problem 3. A TRAPDOOR FUNCTION.

It should be intuitive that multiplying two large primes is a lot easier than factoring their product. For instance, consider the two largest primes in the list (2.1), 89 and 97. Computing their product 8633 by hand should take only a few seconds. However, given the integer 8633, how long will it take one to discover its factors? Multiplication is a *trapdoor function*, easy to do but very difficult to un-do. This very interesting phenomenon is of great theoretical interest, and has applications in cryptography. To get a feeling for the behaviour of this trapdoor function, multiply together two large primes (e.g., two Mersenne primes M_p for p taken from the list (2.2)), and then attempt to un-do the multiplication by means of `ifactor`. Monitor the computation time with the function `time` (see on-line documentation).

MATHEMATICS and COMPUTERS: Worksheet 2

Solution

SOLUTION TO PROBLEM 2:

(a)

```
> ithprime(46),ithprime(47);
```

199, 211

(b)

```
> ithprime(78),ithprime(79);
```

397, 401

```
> ithprime(109),ithprime(110);
```

599, 601

```
> ithprime(139),ithprime(140);
```

797, 809

```
> ithprime(168),ithprime(169);
```

997, 1009

x	$\pi(x)$	$\pi(x) - \pi(x - 200)$
200	46	46
400	78	32
600	109	31
800	139	30
1000	168	29

The density of primes decreases slowly. Indeed, we have

$$\frac{\pi(200)}{200} \simeq 0.230, \quad \frac{\pi(400)}{400} \simeq 0.195, \quad \frac{\pi(600)}{600} \simeq 0.182, \quad \frac{\pi(800)}{800} \simeq 0.174, \quad \frac{\pi(1000)}{1000} \simeq 0.148.$$

This indicates that p_k grows a bit faster than k . It turns out that, on average, p_k grows as $k \log(k)$, but this is fact, known as the *prime number theorem*, is quite difficult to prove.

Now that we know the values in the above table, producing a *computer-assisted proof* that they are correct can be done very effectively. For instance, to verify that $\pi(200) = 46$, it suffices to type

```
> n:=46:x:=200:
> ithprime(n)<= x and ithprime(n+1)>x;
```

true

etc. We can do better though. Let us construct a *function*, called **Ck**, that checks if $\pi(x) = n$; such function will save us a lot of typing.

```
> Ck:=(x,n)->ithprime(n)<= x and ithprime(n+1)>x:
> Ck(200,46),Ck(200,45);
```

true, false

It works. Now we can verify the correctness of the above table with a single statement

```
> Ck(200,46) and Ck(400,78) and Ck(600,109) and Ck(800,139) and Ck(1000,168);
```

true

Computer-assisted proofs —of which this is a very simple example— are now trendy in mathematics.

Sequences

A *sequence* f is a rule which associates to every natural number n a unique quantity $f(n)$. For instance, let us consider the function f which associates to every natural number n the expression $2n - n^2$. We write

$$f := n \mapsto 2n - n^2 \quad n \geq 0. \quad (3.1)$$

The elements of this sequence are the integers $f(0), f(1), f(2), f(3) \dots = 0, 1, 0, -3, \dots$. For sequences it is customary to write f_n in place of $f(n)$ (this somewhat confusing notation is very established). So we write out the elements of the sequence as

$$f_0, f_1, f_2, \dots$$

In Maple we represent sequences as functions, using the notation (3.1)

```
> f:=n->2*n-n^2;
```

$$f := n \rightarrow 2n - n^2$$

```
> f(0),f(1),f(2),f(3);
```

$$0, 1, 0, -3$$

We can easily generate the elements of a sequence with the function `seq`.

```
> seq(f(n),n=0..10);
```

$$0, 1, 0, -3, -8, -15, -24, -35, -48, -63, -80$$

The intrinsic function `seq` has two arguments. The first is the Maple expression $f(n)$, while the second specifies the variable n as well as the list of integers from 0 to 10. The variable n is sequentially assigned each value in the list, which is in turn substituted in the first expression. So the above `seq` command is equivalent to the expression $f(0), f(1), \dots, f(10)$.

Example. We consider again the sequence of primes $p_1 = 2, p_2 = 3, p_3 = 5, \dots$. We have seen that the k th term p_k is represented in Maple as `ithprime(k)`. Using `seq`, we can reproduce all 25 primes less than 100 effortlessly (cf. (2.1))

```
> seq(ithprime(k),k=1..25);
```

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

Example. We construct sequence whose elements are rational numbers (fractions)

```
> f:=n->(n^2-n^3)/(n^2+2):
> seq(f(n), n=1..5);
```

$$0, -\frac{2}{3}, -\frac{18}{11}, -\frac{8}{3}, -\frac{100}{27}$$

Note that Maple automatically simplifies fractions

$$f(4) = \frac{4^2 - 4^3}{4^2 + 2} = \frac{16 - 64}{16 + 2} = \frac{-48}{18} = -\frac{8}{3}.$$

Example. The terms of a sequence can be just about anything

```
> seq([a, [a+beta^a]], a=1..5);
```

$$[1, [1 + \beta]], [2, [2 + \beta^2]], [3, [3 + \beta^3]], [4, [4 + \beta^4]], [5, [5 + \beta^5]]$$

Plotting a sequence

Visualizing a sequence requires plotting points on the plane. Indeed, if $a_1, a_2, \dots, a_n$ are elements of a sequence, they are represented by the points

$$(1, a_1), (2, a_2), \dots, (n, a_n)$$

expressed in cartesian coordinates.

Maple has an all-purpose plotting function called `plot`, which can plot — among other things — a list points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n).$$

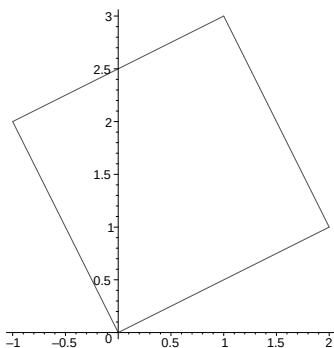
The latter must be given in the following form

$$[[x_1, y_1], [x_2, y_2], \dots, [x_n, y_n]]$$

Note that all brackets are square, and that the list of points is itself enclosed in brackets.

The following commands generate a square, with vertices at the points $(0, 0), (2, 1), (1, 3), (-1, 2)$.

```
> Square:=[[0,0],[2,1],[1,3],[-1,2],[0,0]]:
> plot(Square,scaling=CONSTRAINED);
```



The function `plot` connects points with a segment, so the the vertex $(0,0)$ was entered twice, lest one side would be missing. The option `scaling=CONSTRAINED` ensures that the scale on the two axes is the same (see what happens if you omit it). If you want to plot just points, without connecting them by a segment, use the option `style=POINT`.

Problem 1. PRIME GAPS.

Consider the sequence $p_2 - p_1, p_3 - p_2, p_4 - p_3, \dots$ where p_k is the k -th prime number. This is the sequence of *gaps* between consecutive primes, which is quite mysterious:

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2, \dots \quad (3.2)$$

For notational convenience, we define $g_k := p_{k+1} - p_k$; thus g_k is the k th gap. Plot the first 50 elements of the sequence g_k , hence identify the points at which large gaps first occur. Which is the smallest prime followed by more than 10 composite integers? (Equivalently, what is the smallest k for which g_k is greater than 10?)

Identify large gaps. What is the largest gap you can find? What would you conjecture about the occurrence of small gaps?

Problem 2. A CRAZY SEQUENCE OF FUNCTIONS.

Consider the *sequence of functions* $w_1(x), w_2(x), w_3(x), \dots$, where

$$\begin{aligned} w_1(x) &= \sin(x) \\ w_2(x) &= \sin(x) + \frac{\sin(4x)}{4} \\ w_3(x) &= \sin(x) + \frac{\sin(4x)}{4} + \frac{\sin(9x)}{9} \\ w_4(x) &= \sin(x) + \frac{\sin(4x)}{4} + \frac{\sin(9x)}{9} + \frac{\sin(16x)}{16} \\ &\vdots \\ w_n(x) &= \sum_{k=1}^n \frac{\sin k^2 x}{k^2} = \sin(x) + \frac{\sin(2^2 x)}{2^2} + \dots + \frac{\sin(n^2 x)}{n^2}. \end{aligned}$$

The individual terms appearing in the above sums should be easy to understand, since they are a variant of the familiar sine function. For instance

$$\frac{1}{10^2} \sin(10^2 x)$$

is a function featuring 100 complete oscillations in the interval $[0, 2\pi]$, with a very small amplitude: $1/10^2 = 0.01$. However, it is hard to guess what could be the effect of adding n such functions, particularly for large n .

- (a) Construct some functions in the above sequence, and then plot them in the interval $[0, \pi]$ or $[0, 2\pi]$. (Why is it sufficient to consider this range of x -values?) Try to conjecture what happens when n is large, and in particular as $n \rightarrow \infty$.

To sum the n terms comprising $w_n(x)$, use the function `add`, which closely resembles `seq`. So, for instance, one may construct $w_3(x)$ as follows

```
> w3:=add(sin(k^2*x)/k^2,k=1..3);
```

We have used the symbolic name `w3` for $w_3(x)$. To plot such function in the desired interval, use `plot`, as usual (`plot(w3,x=0..Pi);`). It is possible to restrict both x and w values, e.g., `plot(w3,x=1..2,w=0..1);`.

- (b) Construct the derivative of $w_n(x)$, using `diff` (see Worksheet 1). Then plot it and analyze it in detail. What happens when n becomes large? Try to conjecture what happens as $n \rightarrow \infty$. (This is quite amazing!)

MATHEMATICS and COMPUTERS: Worksheet 3

Solution

SOLUTION TO PROBLEM 1:

We define the gap function, and then we test it — cf. equation (3.2)

```
> g:=k->ithprime(k+1)-ithprime(k):  
> seq(g(k),k=1..10);
```

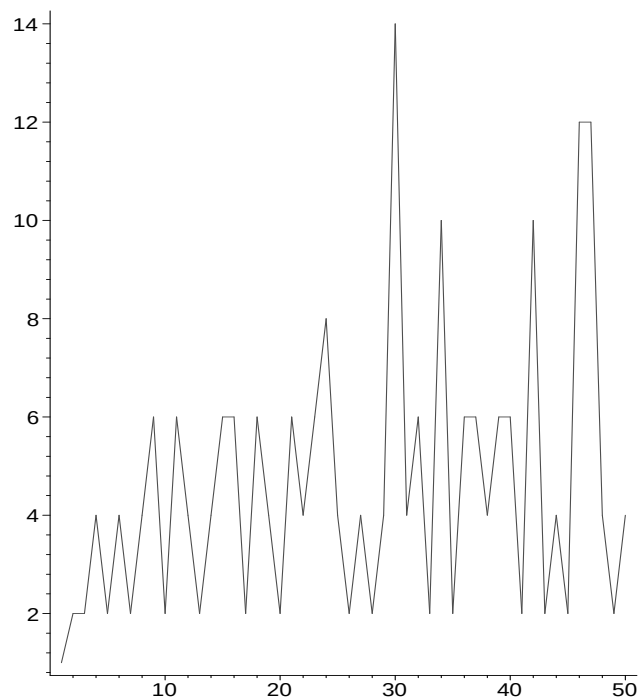
1, 2, 2, 4, 2, 4, 2, 4, 6, 2

(Recall that the colon `:` at the end of a command prevents the output from being shown.) Apart from g_1 , all other values are even — this is not a surprise. Now we collect the data for the plot, i.e., $[[1, g_1], [2, g_2], \dots]$, and we store them in a variable called **Gaps**

```
> Gaps:=[seq([k,g(k)],k=1..50)]:
```

Next we plot it

```
> plot(Gaps);
```

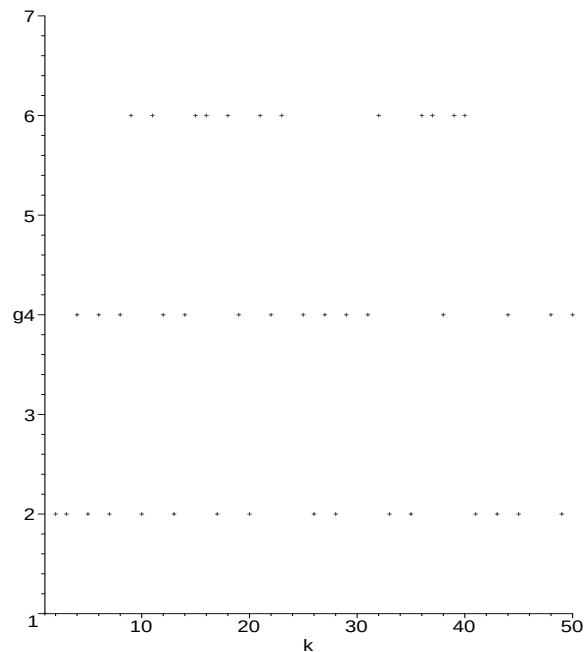


The largest gap (14) in this range occurs at $k = 30$. We verify this directly

```
> ithprime(30),ithprime(31);
```

Small gaps seems to occur quite often. To visualize them, we restrict the g -values, so as to hide from view the larger gaps, and we also plot the data as isolated points

```
> plot(Gaps,k=1..50,g=1..7,style=POINT);
```



Among the first 50 gaps, 16 are of size 2, 15 are of size 4, and 12 are of size 6. If you wish to push the experiment further, you will need more data, e.g., 500 or 1000 gaps. Where is now the largest gap? How frequent are the small gaps? If you analyze these data carefully, you should persuade yourself that

- (i) Occasionally, larger and larger gaps occur. It seems plausible to conjecture that arbitrarily large gaps will eventually be found (this is not too difficult to prove).
- (ii) Small gaps seem common. The famous *twin prime conjecture* states that the smallest gap ($g_k = 2$) occur infinitely often. So far nobody has been able to prove it.

Let us be daring: we conjecture that for any given positive even integer a , there are infinitely many gaps of size a . This is a sweeping generalization of the above conjectures.

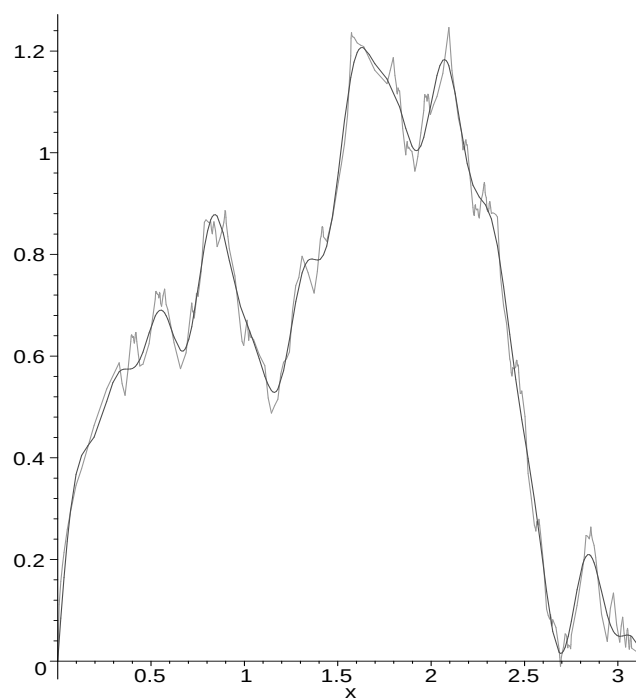
SOLUTION TO PROBLEM 2:

(a) To save typing, we define a function

```
> W:=n->add(sin(k^2*x)/k^2,k=1..n):  
> W(5);
```

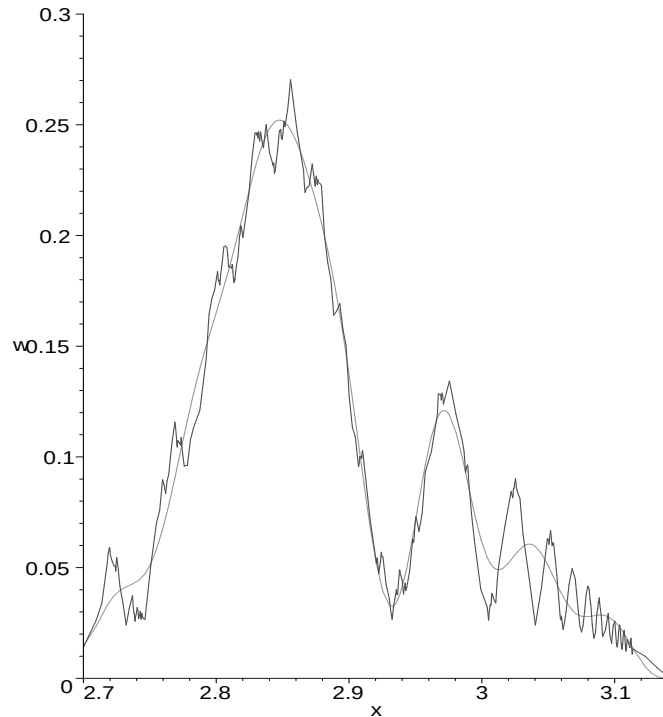
$$\sin(x) + \frac{1}{4} \sin(4x) + \frac{1}{9} \sin(9x) + \frac{1}{16} \sin(16x) + \frac{1}{25} \sin(25x)$$

```
> w5:=W(5):  
> w50:=W(50):  
> plot({w5,w50},x=0..Pi);
```



(Note how we displayed two functions in the same plot.) We see that w_{50} is not very dissimilar from w_5 , but a lot more rugged. To explore this phenomenon, we increase the values of n and magnify the plot

```
> w10:=W(10):  
> w100:=W(100):  
> plot({w10,w100},x=2.7..Pi,w=0..0.3);
```



We see a similar structure on a smaller scale. At this level of magnification, w_{10} appear smooth but w_{100} does not. We expect that also w_{100} will appear smooth if we magnify enough (try it!).

For large n , the function $w_n(x)$ seems to attain a limiting form w_∞ (it *converges*, in jargon). This makes sense; for instance, from the definition, we see that

$$w_{100}(x) = w_{99}(x) + \frac{1}{100^2} \sin(100^2 x)$$

and so the difference between $w_{99}(x)$ and $w_{100}(x)$ is tiny, because the extra term has amplitude $1/10000$. However, the extra term also features 10000 complete oscillations between 0 and 2π , which account for the extra small scale structure. By repeating the above reasoning ad infinitum, we conclude that w_∞ must feature ruggedness on every scale!

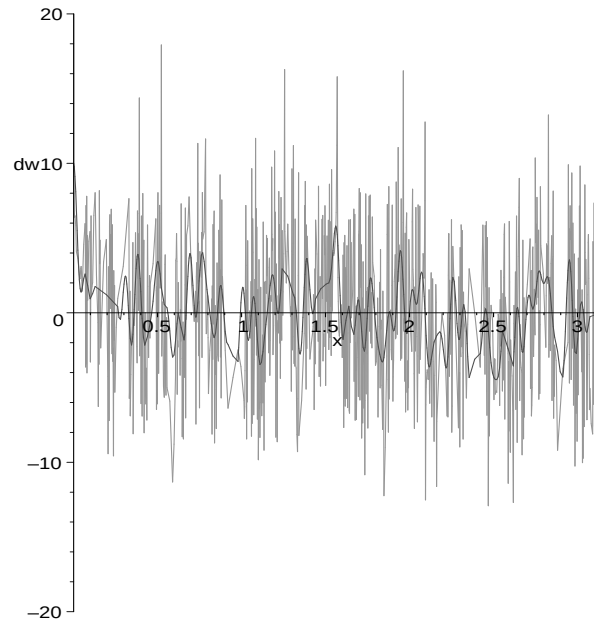
(b) We use `diff` to differentiate the functions $w_k(x)$ constructed above

```
> dw5:=diff(w5,x):
> dw10:=diff(w10,x):
> dw50:=diff(w50,x):
> dw5;
```

$$\cos(x) + \cos(4x) + \cos(9x) + \cos(16x) + \cos(25x)$$

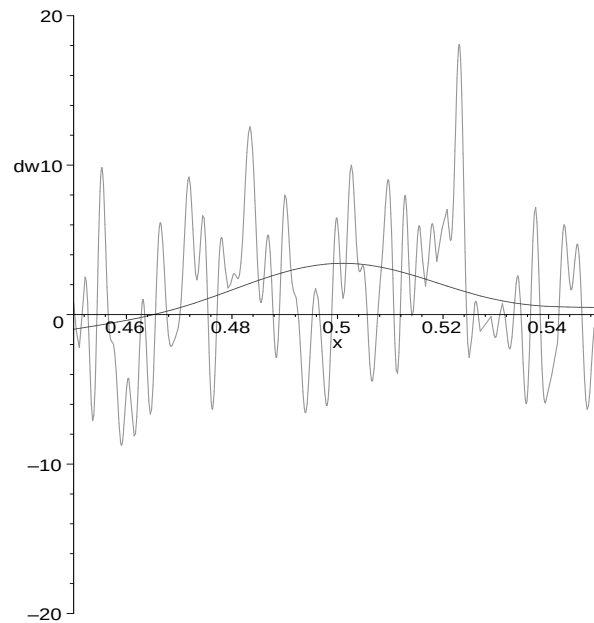
We should have guessed it: the denominators cancel out, so all terms now have equal amplitude, each displaying a rapidly increasing number of oscillations. Adding together several functions of comparable amplitude but with out of phase oscillations can cause a big mess, because some oscillations will add up, other will cancel out, unpredictably

```
> plot({dw10,dw50},x=0..Pi,w=-20..20);
```



Note that the amplitude of $dw_n(x)/dx$ now grows with n . A magnification shows what's going on

```
> plot({dw10,dw50},x=0.45..0.55,dw=-20..20);
```



The larger n , the larger and wilder are the fluctuations of the derivative of $w_n(x)$, which seems to be going all over the place, and have no limiting form. One cannot see how $w_\infty(x)$ could possibly have a derivative!

You have just met a function which is continuous (it has no jumps) but cannot be differentiated! This is the famous *Weierstrass function*. For more info, see

<http://www.mathworld.wolfram.com/WeierstrassFunction.html>