Anomalous transport in a model of Hamiltonian round-off

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Abstract

We study the propagation of round-off error near the periodic orbits of a linear area-preserving map —a planar rotation by a rational angle— which is discretized on a lattice in such a way as to retain invertibility. We consider the round-off error probability distribution as a function of time t, and we show that for each t this is an algebraic number, which can be calculated exactly. We prove that its kth moment increases asymptotically as $t^{1+(k-2)/\delta}\,G(t)$, where δ is the fractional dimension of a self-similar set related to periodic orbits of long-period, while G is a bounded function, periodic in the logarithm of t. This implies the diffusion coefficient displays bounded variations, while all higher order transport coefficients diverge, resulting in anomalous transport. This results contrasts with the case of irrational rotations, where the existence of a central limit theorem has been recently established [15].

1 Introduction

The extent to which round-off errors in computer representations of dynamical systems can be treated as noise, is an issue of considerable importance in theory and applications. In this paper we consider the transport process generated by non-dissipative (Hamiltonian) round-off errors near the periodic orbits of an area-preserving map. In a simple but nontrivial case —a planar rotation by a rational angle— we provide rigorous results characterizing the probabilistic nature of round-off fluctuations. Our main finding is that the round-off transport is anomalous, and therefore not consistent with a simple stochastic model. This result contrasts with the case of irrational rotations, where the existence of a diffusion process was first conjectured in a special case [14], and then proved in a quite general setting [15].

When an invertible dynamical system is discretized —by discretizing coordinates— it may happen that the discrete system be still invertible, acting as a permutation of the phase points.

Such discretizations are desirable for symplectic maps, because in this case the round-off may be regarded as a canonical perturbation of the continuum system. We then speak of *Hamiltonian round-off*, which features fluctuations without dissipation.

Because a symplectic map preserves volume, a Hamiltonian round-off scheme will typically entail a uniform distribution of the discrete points. This may be achieved, for instance, by replacing each coordinate axis by a set of equally spaced points: $\mathbf{R} \mapsto \frac{1}{n}\mathbf{Z}$, for some (large) integer n. By constrast, the floating-point discretization —used in countless numerical experiments— is highly non-uniform (the representable points cluster exponentially near zero), and leads invariably to irreversible dynamics, with long transients and short limit cycles [7].

It must be noted however, that uniformity alone does not suffice to ensure invertibility. Various Hamiltonian round-off schemes on a uniformly discretized phase space have been considered, in a variety of contexts [12, 5, 10, 9, 13, 7, 14, 11]. Discretizations which are uniform but not canonical have also been considered [4, 6].

We consider a model of Hamiltonian round-off for a linear symplectic map of the plane with rational winding number p/q [14, 11]

$$\Phi : \mathbf{Z}^2 \mapsto \mathbf{Z}^2 \qquad \Phi : (x,y) \mapsto (\lfloor \lambda x \rfloor - y,\, x) \qquad \lambda = 2\cos(2\pi p/q),\ p,q \in \mathbf{Z}. \tag{1}$$

The floor function $\lfloor\cdot\rfloor$ ensures that the lattice $\mathbf{Z}^2$ be Φ-invariant. One immediately verifies that the discrete mapping Φ is invertible. Replacing the floor function with the identity function, we recover the continuum map $\Psi : (x,y) \mapsto (\lambda x - y, x)$ with q-periodic dynamics, of which Φ is meant to be a discrete approximation. Here the discretization length is fixed, and the *semi-continuum limit* (vanishing discretization) corresponds to motions at infinity. Because the continuum dynamics is regular and well-understod, one can isolate the effect of round-off errors very clearly.

This model was studied in detail in the simplest non-trivial case ($q = 5$) [11], where we showed that all orbits are periodic and are organized in phase space into a quasi-periodic structure with asymptotic scaling properties (Figure 1). In the present work we turn our attention to the statistical side of the problem, seeking to characterize the time-evolution of round-off errors in term of a probability distribution. Given that the continuum map Ψ^q is the identity, how much does on average the discrete map Φ^q differ from the identity, after t iterations? To answer this question, we let $P(\mathbf{z}, t)$ be the probability that a randomly chosen point on $\mathbf{Z}^2$ be displaced by an integer vector $\mathbf{z} = (x, y)$ after t iterations of the mapping Φ^q. Our main concern is the study of the asymptotic (large t) properties of the probability P, for the rotation number $\nu = 1/5$.

The phase portrait of Φ^5 (Figure 1) shows the coexistence of fixed points (where the round-off errors cancel out) and orbits of increasing period. If the propagation of round-off errors were uniformly dominated by a finite set of orbital periods, there would be no diffusive behavior at all. It turns out however, that the finer the discretization (i.e., the further away from the origin), the longer the allowed periods. A subtle competition exists between the increase in the period of the orbits and the decrease of their density in the plane. Thus even if the probability distribution is *recurrent* —there exist times at which all but an arbitrarily small fraction of the probability is concentrated at the origin— nevertheless the moments of $P(\mathbf{z},t)$ grow according to a power-law.

Probabilities are calculated with respect to an invariant measure on $\mathbf{Z}^2$, which is the two-dimensional version of the *density* of a set of integers (obtained by counting the elements of the set which are smaller than n, and then letting n go to infinity —see section 2). This measure is natural for the problem under consideration (see also [15]), since it guarantees that the limit of small discretization is taken *before* the time is allowed to go to infinity. In this way one simulates the situation in which asymptotic data are extracted from segments of very long discrete periodic orbits —a common feature in computer experiments.

Specifically, let χ be a real-valued test function on $\mathbf{R}^2$ (a *random variable,* in the parlance of probability theory). We consider its expectation value $\langle\chi\rangle_t$ with respect to the density $P(\mathbf{z},t)$

$$\langle\chi\rangle_t \;=\; \sum_{\mathbf{z}\in\mathbf{Z}^2} \chi(\mathbf{z})\, P(\mathbf{z},t). \tag{2}$$

The quantity $\langle\chi\rangle_t$ is finite, since P has finite support f or all t.

The main result of this paper is summarized by the following theorem

Theorem. *Let $\chi:\mathbf{R}^2\mapsto\mathbf{R}$ be a homogeneous function of degree κ*

$$\chi(\rho\mathbf{z}) = \rho^\kappa\chi(\mathbf{z}), \qquad\qquad \rho>0,$$

where κ is a real number such that $\kappa>2-\delta$, with $\delta=\ln 2/\ln\tau$. In addition, let χ be continuous, piecewise differentiable, and with bounded gradient in a neighbourhood of the origin. Then, the expectation value of χ for the mapping (1), with $p=1$ and $q=5$ is given by

$$\langle\chi\rangle_t \;=\; t^\beta\left(G(t)+E(t)\right),$$

where

$$\beta=\beta(\kappa)=1+\frac{\kappa-2}{\delta},$$

G is a bounded function of t such that $G(4\,t)=G(t)$, for all t, while the error term $E(t)$ vanishes as $t\to\infty$.

In this context 'piecewise differentiable' means that differentiability may fail on a finite set of lines through the origin. The real parameter δ is the box-counting dimension of the "orbit at infinity" —the limit of rescaled orbits of increasing period— and the function G can be expressed in terms of the geometric properties of such orbit (see Figure 9).

The above theorem is a special case of more general results which will be presented in Section 5 as theorems 5.3 and 5.8. In the former we provide detailed information on the functions G and E, in the latter we relax the homogeneity condition on χ.

To see that the above result implies anomalous transport, we let $\chi(\mathbf{z}) = \|\mathbf{z}\|$ be a norm on $\mathbf{Z}^2$, which is homogeneous of degree one (such as the 1-norm $\|(x,y)\| = |x| + |y|$, which we shall use in this paper). We consider the exponential generating function of the moments $\mu_i(t) = \langle \chi^i \rangle_t$ of χ with respect to the density $P(\mathbf{z}, t)$ (cf. (2))

$$M_t(s) \;=\; \big\langle \exp(s \cdot \chi) \big\rangle_t \;=\; 1 + \sum_{i=1}^{\infty} \mu_i(t) \frac{s^i}{i!} \qquad t \geq 0.$$

Transport is characterized by the long-time behaviour of the *cumulants* $\kappa_i(t)$ of χ, generated by the logarithm of M_t (see [3], chapter 3)

$$K_t(s) \;=\; \ln(M_t(s)) \;=\; \sum_{i=1}^{\infty} \kappa_i(t) \frac{s^i}{i!} \qquad t \geq 0.$$

The i-th cumulant is a polynomial in the first i moments, with rational coefficients, e.g.,

$$\begin{aligned}
\kappa_1 &= \mu_1 \\
\kappa_2 &= \mu_2 - \mu_1^2 \\
\kappa_3 &= \mu_3 - 3\mu_1\mu_2 + \mu_1^3 \\
\kappa_4 &= \mu_4 - 3\mu_2^2 - 4\mu_1\mu_3 + 12\mu_2\mu_1^2 - 6\mu_1^4
\end{aligned}$$

The ith transport coefficients g_i is then defined as

$$g_i(t) \;=\; \lim_{t \to \infty} \frac{\kappa_i(t)}{t}.$$

The existence of the above limit depends on a delicate cancellation of the nonlinear time-dependence of the moments. The fractional exponent β appearing in the leading term of $\langle \chi \rangle_t$ prevents this cancellation from taking place, for all but (possibly) one value of i. But even in that case the limit does not exist —not even in an average sense— due to the logarithmic oscillations of $G(t)$.

Therefore the round-off transport for the model (1) is anomalous in a very strong sense. Specifically, the zeta function (see [1, 16, 2] and references therein)

$$\mathcal{G}(s) \;=\; \lim_{t\to\infty} \frac{1}{t}\, K_t(s) \tag{3}$$

whose lack of smoothness at the origin signals anomalous transport, is not even defined in our case. (The existence of a central limit theorem in the case of irrational rotations [15], raises the natural question of the structure of the transition from periodic to quasi-periodic round-off dynamics. This problem is currently under investigation.)

We summarize the content of this article. In section 2 we introduce the *local phase space* $\mathbf{L}^2$: a collection of algebraic integers dense in the two-dimensional torus, which is placed in bi-unique correpondence with $\mathbf{Z}^2$ [11]. We then consider an invariant probability measure on $\mathbf{Z}^2$ and we relate it to a natural invariant measure on $\mathbf{L}^2$, corresponding to areas of polygons.

In section 3 we specialize our analysis to the case of rotation number $\nu = 1/5$. We first introduce round-off error probability distribution $P(\mathbf{z}, t)$. Using a class of finite partitions of the local phase space, we then express P in terms of a *finite sum* over periodic orbits and establish its recurrent character. We show that for any $\mathbf{z}$ and t, the probability $P(\mathbf{z}, t)$ is an algebraic number of the form $(m + n\tau)/2$, where m and n are integers, and τ is the golden mean $(1 + \sqrt{5})/2$.

In section 4 we study the asymptotic scaling properties of periodic orbits in terms of a class of *deterministic walks* on the plane, constructed recursively from a finite collection of elementary steps. The strategy is to obtain as approximate representation of the round-off porbits in terms of a family of self-similar sets with simpler scaling properties, and then bound the relative error. The main theorems are proved in section 5.

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2 Invariant measure

We begin to review the local representation of the discrete phase space [11]. Further detail will be given in section 3, and the proofs will be found in [11].

Let $\tau = (1 + \sqrt{5})/2$, let $\lambda = \tau^{-1}$ and let $\mathbf{Z}[\tau]$ be the collection of numbers of the form $\zeta = m + n\tau$, with m and n integer in $\mathbf{Z}$. Then the mapping

$$\eta(x) = \tau\,(\lambda x - \lfloor \lambda x \rfloor) \tag{4}$$

defines a bi-unique correspondence between the integers $\mathbf{Z}$ and the elements of the set

$$\mathbf{L} \;=\; \mathbf{Z}[\tau] \cap [0, \tau).$$

We map $\mathbf{Z}^2$ into $\mathbf{L}^2$ via η. We find that the mapping Φ is conjugate to $\Phi^* = \eta \circ \Phi \circ \eta^{-1}$, where

$$\Phi^* : \mathbf{L}^2 \mapsto \mathbf{L}^2 \qquad \Phi^* : (\zeta_1, \zeta_2) \;\mapsto\; (-\tau\zeta_1 - \zeta_2,\, \zeta_1)\, (\mathrm{mod}\ \tau). \tag{5}$$

The set $\mathbf{L}^2$ will be called the *local* phase space, and the map Φ^* the local mapping.

The *discrete vector field* associated with the mapping Φ at time t is given by

$$\mathbf{v}_t : \mathbf{Z}^2 \mapsto \mathbf{Z}^2 \qquad \mathbf{v}_t : \mathbf{z} \mapsto \Phi^{5t}(\mathbf{z}) - \mathbf{z}. \tag{6}$$

If $\{\mathbf{z}_t\}$ is an orbit of Φ, then $\mathbf{z}_{5t} = \mathbf{z}_0 + \mathbf{v}_t(\mathbf{z}_0)$. For each t the level sets of $\mathbf{v}_t$ have local images consisting of a finite collection of disjoint convex polygons. This representation is the basis of the analysis developed in reference [11].

Let now A be any set of integers, and let

$$A(n) \;=\; \#\{x \in A, |x| < |n|\}.$$

The *density* $\mathcal{D}_1(A)$ of A in $\mathbf{Z}$ is defined as

$$\mathcal{D}_1(A) \;=\; \lim_{n\to\infty} \frac{A(n)}{2n}$$

provided that the limit exists. If A and B have a density, one verifies that

i) $0 \le \mathcal{D}_1(A) \le 1$ (with $\mathcal{D}_1(\mathbf{Z}) = 1$).
ii) $\mathcal{D}_1(\mathbf{Z} \setminus A) = 1 - \mathcal{D}_1(A)$.
iii) if $A \cap B = \emptyset$, then $\mathcal{D}_1(A \cup B) = \mathcal{D}_1(A) + \mathcal{D}_1(B)$.

Note that $A \cap B$ need not have a density, so that the sets with a density do not constitute the domain of a measure. However, we are only interested in certain quasi-periodic subsets of the integers, for which $\mathcal{D}_1$ is a measure which can be computed from a natural measure $\mathcal{M}_1$ in $\mathbf{L}$.

By an *interval* in $\mathbf{L}$ we mean the intersection of $\mathbf{L}$ with a real interval. Let $\mathcal{S}$ be the collection of all subsets of $\mathbf{L}$ that consist of finitely many intervals. The set $\mathcal{S}$ is closed under intersection and symmetric difference, and has $\mathbf{L}$ as identity (it is an algebra of sets). For $I \in \mathcal{S}$ (or I a countable collection of elements of S) we define

$$\mathcal{M}_1(I) = \frac{1}{\tau}\,|\bar{I}|,$$

where the bar denotes closure and $|\cdot|$ is the Lebesgue measure. Then $\mathcal{M}_1(\mathbf{L}) = 1$, $0 \leq \mathcal{M}_1(I) \leq 1$, and $\mathcal{M}_1$ is additive, giving a probability measure on $\mathcal{S}$. This measure is not countably additive, e.g., $\mathbf{L}$ has positive measure but consists of countably many disjoint sets of zero measure.

The mapping η^{-1} (see equation (4)) carries $\mathcal{M}_1$ into a measure in $\mathbf{Z}$, which we now show to be equal to $\mathcal{D}_1$.

Lemma 2.1. *Let I be a countable union of disjoint intervals in $\mathbf{L}$. Then $\mathcal{D}_1(\eta^{-1}(I)) \geq \mathcal{M}_1(I)$. Equality holds for any finite union of intervals.*

Proof: Let I consist of a single interval with endpoints ζ_l and ζ_r (which may or may not belong to I) and let $\alpha_{l,r} = \zeta_{l,r}/\tau$. Then, $m \in \eta^{-1}(I)$ precisely when the fractional part $\{m\lambda\}$ of $m\lambda$ belongs to $[\alpha_l, \alpha_r]$ (with possible exceptions at the endpoints). Our result then follows from Weil's theorem ([8], theorem 445), which states that for irrational λ the frequency of the points $\{m\lambda\}$, $m = 0, 1, 2, \ldots$ in an interval is equal to the length of that interval. (For negative m we use $1 - \lambda$ in place of λ.)

Let now I be a finite union of disjoint intervals: $I = \cup_{k=1}^{s} I_k$. Then $\mathcal{M}_1(I) = \sum_{k=1}^{s} \mathcal{M}_1(I_k)$, from additivity. Let $A_k = \eta^{-1}(I_k)$, and let $A = \eta^{-1}(I)$. Because η is a bijection, the sets A_k are disjoint, so that $A(n) = \sum_{k=1}^{s} A_k(n)$. Then

$$\begin{aligned} \mathcal{D}_1(\eta^{-1}(I)) &= \lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{s} A_k(n) = \sum_{k=1}^{s} \lim_{n\to\infty} \frac{A_k(n)}{2n} \\ &= \sum_{k=1}^{s} \mathcal{D}_1(A_k) = \sum_{k=1}^{s} \mathcal{M}_1(I_k) = \mathcal{M}_1(I). \end{aligned} \tag{7}$$

Let us define

$$f_s(n) = \sum_{k=1}^{s} \frac{A_k(n)}{2n} \qquad s \geq 1.$$

Then if I is a countable union of intervals, we have

$$\mathcal{M}_1(I) = \lim_{s\to\infty} \lim_{n\to\infty} f_s(n) \qquad \mathcal{D}_1(\eta^{-1}(I)) = \lim_{n\to\infty} \lim_{s\to\infty} f_s(n).$$

For any n, we have that $f_s(n) \leq f_{s+1}(n)$, whence $f_s(n) \leq \lim_{s\to\infty} f_s(n)$ so that

$$\mathcal{M}_1(I) \leq \mathcal{D}_1(\eta^{-1}(I)). \tag{8}$$

□

In the case of countable unions, the inequality (8) may be strict, since the sequence f_s need not be uniformly continuous at infinity (even when all I_k have positive Lebesgue measure).

The measure $\mathcal{M}_1$ induces the product measure $\mathcal{M}$ in $\mathbf{L}^2$. Likewise, in $\mathbf{Z}^2$ we replace $\mathcal{D}_1$ with the product density

$$\mathcal{D}(A) = \lim_{n\to\infty} \frac{1}{(2n)^2} \#\{(x,y) \in A, |x|, |y| < |n|\}. \tag{9}$$

We have the following two-dimensional analogue of lemma 2.1

Lemma 2.2. *Let Π be the union of countably many disjoint polygons in $\mathbf{L}^2$. Then $\mathcal{D}(\eta^{-1}(\Pi)) \geq \mathcal{M}(\Pi)$. Equality holds for any finite union of polygons.*

Proof: When Π is a direct product of two intervals (a rectangle with sides parallel to the coordinate axes) the result follows from lemma 2.1. It then extends to any elementary set (finite union of such disjoint rectangles), since $\mathcal{D}$ is additive. It is now sufficient to prove the lemma in the case in which Π is a triangle. For every $\epsilon > 0$ we can clearly choose two elementary sets $\Pi_{1,2}$ such that $\Pi_1 \subset \Pi \subset \Pi_2$ and

$$\mathcal{M}(\Pi_2) - \mathcal{M}(\Pi_1) < \epsilon.$$

Let $A_{1,2} = \eta^{-1}(\Pi_{1,2})$ and $A = \eta^{-1}(\Pi)$. Then $A_1 \subset A \subset A_2$, so that $\mathcal{D}(A)$ —if it exists— must satisfy the inequality

$$\mathcal{D}(A_2) \leq \mathcal{D}(A) \leq \mathcal{D}(A_1).$$

But $\mathcal{M}(\Pi_{1,2}) = \mathcal{D}(A_{1,2})$, so that

$$\mathcal{D}(A_2) - \mathcal{D}(A_1) < \epsilon.$$

Since ϵ was arbitrary, it follows that $\mathcal{M}(\Pi)$ and $\mathcal{D}(A)$ exist and are the same.

In the case of countable union of polygons, the proof is analogous to that of lemma 2.1. $\square$

When dealing with measure, we shall restrict ourselves to finite collections of polygons. Since the local dynamics is area-preserving, and transforms a finite collection of polygons into a finite collections of polygons, the measure $\mathcal{M}$ is ϕ^*-invariant.

3 Probability distribution

We are interested in the probability $P(\mathbf{z}, t)$ that a randomly selected point in $\mathbf{Z}^2$ be displaced by the vector $\mathbf{z}$ under $5t$ applications of the round-off mapping Φ. In terms of the vector field $\mathbf{v}_t$ and the density $\mathcal{D}$ (see equations (6) and (9), respectively), P is defined as

$$P(\mathbf{z}, t) = \mathcal{D}(\mathbf{v}_t^{-1}(\mathbf{z})).$$

It follows from the definition that $P(\mathbf{z},t)$ is supported on the (finite) set $\mathbf{v}_t(\mathbf{Z}^2)$. The local image of $\mathbf{v}_t^{-1}(\mathbf{z})$ consists of finitely many polygons [11]. Then, from lemma 2.2

$$P(\mathbf{z},t) \;=\; \mathcal{M}(\eta(\mathbf{v}_t^{-1}(\mathbf{z}))). \tag{10}$$

For $t=0$ the vector field vanishes identically, so that

$$P(\mathbf{z},0)=\begin{cases} 1, & \mathbf{z}=(0,0)\\ 0, & \mathbf{z}\neq(0,0).\end{cases} \tag{11}$$

The probability at $t=1$ is supported on $\mathbf{v}_1(\mathbf{Z}^2)=\{(0,0),\pm(1,0),\pm(0,1),\pm(1,1)\}$. The corresponding densities are computed from (10) to give

$$P(\mathbf{z},1)=\begin{cases} 14\tau-22=0.652\ldots & \mathbf{z}=(0,0)\\ (5-3\tau)/2=0.073\ldots & \mathbf{z}=\pm(1,0),\ \pm(0,1)\\ (13-8\tau)/2=0.028\ldots & \mathbf{z}=\pm(1,1)\end{cases} \tag{12}$$

From the first of equations (10) we note that almost two-thirds of the points of $\mathbf{Z}^2$ are fixed points of Φ^5, constituting an inert background which does not participate in the evolution of $P(\mathbf{z},t)$. Furthermore, all probabilities are in $\frac{1}{2}\mathbf{Z}[\tau]$, since they can be expressed as finite sums of areas of right triangles with vertices in $\mathbf{Z}[\tau]^2$.

At each successive time step, the density located at $\mathbf{z}$ either translates to a single neighbouring lattice site, or fragments into two neighbouring sites. The probability flows on closed loops, corresponding to periodic orbits. These loops intersect, so that the probability may flow to a given site from several neighbouring ones. A graphical representation of the probability distribution for $t=50$ is given in Figure 2.

3.1 Periodic orbits expansion and recurrence

We represent P in terms of periodic orbits. The sequence T_n of their minimal periods is given by [11]

$$T_n=\frac{1}{3}\left(5\cdot 2^n-2\cdot(-1)^n\right) \qquad n\geq 0. \tag{13}$$

The integer n will be called the *rank* of an orbit, and the local image of the orbits of rank n will be denoted by Π_n. The orbits are organized into two families corresponding to n even and odd, respectively. We define the *level* of an orbit as the integer $\lceil n/2\rceil$, which labels the orbit within the corresponding family. There is a single orbit of level 0 and two orbits of each positive level.

One finds that Π_0 (fixed points) consists of three pentagons, while for $n > 0$ Π_n consists of T_n congruent pentagons π_n of measure

$$\mathcal{M}(\pi_n) = \frac{\sqrt{5}}{2}\,\lambda^{2n+6} \qquad\qquad n \geq 1. \tag{14}$$

Thus the probability $P_n = \mathcal{M}(\Pi_n)$ that a point in $\mathbf{Z}^2$ be periodic of rank n is equal to $T_n \cdot \mathcal{M}(\pi_n)$. From (12), (13) and (14) we obtain

$$P_0 = 14\tau - 22 \qquad\qquad P_n = \frac{\sqrt{5}}{6}\,\lambda^{2n+6}\,(5\cdot 2^n - 2\cdot(-1)^n) \quad n \geq 1. \tag{15}$$

We express P as a sum over periodic orbits. Let

$$\Theta_t(\mathbf{z}) \;=\; \eta(v_t^{-1}(\mathbf{z})),$$

and let $Q_n(\mathbf{z},t)$ be the conditional probability that two points on the same orbit of rank n be separated by a vector $\mathbf{z}$ after t iterations of Φ^5. Then

$$Q_n(\mathbf{z},t) \;=\; \frac{\mathcal{M}(\Pi_n\,\cap\,\Theta_t(\mathbf{z}))}{\mathcal{M}(\Pi_n)}.$$

Because the Π_n are disjoint sets whose union is $\mathbf{L}^2$, from (10) we have

$$\begin{aligned}
P(\mathbf{z},t) \;&=\; \mathcal{M}(\Theta_t(\mathbf{z})) \;=\; \mathcal{M}\big(\cup_{n=0}^{\infty}\Pi_n\,\cap\,\Theta_t(\mathbf{z})\big) && (16)\\
&=\; \mathcal{M}\big(\cup_{n=0}^{\infty}\,(\Pi_n\cap\Theta_t(\mathbf{z})\big) \;\geq\; \sum_{n=0}^{\infty}\mathcal{M}(\Pi_n\cap\Theta_t(\mathbf{z})) && (17)\\
&=\; \sum_{n=0}^{\infty}\mathcal{M}(\Pi_n)\,\frac{\mathcal{M}(\Pi_n\,\cap\,\Theta_t(\mathbf{z}))}{\mathcal{M}(\Pi_n)} \;=\; \sum_{n=0}^{\infty} P_n\,Q_n(\mathbf{z},t) && (18)
\end{aligned}$$

The inequality results from the lack of countable additivity for $\mathcal{M}$. To prove equality, we sum both sides over $\mathbf{z}$. This sum is finite —$P(\mathbf{z},t)$ has finite support— and the infinite sum in (17) is absolutely convergent. Thus

$$\begin{aligned}
1 \;&=\; \sum_{\mathbf{z}} P(\mathbf{z},t) \;\geq\; \sum_{\mathbf{z}}\sum_{n=0}^{\infty}\mathcal{M}(\Pi_n\cap\Theta_t(\mathbf{z}))\\
&=\; \sum_{n=0}^{\infty}\sum_{\mathbf{z}}\mathcal{M}(\Pi_n\cap\Theta_t(\mathbf{z})) \;=\; \sum_{n=0}^{\infty}\mathcal{M}(\Pi_n\cap(\cup_{\mathbf{z}}\Theta_t(\mathbf{z})))\\
&=\; \sum_{n=0}^{\infty}\mathcal{M}(\Pi_n\cup\mathbf{L}^2) \;=\; \sum_{n=0}^{\infty}\mathcal{M}(\Pi_n) = 1. && (19)
\end{aligned}$$

Equality must then hold in (17) for each value of $\mathbf{z}$. So we have proved the following orbit decomposition formula

Lemma 3.1. *For all $\mathbf{z} \in \mathbf{Z}^2$ and $t \geq 0$ we have*

$$P(\mathbf{z}, t) \;=\; \sum_{n=0}^{\infty} P_n\, Q_n(\mathbf{z}, t). \tag{20}$$

The function $Q_n(\mathbf{z}, t)$ is periodic in t with minimal period T_n, and is supported at the origin precisely when t is a multiple of T_n. From (15) one finds that the density of all orbits of rank less than n tends to 1 as $n \to \infty$. It then follows from lemma 3.1 that $P(\mathbf{z}, t)$ is *recurrent*, that is, for any $\epsilon > 0$ the inequality

$$P(\mathbf{0}, t) > 1 - \epsilon$$

is satisfied by infinitely many values of t. A sequence $\{t_k\}$ of recurrence times is constructed, for instance, by taking t_k to be k times the least common multiple of $T_0, T_1, \ldots, T_r$. The larger r, the closer the recurrence. Note that recurrence for P does not imply recurrence for its moments, as we shall see.

3.2 Finite partitions

In this section we reduce (20) to a finite sum, with time-dependent coefficients (see Proposition 3.4). This simplification results from the fact that, for any finite t, the number of possible orbit segments is finite.

The derivation of the finite orbit decomposition formula is based on the properties of a recursive sequence of finite partitions of the local phase space

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \ldots \tag{21}$$

each a refinement of the previous one. Every element of $\mathcal{P}_l$ will contain a periodic pentagon of level not exceeding l.

The partition $\mathcal{P}_1$ consists of the connected components of the local image of the level sets of the vector field (see Figure 3 and [11]). Its invariant part comprises 3 pentagons (zero vector field), and its non-invariant part 10 triangles (non-zero vector field). We agree that the region corresponding to the vector field $\mathbf{z} = (-1, 0)$ —the *fundamental domain* $\mathcal{F}_1$— be subdivided into two triangles Δ_1 and Δ_2 by the *discontinuity line* $\zeta_1 = 1$.

It is useful to construct the non-invariant part of $\mathcal{P}_1$ dynamically. We let ∇_1 and ∇_2 be the domains located below and above, respectively of the line $\zeta_2 = (1 - \tau)\, \zeta_1 + 2$. The non-invariant part of $\mathcal{P}_1$ then consists of

$$\Delta_1, \quad \Delta_2, \qquad \nabla_1^{-k} = \Phi^{*-k}(\nabla_1), \;\; k = 1, 2, 3, \qquad \nabla_2^{-k} = \Phi^{*-k}(\nabla_2), \;\; k = 1, \ldots, 5. \tag{22}$$

One verifies that $\Phi^*(\Delta_1)$ and $\Phi^*(\Delta_2)$ are distinct elements of the 1-partition, with distinct values of the field. The same is true for $(\Phi^*)^{-1}(\nabla_1)$ and $(\Phi^*)^{-1}(\nabla_2)$.

To construct the refined partitions $\mathcal{P}_l$ for $l > 1$, we first define the similarity transformation with fixed point $(1, \tau)$ and linear scale factor λ^2

$$\Lambda : (\zeta_1, \zeta_2) \mapsto (\lambda^2 \zeta_1 + \lambda, \lambda^2 \zeta_2 + 1). \tag{23}$$

Next we define level l fundamental domain $\mathcal{F}_l = \Lambda^{l-1}\mathcal{F}_1$, which contains the triangles Δ_{2l}, Δ_{2l-1}, ∇_{2l} and ∇_{2l-1} defined similarly.

For every $l \geq 1$, the set $\mathcal{F}_l$ is recurrent, with return map Υ_l. Its dynamics is described by the following lemma (see Figure 4)

Lemma 3.1.

i) The return map Υ_l of $\mathcal{F}_l$ is given by

$$\Upsilon_l\, \zeta = \begin{cases} \Phi^{*T_{2l}}\, \zeta & \text{if } \zeta \in \Delta_{2l} \\ \Phi^{*T_{2l-1}}\, \zeta & \text{if } \zeta \in \Delta_{2l-1} \end{cases} \qquad l \geq 1, \tag{24}$$

where T_n is the period of the rank n orbits (13). Moreover

$$\Upsilon_l = \Lambda^{l-1} \circ \Upsilon_1 \circ \Lambda^{1-l} \qquad l > 1, \tag{25}$$

where Λ is given by (23).

ii) For $l \geq 1$ we have

$$\Upsilon_l(\Delta_{2l}) = \nabla_{2l}, \qquad \Upsilon_l(\Delta_{2l-1}) = \nabla_{2l-1} \tag{26}$$

iii) For each $n \geq 1$ the domains Δ_n contain a unique periodic pentagon π_n of rank n, with the property that

$$\frac{\mathcal{M}(\Delta_{2l-1})}{\mathcal{M}(\pi_{2l-1})} = \frac{\tau^3}{\sqrt{5}} = \rho; \qquad \frac{\mathcal{M}(\Delta_{2l})}{\mathcal{M}(\pi_{2l})} = \frac{\tau^4}{\sqrt{5}} = \sigma. \tag{27}$$

(The same is true for ∇_n, from ii).)

iv) The following mappings are continuous on their respective domains of definition

$$\begin{array}{lll} \Phi^{*k} & k = 1, \ldots, T_{2l} & \text{on} \quad \Delta_{2l} \\ \Phi^{*k} & k = 1, \ldots, T_{2l-1} & \text{on} \quad \Delta_{2l-1} \\ \Phi^{*-k} & k = 1, \ldots, T_{2l} & \text{on} \quad \nabla_{2l} \\ \Phi^{*-k} & k = 1, \ldots, T_{2l-1} & \text{on} \quad \nabla_{2l-1}. \end{array} \tag{28}$$

Proof: All the above statements can be proved by induction. We prove equations (28). Direct computation shows that they hold for $l = 1$. Assume them true for some $l \geq 1$. Using (25) one verifies directly that Φ^{*k} is continuous on $\Delta_{2\,l+2}$ for the values of k corresponding to the return times to $\mathcal{F}_l$ up to and including $T_{2(l+1)}$. Similarly, one checks the validity of the inclusion relations

$$\Upsilon_l^k(\Delta_{2\,l+2}) \subset \Delta_{2\,l}, \quad k=0,1,4 \qquad \Upsilon_l^k(\Delta_{2\,l+2}) \subset \Delta_{2\,l-1}, \quad k=2,3, \qquad \nabla_{2\,l+2} \subset \nabla_{2\,l-1}.$$

which together with the induction hypothesis establish the continuity of Φ^{*k} on $\Delta_{2\,l+2}$ at all intermediate times. This proves the first equation of part iv). The other domains are dealt with similarly. □

We now construct $\mathcal{P}_{l+1}$ for $l>1$. One verifies using (25) that the Υ_l-invariant pentagons π_{2l-1} and π_{2l}, and the 8 triangles

$$\Upsilon_l^{-k}(\Delta_{2\,l+2}),\ k=0,1,2 \qquad \Upsilon_l^{-k}(\Delta_{2\,l+1}),\ k=0,1,2 \qquad \Upsilon_l^{-k}(\nabla_{2\,l+2}),\ k=3,4$$

define a partition of $\mathcal{F}_l$, shown in Figure 5. This partition is first extended recursively to a finite partition of $\mathcal{F}_1$, and then, using $(\Phi^*)^{-1}$, to a partition $\mathcal{P}_{l+1}$ of the entire local phase space. One proves by induction that $\mathcal{P}_l \subset \mathcal{P}_{l+1}$, that is, that every element of $\mathcal{P}_l$ is the union of elements of $\mathcal{P}_{l+1}$.

Let now $P \in \mathcal{P}_l$. We consider the smallest time $u = u(P)$ (possibly infinite) for which the interior of $\Phi^{*u}(P)$ intersects the discontinuity line. Taking the minimum of $u(P)$ over all elements of the partition, we obtain a positive integer $u^* = u^*(\mathcal{P}_l)$ called the *critical time* of the partition of level l. Because of (21), we have that u^* is a non-decreasing function of l.

The following result characterizes these partitions and their critical times.

Lemma 3.2. *Let $l \geq 1$ and let $\mathcal{P}_l$ be the partition of level l. Then*
i) Each component of $\mathcal{P}_l$ contains a unique periodic pentagon of level not greater than l.
ii) The critical time $u^*(l)$ is given by

$$u^*(1)=1; \qquad u^*(l)=T_{2l-1} \qquad l>1. \tag{29}$$

Proof. From Lemma 3.1 iii), we have that for each $l \geq 1$, $\Delta_{2\,l}$ and $\nabla_{2\,l}$ contain a unique orbit of rank $2l$, while $\Delta_{2\,l-1}$ and $\nabla_{2\,l-1}$ contain a unique orbit of rank $2l-1$. From Lemma 3.1 iv) it follows that the same is true for all their pre-images in $\mathcal{P}_l$. For $l=1$ these pre-images exaust $\mathcal{P}_1$, except for the stationary part, which consists of three components of rank 0. This proves i) for $l=1$. For $l \geq 1$, we note that the only components of $\mathcal{P}_{l+1}$ in $\mathcal{F}_l$ that cannot be generated dynamically from the triangles Δ_i are two pentagons of rank $2l-1$ and $2l$. Part i) above now follows by induction.

As to part *ii*), we see from (22) that equation (29) holds for $l = 1$. To prove it for $l > 1$ we observe that equations (24) and (26) give the bounds

$$u(\Delta_{2l}) \leq T_{2l} \qquad u(\Delta_{2l-1}) \leq T_{2l-1}. \tag{30}$$

We now note that by construction, the mapping $\Phi^{*u(\Delta_{2l})+5}$ is discontinuous on Δ_{2l}, and similarly for Δ_{2l-1}. Lemma 3.1 *iv*) then shows that $u(\Delta_{2l}) > T_{2l} - 5$ and $u(\Delta_{2l-1}) > T_{2l-1} - 5$. Equality in (30) follows from equations (22), (26), and the fact that ∇_{2l} and ∇_{2l-1} are contained in ∇_2, for $l > 1$. An analogous continuity argument, together with the inclusions

$$\Upsilon_{l-1}^{-1}(\nabla_{2l}) \subset \Delta_{2l-2} \qquad \Upsilon_{l-1}^{-2}(\nabla_{2l}) \subset \Delta_{2l-3} \qquad \Upsilon_{l-1}^{-3}(\nabla_{2l}) \subset \Delta_{2l-3}$$

gives

$$u(\Upsilon_{l-1}^{-3}(\nabla_{2l})) = T_{2l-2} + 2\,T_{2l-3} = T_{2l-1}. \tag{31}$$

We now note that every element p of $\mathcal{P}_l$ either has $u(p) = \infty$, or is a pre-image of one of Δ_{2l}, Δ_{2l-1} or $\Upsilon_{l-1}^{-3}(\nabla_{2l})$, and so it has a larger value of u. Therefore, from (30) (with the equal signs) and (31) we see that the minimum of $u(p)$ is attained at $p = \Delta_{2l-1}$ and $p = \Upsilon_{l-1}^{-3}(\nabla_{2l})$, and the lemma follows. $\square$

The above lemma says that if ζ and ζ' belong to the same component of the partition of level l, then their respective kth iterates correspond to the same value of the vector field for all k smaller or equal to the critical time $u^*(l)$. Moreover, $u^*(l)$ is the largest time for which this property holds for an arbitrary component.

We say that an orbit is a *t-shadow* of another orbit if their respective kth iterates correspond to the same value of the vector field for $k = 0, \ldots, 5(t-1)$. One sees that t-shadowing means identification of orbit segments of length t, up to translation. It then follows from the definition of critical time that for all t such that $5\,(t-1) \leq u^*(l)$, any two orbits belonging to the same component of $\mathcal{P}_l$ are a t-shadow of one another.

We now invert the function u^* by defining

$$l^*(t) = \min_{l \geq 1} \left\{ l \,\middle|\, 5\,(t-1) \leq u^*(l) \right\} \qquad t \geq 1. \tag{32}$$

The integer $l^*(l)$ is called the *critical level* of the partition $\mathcal{P}_l$ at time t.

The following result characterizes shadowing by periodic orbits.

Lemma 3.3. *For any $t \geq 1$, every orbit of Φ^{*5} is a t-shadow of a periodic orbit of level not exceeding the critical level l^*. Moreover*

$$l^*(1) = 1; \qquad l^*(t) = \left\lceil \log_4 \left(6\,t - \frac{34}{5} \right) \right\rceil \qquad t > 1. \tag{33}$$

Proof. The first statement is a corollary of Lemma 3.2 *i*). We must show that (33) follows from the definition (32). For $t = 1$ we have from (29) that $L^*(1) = 1$, in agreement with (33). Let now $t > 1$. Using (29) and (13) we rewrite (32) as

$$l^*(t) = \min_{l \geq 1} \left\{ l \,\middle|\, 5\,(t-1) \leq \frac{1}{3} \left(\frac{5}{2} \cdot 4^l + 2 \right) \right\}$$

which gives (33). □

We can now truncate the sum (20) at $n = 2\,l^*(t)$, to obtain a finite orbit decomposition formula.

Proposition 3.4. *Let $t > 0$ and let $l^* = l^*(t)$ be given by (33). Then for any $l \geq l^*$ we have*

$$P(\mathbf{z}, t) = \sum_{n=0}^{2\,l-2} P_n\, Q_n(\mathbf{z}, t) \; + \; \frac{\tau^3}{\sqrt{5}} P_{2\,l-1}\, Q_{2\,l-1}(\mathbf{z}, t) \; + \; \frac{\tau^4}{\sqrt{5}} P_{2\,l}\, Q_{2\,l}(\mathbf{z}, t). \tag{34}$$

Proof. The terms in (20) corresponding to $n \geq 2\,l - 1$ give the contribution (see (16))

$$\mathcal{M}\big(\cup_{n=2\,l-1}^{\infty} \Pi_n \,\cap\, \Theta_t(\mathbf{z})\big).$$

The set $\cup_{n=2\,l-1}^{\infty} \Pi_n$ is equal to the union of all triangles Δ_i and ∇_i in $\mathcal{P}_l$, and their pre-images, each containing a unique pentagon π_i of level l, from Lemma 3.2 *i*). Because $l \geq l^*$, it follows from Lemma 3.3 that $\Delta_i \cap \Theta_t(\mathbf{z})$ is equal to Δ_i or to the empty set if and only if $\pi_i \cap \Theta_t(\mathbf{z})$ is equal to π_i or to the empty set, respectively. The proposition now follows from the relation of measures (27), and the fact that the pentagon of level l is unique in each triangular components of $\mathcal{P}_l$. □

Because the critical level depends on the choice of the partition, the questions arises as to determining its minimal value over all partitions. The critical time of a partition of level l satisfies the bound $u^*(l) \leq 5\,T_{2\,l}$, since that many iterates easily suffice to resolve the pentagon $\pi_{2\,l}$. This implies that the difference between the critical level of the partition $\mathcal{P}$ (33) and its minimal value is bounded, which suffices for our purposes.

We sketch the construction of minimal partitions. The coarsest partition is the entire local phase space $\mathbf{L}^2$ (with discontinuity set $\zeta_1 \cdot \zeta_2 = 0$). Its preimages define a nested sequence of partitions

$$\mathcal{P}^*{}_u \;=\; \Phi^{*1-u}(\mathbf{L}^2) \qquad u \geq 1, \tag{35}$$

whose critical time is equal to u. These partitions have the property that components and orbit segments are in one-to-one correspondence. We now extract from (35) the subsequence corresponding to the smallest u for which $\mathcal{P}^*{}_u$ is of level l, for each $l \geq 1$. This is the desired miminal sequence of partitions.

In the rest of this paper we shall take advantage of Lemma 3.3 and Proposition 3.4, and choose the following simple upper bound $L(t)$ for the critical level

$$L(t) = \lceil \log_4 (6\,t) \rceil \qquad t \geq 1. \tag{36}$$

4 Asymptotic properties of periodic orbits

The analysis of the long-time behavior of the probability distribution $P(\mathbf{z}, t)$ will be based, via the expansion (34), on the study of the asymptotic behavior of the infinite sequence of periodic orbits. We shall consider the scaling properties of these orbits, and relate their long-time behaviour to the structure of a certain fractal set —a pentagonal analogue of the von Koch snowflake.

4.1 Walks

We study the orbits of Φ^5 as walks on the plane, specified by sequences of elementary steps. The following definitions provide the desired framework.

A *displacement list* $D = (\mathbf{d}_0, \ldots, \mathbf{d}_{M-1})$ is a list of planar displacement vectors (the elementary steps), which may include repetitions, and which are coded by the integers $0, \ldots, M-1$. A *code sequence* $C = (c_0, \ldots, c_{N-1})$ is a sequence of N such displacement codes.

A *walk* is a pair $W = (C, D)$. Its kth *vertex* $\mathbf{r}_k = \mathbf{r}_k(W)$ is the position reached in k steps, starting from the origin, i.e.

$$\mathbf{r}_0(W) = (0, 0), \qquad \mathbf{r}_i(W) = \sum_{j=0}^{i-1} \mathbf{d}_{c_j}, \quad i = 1, \ldots, N. \tag{37}$$

The set of all walks with displacement list D will be denoted by $\mathcal{W}(D)$. Let now D and D' be two displacement lists, and let the *code map* ϕ be a function associating to each code in D a finite code sequence in D'. Then ϕ induces a mapping $\sigma : \mathcal{W}(D) \mapsto \mathcal{W}(D')$ assigning to the walk with code sequence C, the walk with code sequence C' where

$$C' = \phi(c_0) \frown \phi(c_1) \frown \ldots \frown \phi(c_{N-1})$$

and $\frown$ is the concatenation operator

$$(c_0, \ldots, c_k) \frown (c_{k+1}, \ldots, c_n) = (c_0, \ldots, c_n).$$

We call such mapping σ a *substitution.*

For illustration, let us translate the familiar recursive construction of the Koch snowflake into the language of walks. The displacement list contains six elementary steps $D = (\mathbf{d}_0, \ldots, \mathbf{d}_5)$, where

$$\mathbf{d}_k = \left(\cos\frac{2\pi k}{6}, \sin\frac{2\pi k}{6}\right) \qquad k = 0, \ldots, 5.$$

The successive elements of the sequence of Koch polygons are generated from the 3-step walk around an equilateral triangle $\big((0,2,4), D\big)$ by iterating the substitution induced by the code mapping

$$\phi(k) = (k, (k+5)_6, (k+1)_6, k), \qquad k = 0, \ldots, 5,$$

where the subscript m means that the corresponding integer must be reduced modulo m. The associated vertices $\mathbf{r}_k$ are the vertices of an $N_n = 3{\cdot}4^n$ sided polygon with diameter $d_n = d_0{\cdot}3^n, d_0 = 2/\sqrt{3}$, traversed counterclockwise. Note that

$$\frac{N_n}{3} = \left(\frac{d_n}{d_0}\right)^{\delta}, \qquad \delta = \frac{\ln 4}{\ln 3}.$$

In the limit $n \to \infty$, the polygons, rescaled by 3^{-n}, approach (in the sense of Hausdorff metric) a hexagonally symmetric fractal set with box-counting dimension δ.

The construction built on the triangular walk $((0,4,2), D)$ and code map

$$\phi(k) = (k, (k+1)_6, (k+5)_6, k), \qquad k = 0, \ldots, 5$$

yields exactly the same set of polygons, but traversed clockwise.

4.2 Orbits as walks

Our object of study are the large-period orbits of Φ^5, which we represent (up to cyclic permutation) as the walks V_n with displacement list given by the six non-zero values of the vector field

$$D_V = ((1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)).$$

We shall call V_n a *vector field walk.* It turns out however, that simpler renormalization with the same asymptotics can be achieved by considering in place of D_V the following displacement list D of ten step types:

$$D = ((-1,0),(0,-1),(1,0),(0,1),(0,0),(-1,1),(-1,-1),(0,-1),(1,0),(1,1)).$$

Note that one of the types is null, and there are types with identical displacements. The conversion between the D and D_V walks is realized via the substitution

$$\sigma_V : \mathcal{W}(D) \to \mathcal{W}(D_V) \qquad \sigma_V(W_n) = V_n \qquad n = 1,2,\ldots \tag{38}$$

generated by the code map

$$\phi_V(0) = (3),\ \phi_V(1) = (5),\ \phi_V(2) = (0),\ \phi_V(3) = (2),\ \phi_V(4) = \emptyset,$$

$$\phi_V(5) = (3,2),\ \phi_V(6) = (4),\ \phi_V(7) = (5),\ \phi_V(8) = (0),\ \phi_V(9) = (1).$$

(Here the symbol $\emptyset$ denotes the null sequence.)

We now construct the pre-image $W_n \in \mathcal{W}(D)$ of the vector field walk V_n of rank n. The lowest-rank orbits are, up to cyclic permutations of the code sequence

$$W_1 = ((0,1,2,3,4),D)\,, \qquad W_2 = ((5,9,8,7,6),D)\,.$$

The reader will verify that the vertices of the vector field walk $V_1 = \sigma_V(W_1)$ define a square traversed counterclockwise (a 4-cycle). Similarly, those of $V_2 = \sigma_V(W_2)$ define a hexagon traversed clockwise.

All higher-rank orbits (again, up to cyclic permutations) are obtained from W_1 and W_2 by iterated application of the substitution $\sigma : \mathcal{W}(D) \to \mathcal{W}(D)$ induced by

$$\begin{aligned} \phi(k) &= (k,(k+1)_5,5+(k+2)_5), \\ \phi(k+5) &= (k+5,5+(k+4)_5,(k+3)_5,(k+4)_5,k+5), \end{aligned} \qquad k = 0,\ldots,4. \tag{39}$$

Thus (see Figure 6)

$$W_{2l-1} = \sigma^{l-1}(W_1), \qquad W_{2l} = \sigma^{l-1}(W_2). \tag{40}$$

The number of steps $|W_n|$ of W_n is easily calculated as

$$|W_n| = \frac{5}{3}(2^n - (-1)^n) \qquad n \geq 1. \tag{41}$$

We wish to represent a walk $W_n = (C_n, D)$ as a linear superposition of three walks

$$W_{n,\gamma} = (C_n, D_\gamma) \qquad \gamma = 1, 2, 3, \tag{42}$$

each sharing the code sequence $C_n = (c_0, \ldots, c_{|W_n|-1})$ of W_n. We decompose the elementary steps $\mathbf{d}_k(D)$ as follows

$$\mathbf{d}_k(D) = \sum_{\gamma=1}^{3} \xi_\gamma \mathbf{d}_k(D_\gamma), \qquad k = 0, \ldots, 9. \tag{43}$$

The displacement lists of each component and the coefficients ξ_γ, are given in Table 2.

Type	D	(D_1)	(D_2)	(D_3)
0	$(-1,0)$	$(0,-1)$	$(-1,1)$	$(0,1)$
1	$(0,-1)$	$(1,0)$	$(-1,-1)$	$(-1,0)$
2	$(1,0)$	$(\lambda,1)$	$(0,-1)$	$(\tau,-1)$
3	$(0,1)$	$(-\lambda,\lambda)$	$(1,0)$	$(-\tau,\tau)$
4	$(0,0)$	$(-1,-\lambda)$	$(1,1)$	$(1,-\tau)$
5	$(-1,1)$	$(0,\tau)$	$(-1,-1)$	$(0,\lambda)$
6	$(-1,-1)$	$(-\tau,0)$	$(1,-1)$	$(-\lambda,0)$
7	$(0,-1)$	$(-1,-\tau)$	$(1,1)$	$(1,-\lambda)$
8	$(1,0)$	$(1,-1)$	$(0,1)$	$(-1,1)$
9	$(1,1)$	$(\tau,1)$	$(-1,0)$	$(\lambda,-1)$
ξ_γ	—	$\tau^2/\sqrt{5}$	1	$\lambda^2/\sqrt{5}$

Table 1: Data for the substitutions $\sigma_\gamma, \gamma = 1, 2, 3$.

The coefficients ξ_γ were chosen to satisfy the property that for any integer $m > 0$ the vertex

$$\mathbf{r}_m(W_{n,\gamma}) = \sum_{k=0}^{m-1} \mathbf{d}_{c_{k,n}}(D_\gamma) \qquad \gamma = 1, 2, 3$$

is an eigenvector of the substitution σ (see Section 4.2) with eigenvalues τ^2, 1 and λ^2, respectively. We omit the details of the solution of this eigenvalue problem, since it is easier explicitly to verify (43) for each of the ten listed types.

The usefulness of the above decomposition becomes apparent when we consider the asymptotic long-period scaling of the sequence of periodic orbits of Φ^5. In the next section we shall find that for $n \to \infty$, the position vectors of W_n, rescaled by a factor proportional to λ^n, tend to the corresponding position vectors of $W_{n,1}$ —the "asymptotic part" of W_n. In section 4.4, we shall demonstrate that the remainder, coming from $W_{n,2}$ and $W_{n,3}$, describes fluctuations whose relative magnitude tends exponentially to zero (of order λ^n).

4.3 Dominant terms

Postponing the question of convergence, we now establish an important property of the purported asymptotic orbits $W_{n,1}$. We shall show that its vertices are embedded in a $5\times 4^{\lceil n/2\rceil}$-sided polygon K_n (a pentagonal snowflake, see Figures 7, 8), with the rescaled sequence of polygons converging, in the limit $n\to\infty$, to a fractal set of dimension $\delta=\ln 2/\ln\tau$.

The set K_n originates from a displacement list with only 5 steps (that of $W_{n,1}$ has 10), it has simpler substitution rules than $W_{n,1}$, and a simpler expression for its period. Its resemblance to the hexagonal Koch construction described in section 4.1 is very close.

The construction of K_n is based on the displacement lists

$$\begin{aligned} D_- &= ((0,-\lambda^2),(\lambda^2,0),(\lambda^3,\lambda^2),(-\lambda^3,\lambda^3),(-\lambda^2,-\lambda^3)) \\ D_+ &= \big((0,\lambda),(-\lambda,0),(-\lambda^2,-\lambda),(\lambda^2,-\lambda^2),(\lambda,\lambda^2)\big)\,. \end{aligned}$$

(Note that $D_+=-\lambda\,D_-$.) Defining the substitutions $\sigma_\pm:\mathcal{W}(D_\pm)\to\mathcal{W}(D_\pm)$ from the code maps

$$\begin{aligned} \phi_-(k) &= (k,(k+1)_5,(k+4)_5,k), \\ \phi_+(k) &= (k,(k+4)_5,(k+1)_5,k). \end{aligned} \qquad k=0,\dots,4, \tag{44}$$

we construct the snowflakes K_n as the walks

$$K_{2l-1}=\sigma_-^l(K_{0-}) \qquad K_{2l}=\sigma_+^l(K_{0+}), \qquad l=1,2,\dots$$

with

$$K_{0-}=((0,1,2,3,4),D_-) \qquad K_{0+}=((0,4,3,2,1),D_+).$$

To establish that the vertices of $W_{n,1}$ are a subset of the vertices of a walk in $\mathcal{W}(D_-)$, if n is odd, or $\mathcal{W}(D_+)$, if n is even, we first introduce substitutions

$$\sigma_{1\pm}:\mathcal{W}(D)\to\mathcal{W}(D_\pm) \tag{45}$$

with code maps, for $k=0,\dots 4$

$$\begin{aligned} \phi_{1-}(k) &= (k,(k+1)_5,(k+4)_5,k), \\ \phi_{1-}(k+5) &= ((k+2)_5,(k+3)_5,(k+1)_5,(k+2)_5,(k+3)_5,(k+4)_5,(k+2)_5,(k+3)_5), \\ \phi_{1+}(k) &= ((k+3)_5,(k+2)_5), \\ \phi_{1+}(k+5) &= (k,(k+4)_5,(k+1)_5,k). \end{aligned}$$

It then suffices to verify that for $k = 0, \ldots, 4$ the images of the one-step walk $((k), D)$ under the substitutions σ_1, σ_{1+} and σ_{1-} are identical. For example, for $k = 0$ we have

$$\begin{aligned}
\sigma_1((0), D) &= \mathbf{d}_0(D_1) = (0, -1) \\
&= \mathbf{d}_3(D_{1+}) + \mathbf{d}_2(D_{1+}) = \sigma_{1+}((0), D) \\
&= \mathbf{d}_0(D_{1-}) + \mathbf{d}_1(D_{1-}) + \mathbf{d}_4(D_{1-}) + \mathbf{d}_0(D_{1-}) = \sigma_{1-}((0), D).
\end{aligned}$$

We now are in a position to state and prove the main result of this subsection (cf. Figure 8).

Lemma 4.1. *The vertices of $W_{n,1}$ form a subsequence of the sequence of vertices of K_n.*

Proof. It is easy to verify that $\sigma_{1-}(W_1) = K_1$, and $\sigma_{1+}(W_2) = K_2$. To establish the result by induction, it is sufficient to prove the commutation relation (valid on all of $\mathcal{W}(D)$)

$$\sigma_{1\pm}\sigma = \sigma_{\pm}\sigma_{1\pm}.$$

To prove this it is sufficient to prove the commutation of the respective code maps, since the rest is trivial. Moreover, it is enough to verify the code maps for $k = 0$ and $k = 5$, since the other relations are obtained from these by simultaneous cyclic permutations of $0, 1, 2, 3, 4$ and $5, 6, 7, 8, 9$. The explicit verification of the k=0,5 code map commutation relations is an easy exercise. $\square$

4.4 Fluctuations

The following two lemmas provide estimates for the magnitudes of the fluctuating components $W_{n,2}$ and $W_{n,3}$ of W_n.

Lemma 4.2. *$W_{n,2}$ is a walk on the pentagon π with vertices*

$$\mathcal{V}[\pi] = \{(0,0), (-1,1), (-2,0), (-2,-1), (-1,-1)\},$$

and so its vertices are bounded in norm by 3.

Proof. We see immediately from Table 2 that the step types in D_2 labeled by $0, 1, 2, 3, 4$ are the displacements between consecutive vertices of π (counterclockwise), while those labeled by $5, 9, 8, 7, 6$ are the analogous displacements for a clockwise circuit. Moreover, the initial step of W_n is always a step of type 0 for n odd and of type 5 for n even, advancing from $(0,0)$ to $(-1,1)$ for odd n, and from $(0,0)$ to $(-1,-1)$ for even n, in accord with the statement of the lemma. To complete the proof, we must show that if the pth step connects $\mathbf{r}_p \in \mathcal{V}[\pi]$ to $\mathbf{r}_{p+1} \in \mathcal{V}[\pi]$, then step

$(p+1)$ connects $\mathbf{r}_{p+1}$ to $\mathbf{r}_{p+2} \in \mathcal{V}[\pi]$. It is not difficult to establish that the code for step $(p+1)$ is just that for step p transformed by either $f_0(p)$ or $f_1(p)$, where

$$\begin{array}{ll} f_0(k) = (k+1)_5 & f_0(k+5) = 5 + (k+4)_5 \\ f_1(k) = 5 + (k+1)_5 & f_1(k+5) = (k+4)_5 \end{array} \qquad k = 0, \ldots, 4. \tag{46}$$

Consulting Table 2, we see that in the first case one continues around π an additional step in the same sense, whereas in the second case, one reverses direction and returns to the last vertex visited. In both cases, the assertion of the lemma is verified. $\square$

Lemma 4.3. *The vertices of the walk $W_{n,3}$ are uniformly bounded in norm by* 1.

Proof. The treatment here is based on a Koch-type construction, analogous to that developed in section 4.3 (cf. Figure 7). This time we have the displacement lists

$$\tilde{D}_- = ((0,1), (-1,0), (\tau,-1), (-\tau,\tau), (1,-\tau)), \tag{47}$$

$$\tilde{D}_+ = \big((0,\tau), (-\tau,0), (\tau^2,-\tau), (-\tau^2,\tau^2), (\tau,-\tau^2)\big). \tag{48}$$

The substitutions $\tilde{\sigma}_\pm : \mathcal{W}(\tilde{D}_\pm) \to \mathcal{W}(\tilde{D}_\pm)$ are based on the same code map as before, i.e., $\tilde{\phi}_\pm = \phi_\pm$ (see (44)). It is straightforward to show that the vertices of $W_{n,3}$ are a subset of the vertices of the walks obtained by iterated application of $\tilde{\sigma}_\pm$ to the walks

$$\tilde{K}_{0-} = ((0,1,2,3,4), \tilde{D}_-), \qquad \tilde{K}_{0+} = ((0,4,3,2,1,\tilde{D}_+).$$

Moreover it is not difficult (though a bit tedious) to establish that for any $l \geq 1$ all points $\tilde{K}_{2l-1} = \tilde{\sigma}_-^{l-1}\tilde{K}_{0-}$ lie in the square $[-\tau,\tau]^2$, while those of $\tilde{K}_{2l} = \tilde{\sigma}_+^l \tilde{K}_{0+}$ lie in $[-\tau^2,\tau^2]^2$. The proof is recursive: one shows that all points of $\tilde{K}_{2l-1}$ lie in the square of side d_{l-1} centered at $\mathbf{u}_{l-1}$, where

$$d_0 = \tau, \quad d_j = \lambda^2 d_{j-1} + 2,$$

$$\mathbf{u}_0 = (-\lambda^2/2, \tau/2), \quad \mathbf{u}_l = \lambda^2 \mathbf{u}_{l-1},$$

while all points of $\tilde{K}_{2l}$ lie in the square of side d_l and center $\mathbf{u}_l$, where

$$d_0 = 1, \quad d_l = \lambda^2 d_{l-1} + 2\tau,$$

$$\mathbf{u}_0 = (\lambda^3/2, \lambda^3/2), \quad \mathbf{u}_l = \lambda^2 \mathbf{u}_{l-1}.$$

The norm bound follows from the geometrical restrictions on $\tilde{K}_n$, once the appropriate scale factor $\lambda^2/\sqrt{5}$ is included. $\square$

4.5 Estimates

We are now in a position to isolate the asymptotic part of each periodic orbit, with rigorous bounds for the remainder terms. The main result, contained in Lemma 4.4, is that for asymptotically large time t, the position vectors

$$\mathbf{z}_{n,t} = \mathbf{r}_t(V_n) \tag{49}$$

approach the snowflake vertices $\mathbf{r}_{st}(K_n)$, suitably rescaled, where

$$s = s(n) = \begin{cases} 6 & n \text{ odd} \\ 3 & n \text{ even} \end{cases} \tag{50}$$

with remainder terms which are relatively small for n near the critical value $2L(t)$ (cf. (36)).

Since in the present context we do not want to restrict t to be less than the period $|V_n|$, we introduce the *extended walk* $W(i)$ consisting of $\lfloor i/|W| \rfloor$ copies of a walk W followed by a sub-walk of W of length $i \bmod |W|$. For a given positive n and i, we consider the extended walk $W_n(i)$ and associate with it extended walks $V_n(t)$ and $K_n(j)$, with t and j uniquely determined by the conditions

$$\mathbf{r}_t(V_n) = \mathbf{r}_i(W_n), \qquad \mathbf{r}_j(K_n) = \mathbf{r}_i(W_{n,1}). \tag{51}$$

This association is made possible by the existence of the substitutions σ_V and $\sigma_{1\pm}$ (cf. (38) and (45)). The numbers of steps of the associated extended walks are of course not the same, since $|\sigma_V((k), D)|$ and $|\sigma_{1\pm}((k), D)|$ depend nontrivially on the step type k. The mismatch in step counts st, si, and j is one of the main sources of nonleading asymptotic behavior.

Now we turn to the statement and proof of our main estimate.

Lemma 4.4. *For all positive t, n, we have*

$$\| \mathbf{z}_{n,t} - \frac{\tau^2}{\sqrt{5}} \mathbf{r}_{st}(K_n) \| \le \epsilon_1 \, (2^{-n}\, t)^{1/\delta} + \epsilon_2 \, \lceil \frac{n}{2} \rceil^{1/\delta} + \epsilon_3, \tag{52}$$

where

$$\epsilon_1 = \epsilon_0 \left(\frac{4}{3}\right)^{1/\delta}, \qquad \epsilon_2 = \epsilon_0 \; (13)^{1/\delta}, \qquad \epsilon_3 = \epsilon_0 \; (26)^{1/\delta} + 5 + 6\tau/\sqrt{5},$$

with

$$\epsilon_0 = 4\tau^4/\sqrt{5}.$$

Proof. For arbitrary positive integer i, consider the extended walk $W_n(i)$, together with the associated extended walks $V_n(t)$ and $K_n(j)$, with t and j uniquely determined by equations (51). Not all t values are accessible this way (in particular, the appearance of a "5" in the code sequence of W_n corresponds to a jump in t by 2 vector-field steps). We shall postpone treatment of these exceptional t values until the end of the proof.

¿From Lemmas 4.1, 4.2, and 4.5, we have

$$\| \mathbf{z}_{n,t} - \frac{\tau^2}{\sqrt{5}} \mathbf{r}_j(K_n) \| \leq 4, \tag{53}$$

so that the proof reduces to finding a suitable bound on

$$\eta = \mathbf{r}_j(K_n) - \mathbf{r}_{st}(K_n).$$

To replace $\mathbf{r}_j$ by $\mathbf{r}_{st}$ in the estimate on $\| \eta \|$, we apply Lemma 4.5 below. This yields

$$\| \mathbf{r}_j(K_n) - \mathbf{r}_{st}(K_n) \| \leq 4\tau^2 \, |j - st|^{1/\delta} \tag{54}$$

By accounting carefully for the mismatch between step counts, one can derive (we omit details)

$$|j - st| \leq s \left(\frac{2t + 25 \cdot 2^n}{3T_n} + 3 \right).$$

We insert this into (54), and apply the inequality $T_n \geq 3 \cdot 2^{n-1}$, and add on the contribution 53 to get, for all non-exceptional t,

$$\| \mathbf{z}_{n,t} - \frac{\tau^2}{\sqrt{5}} \mathbf{r}_{st}(K_n) \| \leq \epsilon_0 \, \left(\frac{4}{3} \, 2^{-n} \, t + 13 \lceil \frac{n}{2} \rceil + 26 \right)^{1/\delta} + 4. \tag{55}$$

To handle the exceptional time values, we use the fact that the vertex at time t lies just one lattice site below the vertex at time $t+1$, so that

$$\| \mathbf{z}_{n,t} - \mathbf{z}_{n,t+1} \| = 1,$$

and moreover,

$$\| \mathbf{r}_{st+s}(K_n) - \mathbf{r}_{st}(K_n) \| \leq 6\lambda$$

Adding these contributions to that of (55), and exploiting the inequality

$$|X + Y|^\nu \leq |X|^\nu + |Y|^\nu, \qquad 0 < \nu < 1,$$

we end up with the estimate (52). □

We conclude this section by presenting some bounds on the norms of the displacement vectors connecting vertices of K_n.

Lemma 4.5. *For all i, j such that $0 \le i, j < 5 \times 4^l$, and $l = \lceil n/2 \rceil$, we have*

$$\|\mathbf{r}_i(K_n) - \mathbf{r}_j(K_n)\| \le 2\tau^n. \tag{56}$$

If, in addition, l' is the smallest non-negative integer such that $|i - j| \le 4^{l'}$, then

$$\| \mathbf{r}_i(K_n) - \mathbf{r}_j(K_n) \| \le \begin{cases} 4\tau^{2l'-1} & n = 2l - 1 \\ 4\tau^{2l'} & n = 2l. \end{cases} \tag{57}$$

Proof. The generalized Koch construction of K_n is such that all of its vertices, rescaled by a factor λ^{n-1}, lie on or within the convex pentagon connecting $\{(0,0),(0,1),(1,\tau),(\tau,1),(1,0)\}$. This bounding pentagon is contained in the closed square $[0,\tau]^2$. Since the norm of the difference vector between any two points in or on the square does not exceed 2τ, the estimate (56) follows.

To prove (57) we note that the walk between $\mathbf{r}_i(K_n)$ and $\mathbf{r}_j(K_n)$ is confined to at most two adjacent sub-walks $\sigma_-^{l'}((k), D_-)$ or $\sigma_+^{l'}((k), D_+)$. Each of these is a sub-walk of $K_{2l'-1}$ or $K_{2l'}$, respectively, with a bound on displacements equal to $2\tau^{2l'-1}$ (resp. $2\tau^{2l'}$) by (56), and the estimate (57) follows. □

5 Main theorems

The asymptotic behavior of the probability distribution $P(\mathbf{z}, t)$ will be described in terms of the expectation value of a random variable χ with respect to P

$$\langle\chi\rangle_t = \sum_{\mathbf{z}\in\mathbf{Z}^2} P(\mathbf{z}, t)\chi(\mathbf{z}), \tag{58}$$

Since for fixed t the support of $P(\mathbf{z}, t)$ is finite, the sum in (58) is well-defined.

If $\langle\chi\rangle_t$ is to have an identifiable "tail" for $t \to \infty$, then χ must be a suitably well-behaved function on the plane. What turns out to be crucial is the property that it be possible to extract a leading asymptotic part χ_+ which is homogeneous of some positive degree κ such that $|\chi - \chi_+|$ increases more slowly than $\|\mathbf{z}\|^\kappa$. The full set of relevant constraints is stated in the definitions below.

A *homogeneous random variable χ of degree κ* is a homogeneous real function on $\mathbf{R}^2$ of degree $\kappa \in \mathbf{R}^+$

$$\chi(\rho\mathbf{z}) = \rho^\kappa \chi(\mathbf{z})$$

which is continuous, differentiable except —possibly— on a finite set of lines through the origin, and whose gradient (wherever defined) is bounded in some neighbourhood of the origin.

A *random variable of asymptotic degree* κ is a real function χ on $\mathbf{R}^2$ of the form

$$\chi = \chi_+ + \chi_- + \chi_0, \tag{59}$$

where the *leading asymptotic part* χ_+ is a homogeneous random variable of degree κ, the *nonleading asymptotic part* χ_- is continuous and bounded in absolute value by a homogeneous random variable of degree smaller than κ, and where χ_0 is uniformly bounded.

Note that the set of homogeneous random variables includes the functions $x^\nu y^{\kappa-\nu}$ (for $\kappa \geq \nu \geq 0$) and $\|\mathbf{z}\|^\kappa = (|x| + |y|)^\kappa$ frequently used to construct moments of a probability distribution on the plane. We shall demonstrate below that if χ is a random variable of asymptotic degree greater than $2 - \delta$ satisfying the constraints of the definition, then for $t \to \infty$, the expectation value $\langle \chi \rangle_t$ conforms asymptotically to a modulated power law, with rigorous bounds on the nonleading contribution.

To prepare for the statement of the main result, we first note that a continuous function which is homogeneous of positive degree, is necessarily bounded in a neighbourhood of the origin. It then follows that a homogeneous random variable χ satisfies the bounds

$$|\chi(\mathbf{z})| \leq B_0 \, \|\mathbf{z}\|^\kappa \tag{60}$$

$$\| \operatorname{grad}(\chi(\mathbf{z})) \| \leq B_1 \, \|\mathbf{z}\|^{\kappa-1} \tag{61}$$

for some positive constants B_0 and B_1, which can be taken as the supremum of $|\chi|$ and $\|\operatorname{grad}\chi\|$ on the unit disc. By the same token, we have for the nonleading component of χ

$$|\chi_-| \leq B_2 \, \|\mathbf{z}\|^{\kappa'} \tag{62}$$

for some $B_2 \geq 0$ and $\kappa' < \kappa$.

We shall be applying these bounds below in situations where κ is fixed and $\|\mathbf{z}\|$ is uniformly bounded, so that $|\chi|$ and $\|\operatorname{grad}\chi\|$ are also uniformly bounded. The gradient bound will allow us to bound as well the difference

$$\chi(\mathbf{z}_1) - \chi(\mathbf{z}_2) = \int_{\mathbf{z}_1}^{\mathbf{z}_2} \operatorname{grad}\chi \cdot d\mathbf{z},$$

where the integral is taken over the straight line segment connecting $\mathbf{z}_1$ and $\mathbf{z}_2$. Our definition of an homogeneous random variable implies that if both $\mathbf{z}_1$ and $\mathbf{z}_2$ belong to a discontinuity line of $\operatorname{grad}\chi$, the integral can still be computed approaching such line from either side. Otherwise the integrand will feature at most finitely many discontinuities along the path of integration.

Next we introduce the sets $\mathcal{K}_n$ of rescaled position vectors $\mathbf{k}_{n,i}$ for $n = 0, 1, \ldots$ and $i = 0, 1, 2, 3, 4$

$$
\begin{array}{rcl}
\mathbf{k}_{0,i} & = & \lambda^{-1}\mathbf{r}_i(K_{0+}) \ = \ \lambda^{-2}\mathbf{r}_i(K_{0-}) \\
\mathbf{k}_{2l-1,i} & = & \lambda^{2l-2}\mathbf{r}_i(K_{2l-1}) + (0,1) \qquad l = 1, 2, \ldots \\
\mathbf{k}_{2l,i} & = & \lambda^{2l-1}\mathbf{r}_i(K_{2l}) \qquad\qquad l = 1, 2, \ldots \\
\mathcal{K}_n & = & \{\mathbf{k}_{n,0}, \ldots, \mathbf{k}_{n,|K_n|}\}.
\end{array}
$$

We note the pairing of the snowflakes according to level $l = \lceil \frac{n}{2} \rceil$. The position vectors $\mathbf{k}_{2l-1,i}$ differ from the $\mathbf{k}_{2l,i}$ only by a time reversal $i \to |K_n| - i$ followed by a cyclic permutation of indices. Thus

$$\mathcal{K}_{2l-1} = \mathcal{K}_{2l}.$$

Moreover, the sets $\mathcal{K}_n$ are nested with a countable limit set $\mathcal{K}$:

$$\mathcal{K}_0 \subset \cdots \subset \mathcal{K}_n \subset \mathcal{K}_{n+1} \subset \cdots \subset \mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n. \tag{63}$$

Consider now the operation of permuting cyclically the vertices of K_n via the index shift $i \to i + j$. We define the corresponding angular displacement by

$$\alpha_{n,j} = j/|K_n|,$$

and introduce the expectation value of a continuous function χ of the j-step displacement vectors,

$$\mu_n(\alpha_{n,j}) = \frac{1}{|K_n|} \sum_{i=0}^{|K_n|-1} \chi(k_{n,i+\alpha_{n,j}|K_n|} - k_{n,i}). \tag{64}$$

Again their is pairing according to level, with

$$\mu_{2l-1}(\alpha_{2l-1,j}) = \mu_{2l}(-\alpha_{2l,j}),$$

where the negative sign takes into account the fact that the vertices of the two level-l walks are traversed in opposite senses. For all $n \geq 0$, periodicity implies the relations

$$\mu_n(\alpha_{n,j}) = \mu_n(\alpha_{n,j+|K_n|}) = \mu_n(\alpha_{n,j} + 1).$$

The nesting property (63) and the scaling behavior discussed in Section 4 allow us to prove the existence of the $n \to \infty$ limit of μ_n for fixed argument.

Lemma 5.1. *Let χ be a homogeneous random variable of degree κ, and let $\mu_n(\alpha)$ be the expectation value of χ defined by (64). Further, let α be a rational number of the form*

$$\alpha = \frac{j}{5 \cdot 4^{\lceil n/2 \rceil}}, \quad j \in \mathbf{Z}.$$

Then the limit

$$\lim_{m \to \infty} \mu_{n+2m}(\alpha) = \mu((-1)^n \alpha) \tag{65}$$

exists, and moreover

$$|\mu_n(\alpha) - \mu((-1)^n\alpha)| \leq 4\, B_1 (2\tau)^\kappa \lambda^n, \tag{66}$$

where B_1 is the coefficient of the gradient bound (61).

Proof. We start with the embedding property

$$\mathbf{k}_{n,i} = \mathbf{k}_{n+2m, 4^m i}, \tag{67}$$

from which it follows

$$\mu_n(\alpha) = \frac{1}{|K_n|} \sum_{j=0}^{|K_n|-1} \chi(\mathbf{k}_{n+2m, 4^m(i+\alpha|K_n|)} - k_{n+2m,4^m i}).$$

Using Lemma 4.8, we can relate the righthand side of this equation to $\mu_{n+2m}(\alpha)$, bounding the difference with (61):

$$|\mu_n(\alpha) - \mu_{n+2m}(\alpha)| \leq 4\, B_1 (2\tau)^\kappa \lambda^n.$$

Thus, as $m \to \infty$, we have

$$|\mu_{n+2m}(\alpha) - \mu_{n+2(m+1)}(\alpha)| = O(\lambda^{n+2m}),$$

which assures the existence of the limit and the validity of the estimate (66). $\square$

We now state and prove the main result for homogeneous χ.

Theorem 5.2. *Let χ be a homogeneous random variable of degree $\kappa > 2 - \delta$, where $\delta = \ln 2 / \ln \tau$. Then the following holds:*

$$\langle \chi \rangle_t = t^\beta \left(G(t) + O(t^{-a} \log^b t) \right), \tag{68}$$

where

$$\beta = \beta(\kappa) = 1 + \frac{\kappa - 2}{\delta}, \tag{69}$$

and

$$a = \min\left(\frac{\beta}{2}, \frac{1}{\delta}\right) \;>\; 0, \qquad\qquad b = 1 + \frac{1}{\delta}. \tag{70}$$

The function G is a bounded function of t, which satisfies the equation $G(t) = G(4t)$ for all positive integers t. Specifically, $G(t)$ is given by the formulae

$$G(t) = F\left(\frac{1}{10} \cdot 4^{\{\log_4 6\, t\}}\right)$$

where $\{\cdot\}$ denotes the fractional part and F is given by

$$F(\alpha) = \frac{1}{6}\left(\frac{3}{5\alpha}\right)^{\beta} (\lambda\sqrt{5})^{2-\kappa} \sum_{m=0}^{\infty} \frac{c_m}{2^{\beta m}} \mu\left((-2)^m \alpha\right), \tag{71}$$

where μ is given by (65) and

$$c_m = \begin{cases} 1 & m = 0 \\ \lambda & m = 1 \\ \sqrt{5}\,\lambda^4 & m > 1. \end{cases}$$

To prove Theorem 5.2, we first decompose $\langle\chi\rangle_t$ into a finite sum of periodic-orbit contributions. From (34) and (58) we have:

$$\langle\chi\rangle_t = \sum_{\mathbf{z}\in\mathbf{Z}^2} \chi(\mathbf{z}) \sum_{n=1}^{2L(t)} b_n(t)\, P_n\, \mathbf{Q}(\mathbf{z}, t) \;=\; \sum_{n=1}^{2L(t)} b_n(t)\, P_n\, \langle\chi\rangle_{n,t}, \tag{72}$$

where

$$\langle\chi\rangle_{n,t} = \frac{1}{T_n} \sum_{t'=0}^{T_n-1} \chi(\mathbf{z}_{n,t+t'} - \mathbf{z}_{n,t'}). \tag{73}$$

Here we have $b_n(t) = 1$ for $n < 2L(t) - 1$, $b_{2L(t)-1}(t) = \tau^3/\sqrt{5}$, and $b_{2L(t)-1}(t) = \tau^4/\sqrt{5}$ in accordance with Proposition 3.4, while $\mathbf{z}_{n,t}$ was defined in (49).

The proof proceeds in three stages, contained in Lemmas 5.3–5 below. Summing up the bounds on the nonleading contributions will always involve summing and bounding a finite geometric series, from which we extract the leading behaviour with the formula

$$\sum_{m=0}^{N} 2^{cm} \;=\; 2^{\max(c,0)N} S^c_{N+1}, \tag{74}$$

where S^c_N grows at most linearly in N:

$$S^c_N \leq \begin{cases} N & c = 0 \\ \min(N, (1 - 2^{-|c|})^{-1}) & c \neq 0. \end{cases}$$

A complication in arriving at our result is the fact that some of the nonleading contributions decrease exponentially with increasing n, while others increase. It turns out that all contributions can be controlled if we truncate the series (72) as in the following lemma. To simplify the calculations later on, we simultaneously isolate the leading term in the formula (13) for the period T_n.

Lemma 5.3. *We have the following estimate*

$$\left| \langle\chi\rangle_t - \frac{5}{3}\sum_{n=L(t)}^{2L(t)} 2^n b_n(t)\frac{P_n}{T_n}\langle\chi\rangle_{n,t} \right| \le t^\beta \left(h_0(t)t^{-\beta/2} + h_1(t)t^{-\min(\beta,1)}\right), \tag{75}$$

where

$$h_0(t) \;=\; B_0 i\,\lambda^6\,5^{3/2}\,6^{-1+\beta/2}\,8^\kappa\,S_{L(t)}^\beta\,(24)^{\beta/2} \tag{76}$$

$$h_1(t) \;=\; B_0\,\lambda^2 3^{-1}\,8^\kappa 24^{\max(0,\beta-1)}\,S_{2L(t)+1}^{\beta-1}, \tag{77}$$

and B_0 is given by (60).

Proof. From equations (53) and (56) we have, for all $t, t' > 0$,

$$\|\,\mathbf{z}_{n,t+t'} - \mathbf{z}_{n,t'}\,\| \le 8 + 2\tau^{n+2}/\sqrt{5}, \tag{78}$$

and so, from (60) and the fact that $\tau = 2^{1/\delta}$, we have

$$|\langle\chi\rangle_{n,t}| \le B_0\,8^\kappa\,2^{n\,\kappa/\delta}. \tag{79}$$

Thus we see that the left-hand side of (75) is less than or equal to

$$\frac{5}{3}\sum_{n=1}^{L(t)-1} 2^n\frac{P_n}{T_n}|\langle\chi\rangle_{n,t}| + \sum_{n=1}^{2L(t)} b_n(t)\frac{P_n}{T_n}|T_n - \frac{5}{3}2^n|\,|\langle\chi\rangle_{n,t}|$$

$$\le B_0 8^\kappa\left(\frac{1}{6}\lambda^6\,5^{(3-\kappa)/2}\sum_{n=0}^{L(t)-1} 2^{n\,\beta} + \frac{1}{3}\lambda^2\sum_{n=0}^{2L(t)} 2^{n(\beta-1)}\right),$$

with β given by (69). Applying (74) and the inequality $L(t) < \log_4 24t$ (cf. (36)) we arrive at the estimate (75). $\square$

We now move to the second stage of the proof of Theorem 5.2, in which we use Lemma 4.7 to express the leading asymptotic behavior of $\langle\chi\rangle_{n,t}$ in terms of an average of χ over the displacement vectors of the snowflake set $\mathcal{K}_n$.

Lemma 5.4. *Let* $\langle\chi\rangle_{n,t}$ *and* $s(n)$ *be given by (73) and (50), respectively. Then*

$$\left|(\lambda^{n+1}\sqrt{5})^\kappa\langle\chi\rangle_{n,t} - \mu(\alpha_{n,st})\right| \le g_1(t)\lambda^n + g_2(t)2^{-n} + g_3\tau^{2(L(t)-n)}, \tag{80}$$

where

$$\begin{aligned} g_1(t) &= B_1(2\tau)^{\kappa-1}\lambda\left(5+4\tau+(4\epsilon_1+2\epsilon_2\,L(t)^{1/\delta}+2\epsilon_3)/\sqrt{5}\right) && (81)\\ g_2(t) &= \frac{4}{9}B_0(2\tau)^\kappa && (82)\\ g_3 &= B_1(2\tau)^{\kappa-1}(\lambda/\sqrt{5})\,6^{-1/\delta}\,\epsilon_1, && (83)\end{aligned}$$

with B_0, B_1 *and* ϵ_k *as in (60), (61) and Lemma 4.7, respectively.*

Proof. Set

$$\Omega_{n,t} = (\lambda^{n+1}\sqrt{5})\mathbf{z}_{n,t} - \mathbf{k}_{n,st}.$$

Then we may write

$$(\lambda^{n+1}\sqrt{5})^\kappa\langle\chi\rangle_{n,t} = \frac{1}{T_n}\sum_{t'=0}^{T_n-1}\chi(\mathbf{k}_{n,s(t+t')} - k_{n,st'} + \Omega_{n,t+t'} - \Omega_{n,t'}).$$

Using the gradient bound (61) and Lemma 4.7, we obtain

$$\begin{aligned}\left|(\lambda^{n+1}\sqrt{5})^\kappa\langle\chi\rangle_{n,t} \right. &- \left.\frac{1}{T_n}\sum_{t'=0}^{T_n-1}\chi(\mathbf{k}_{n,s(t+t')} - k_{n,st'})\right| \\ &\le B_1\,(2\tau)^{\kappa-1}(\lambda^{n+1}/\sqrt{5})\max_{0\le t'<T_n}(\|\,\Omega_{n,t+t'}\,\| + \|\,\Omega_{n,t'}\,\|) \\ &\le B_1\,(2\tau)^{\kappa-1}(\lambda^{n+1}/\sqrt{5})(4\epsilon_1 + 2\epsilon_2 L(t)^{1/\delta} + 2\epsilon_3).\end{aligned}$$

To complete the proof, we must relate the sum over $\chi(\mathbf{k}_{n,s(t+t')} - k_{n,st'})$ to $\mu(\alpha_{n,st})$. This is done using Lemma 4.8, the $|\chi|$ bound (60), and Lemma 5.1 to obtain, respectively, the three estimates

$$T_n^{-1}\left|\sum_{t'=0}^{T_n-1}\chi(\mathbf{k}_{n,s(t+t')} - \mathbf{k}_{n,st'}) - \frac{1}{s}\sum_{j=0}^{|K_n|-1}\chi(\mathbf{k}_{n,st+j} - \mathbf{k}_{n,j})\right| \le 5B_1(2\tau)^{\kappa-1}\lambda^{n+1} \tag{84}$$

$$\left|\frac{1}{T_n s}\sum_{j=0}^{|K_n|-1}\chi(\mathbf{k}_{n,st+j} - \mathbf{k}_{n,j}) - \mu_n(\alpha_{n,st})\right| \le \frac{2}{3T_n}B_0(2\tau)^\kappa \le \frac{4}{9}B_0(2\tau)^\kappa \tag{85}$$

$$\left|\mu_n(\alpha_{n,st}) - \mu((-1)^n\alpha_{n,st})\right| \le 4\,B_1(2\tau)^\kappa\lambda^n. \tag{86}$$

Combining the contributions of (84), (85) and (86), we get (80). $\square$

To complete the proof of Theorem 5.2, we first insert the expression for $\langle\chi\rangle_{n,t}$ given by Lemma 5.4 into the inequality of Lemma 5.3, to obtain an estimate for

$$\Big| \langle\chi\rangle_t - \sum_{n=L(t)}^{2L(t)} b_n(t)(\frac{5}{3}2^n)\frac{P_n}{T_n}\left(\frac{\tau^{n+1}}{\sqrt{5}}\right)^{\kappa} \mu((-1)^n\alpha_{n,st})\Big|. \tag{87}$$

We insert the expressions for $b_n(t)$ and P_n/T_n, and write

$$(-1)^n\alpha_{n,st} = (-2)^{2L(t)-n}\alpha_t,$$

where

$$\alpha_t = \frac{3t}{5\cdot 4^{L(t)}} = \frac{1}{10}\cdot 4^{\{\log_4 6t\}}.$$

Finally, we change the summation variable from n to $m = 2L(t) - n$ and extend the sum over m to infinity, adding a new term to the bound on nonleading contributions which grows asymptotically as $t^{\beta/2}$ (the calculation follows the same pattern as in the proof of Lemma 5.3 and we omit the details). With these transformations, the sum in (87) assumes the form $t^\beta F(\alpha_t)$. Summing, with the aid of (74) the various bounds on nonleading contributions from Lemmas 5.3 and 5.4 gives us the overall estimate of the following lemma, from which Theorem 5.2 follows immediately.

Lemma 5.5. *The following holds*

$$\begin{aligned} |t^{-\beta}\langle\chi\rangle_t - F(\alpha_t)| \;&\leq\; h_1(t)\,t^{-\min(\beta,1)} + h_2(t)\,t^{-\beta/2} + h_3(t)\,t^{-\min(\frac{\beta}{2}+\frac{1}{2\delta},\frac{1}{\delta})} + \\ & \quad\; h_4(t)\,t^{-\min(\frac{\beta}{2}+\frac{1}{2},1)} + h_5(t)\,t^{-\min(\frac{\beta}{2},\frac{1}{\delta})}, \end{aligned} \tag{88}$$

where

$$h_2(t) = h_0(t) + B_0(2\tau)^\kappa\,(24)^{\frac{\beta}{2}}\,(2^\beta - 1)^{-1} r_\kappa, \tag{89}$$

$$h_3(t) = g_1(t)\,(24)^{\max(\frac{\beta}{2}-\frac{1}{2\delta},\beta-\frac{1}{\delta})}\,S_{L(t)+1}^{\beta-\frac{1}{\delta}} r_\kappa, \tag{90}$$

$$h_4(t) = g_2(t)\,(24)^{\max(\frac{\beta}{2}-\frac{1}{2},\beta-1)}\,S_{L(t)+1}^{\beta-1} r_\kappa, \tag{91}$$

$$h_5(t) = g_3\,(24)^{\max(\frac{\beta}{2},\beta-\frac{1}{\delta})}\,S_{L(t)+1}^{\beta-\frac{2}{\delta}} r_\kappa. \tag{92}$$

and

$$r_\kappa = \frac{1}{6}\lambda^3(\lambda\sqrt{5})^{3-\kappa}.$$

We now generalize Theorem 5.2 to an arbitrary random variable of asymptotic degree κ.

Theorem 5.6. *Let χ be a random variable of asymptotic degree $\kappa > 2-\delta$, where $\delta = \ln 2/\ln\tau$. Let χ_+, χ_-, and χ_0 be, respectively, the leading, nonleading, and bounded asymptotic parts of χ. Let κ' be the exponent associated with the bound on χ_- as in (62). Then the following holds:*

$$\langle\chi\rangle_t = t^\beta \left(G(t) + O(t^{-a}\log^b t)\right), \tag{93}$$

where $G(t)$, β, and b are defined in the statement of Theorem 5.3, as applied to χ_+, and

$$a = \min\left(\frac{\beta}{2}, \frac{1}{\delta}, \frac{\kappa-\kappa'}{\delta}\right).$$

The only ingredient that is not an immediate consequence of Theorem 5.2 is the bound on the asymptotic behavior of χ_-. But this follows from a general result which we state as a separate proposition.

Proposition 5.7. *Let χ be a random variable of asymptotic degree κ, and let $B > 0$ be such that*

$$|\chi(\mathbf{z})| \leq B\,\|\mathbf{z}\|^\kappa.$$

Then

$$|\langle\chi\rangle_t| \;\leq\; (24\,t)^{\max(0,\beta)}\,B\,\lambda^2\,8^\kappa\,S^\beta_{2L(t)+1},$$

where β is given by (69).

Proof. With the estimates (cf. (13), (79) and the definition of $b_n(t)$ after equation (73))

$$T_n \leq 2^{n+1}, \qquad b_n(t) \leq \frac{\tau^4}{\sqrt{5}}, \qquad |\langle\chi\rangle_{n,t}| \leq B 8^\kappa 2^{n\,\kappa/\delta},$$

we obtain, from (72),

$$|\langle\chi\rangle_t| \leq B\lambda^2 8^\kappa \sum_{n=1}^{2L(t)} 2^{\beta n}.$$

Applying the geometric series formula (74), we get the desired result. □

The calculation of $\langle\chi\rangle_t$ can be done in polynomial time (proportional to t). This follows from the fact that we only need to include contributions from periodic orbits of lengths T_n for $n \leq 2L(t)$, where T_n is exponential in n, while $L(t)$ is logarithmic in t. For the longest of these we must calculate a number of displacement vectors which is roughly between 10 and 40 times t, with a comparable number from the totality of the shorter orbits. For t so large that the exact calculation time is prohibitive, our results give us a way of approximating $\langle\chi\rangle_t$ with a relative error of order

$O(t^{-a}\log^b t)$, with a and b given in Theorems 5.2 and 5.6. Moreover, for the case of monomial moments $\langle x^r y^{\kappa-r}\rangle_t$, the computation of the leading term $t^\beta G(t)$ is extremely efficient, since we may exploit linear recursion relations for the 2-point correlation function of the snowflake sets $\mathcal{K}_{2l}$. We have used this method to compute numerically $G(t)$ for $\chi(\mathbf{z}) = x^2$, for all values of t with $L(t) = 7$, i.e., $683 \leq t \leq 2730$. The results are plotted in Figure 9.

Figure captions

Figure 1. Orbits of Φ^5 on $\mathbf{Z}^2$. The dots represent fixed points.

Figure 2. The probability distribution $P(\mathbf{z}, t)$ for $t = 50$. Each circle denotes a lattice point $\mathbf{z} = (x, y) \in \mathbf{Z}^2$ belonging to the support of P. The latter is contained in the square $-23 \le x, y \le 23$. The probability at $\mathbf{z}$ is characterized by the area of the corresponding circle, which is proportional to the logarithm of $1/P(\mathbf{z}, 50)$. For clarity, we have omitted the largest circle located at the origin, representing the fixed-point probability.

Figure 3. The level-1 partition $\mathcal{P}_1$, whose components correspond to specific values of the vector field. The symbol ∇_i^{-k} denotes the kth pre-image of ∇_i under the local map Φ^*, as defined in (22). The triangles ∇_1 and ∇_2 —the images of Δ_1 and Δ_2 under Υ_1— are shown in Figure 4.

Figure 4. Dynamics in $\mathcal{F}_l$, the fundamental domain of level l.

Figure 5. Partition $\mathcal{P}_{l+1}$ on the fundamental domain $\mathcal{F}_l$.

Figure 6. Substitutions $\sigma : W_1 \to W_3$ and $\sigma : W_2 \to W_4$. The substitution rule for each of the 10 step types in the list D is apparent in this diagram. The graphical representation makes clear the relation between each W_n and the correponding vector-field walk V_n. Specifically, step type 5 is represented by two concatenated arrows, and step type 4 (null step) is represented by a dot.

Figure 7. Substitutions $\sigma_- : K_{0-} \to K_1$ and $\tilde{\sigma}_- : \tilde{K}_{0-} \to \tilde{K}_1$, from which the general action of σ_- and $\tilde{\sigma}_-$ can be inferred. The action of σ_+ and $\tilde{\sigma}_+$ is similar.

Figure 8. Relationship among $W_n, V_n, W_{n,1}, K_n$ for $n = 5$. In *(a)*, a polygonal line traces the 54-step walk V_5 starting at the point labeled "0". The dots are the 55 vertices of W_5, including the 6 null steps indicated by off-line dots. In *(b)*, the snowflake K_5 is shown as a 320-sided polygon, with dots indicating the vertices of $W_{5,1}$, a subset of the vertices of K_5. The 55 vertices of $W_{5,1}$ correspond, in order, to the vertices of W_5 shown in *(a)*.

Figure 9. The function $G(t)$ for $\chi = x^2$, plotted versus $\log_4 t$ for $683 \le t \le 2730$. These are all the t values for which $L(t) = 7$.

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