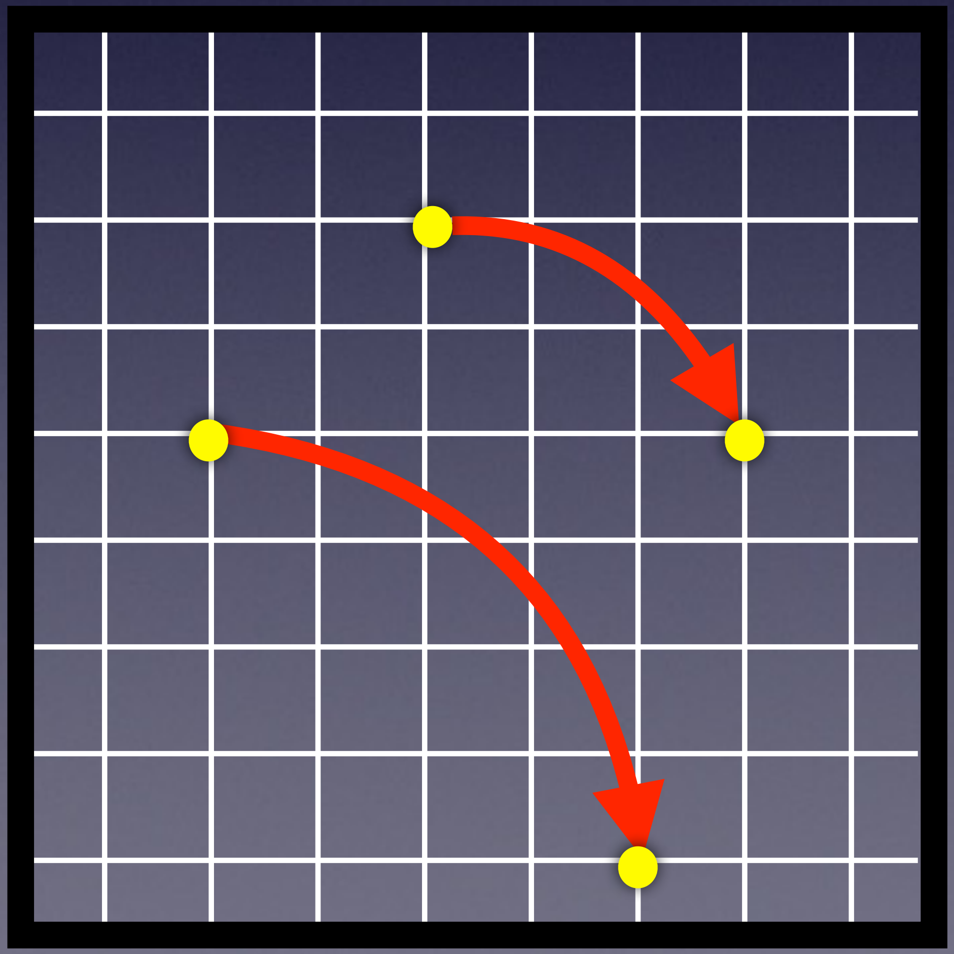
Nonlinear rotations
on lattices

Franco Vivaldi
Queen Mary, University of London

with Fairuz Alwani



Smooth area-preserving maps

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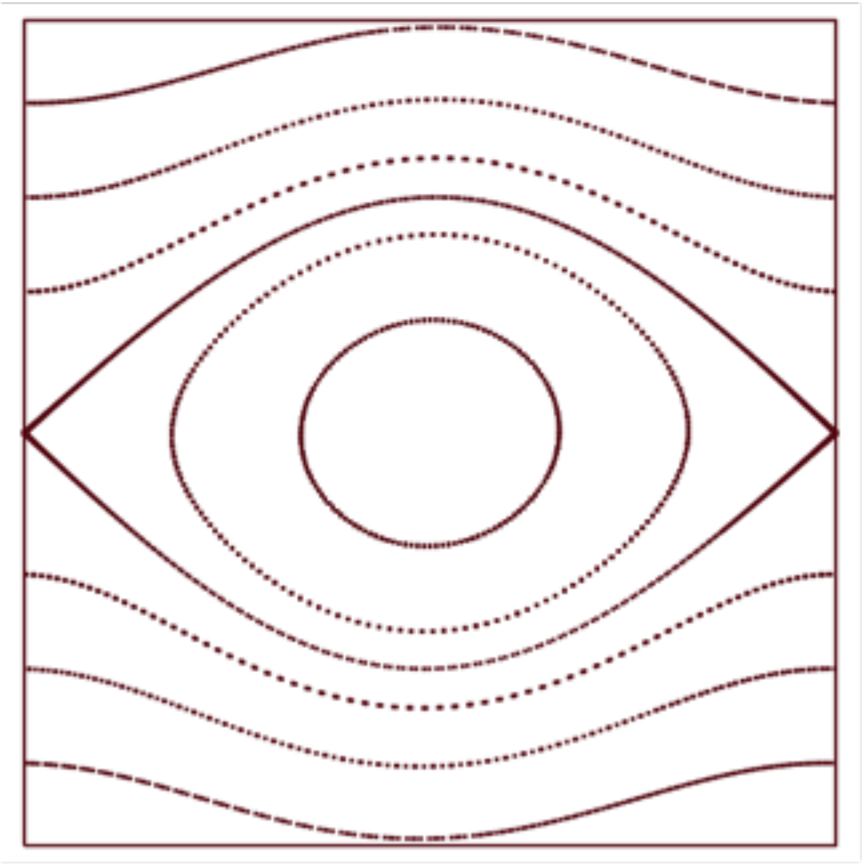
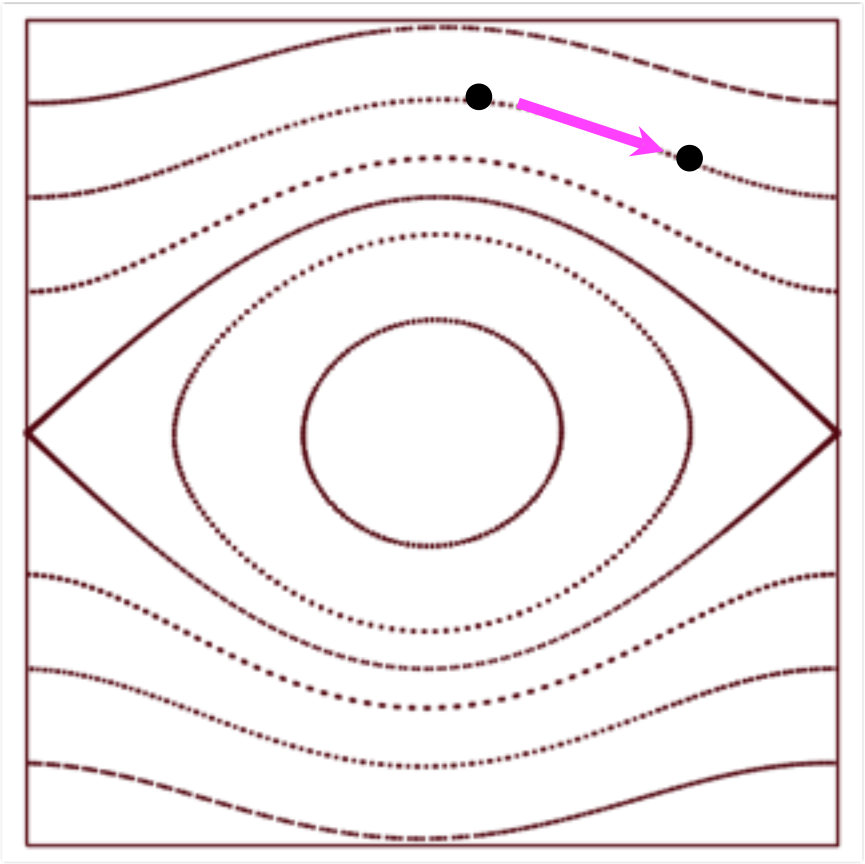


integrable

Smooth area-preserving maps

foliation by
invariant curves



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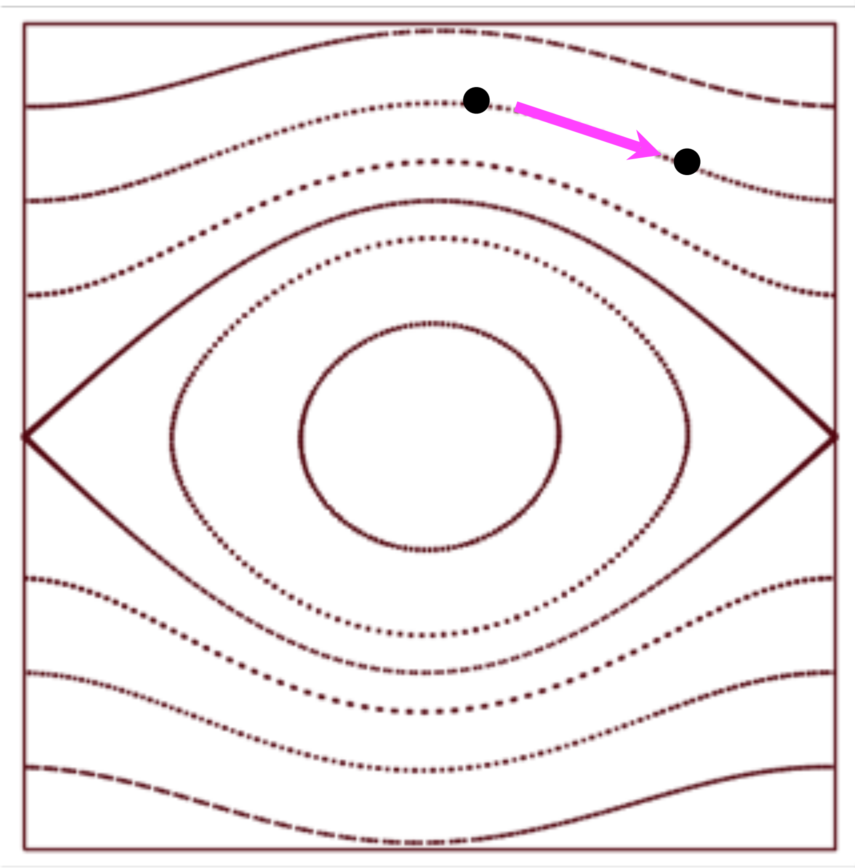

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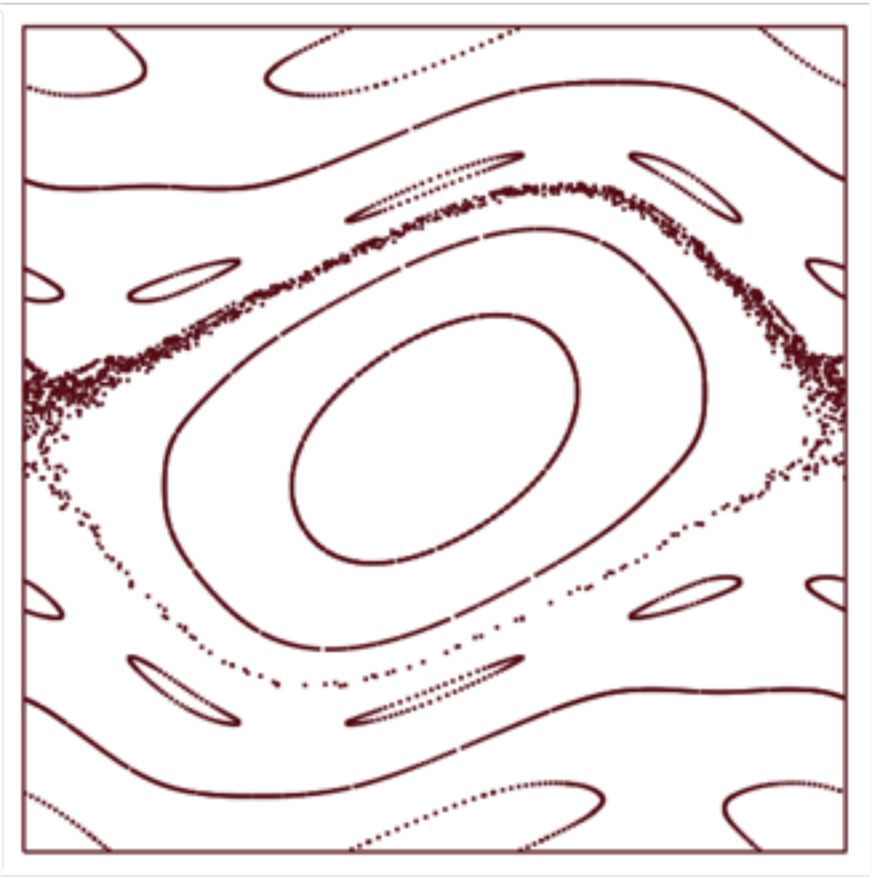

near-integrable: stable

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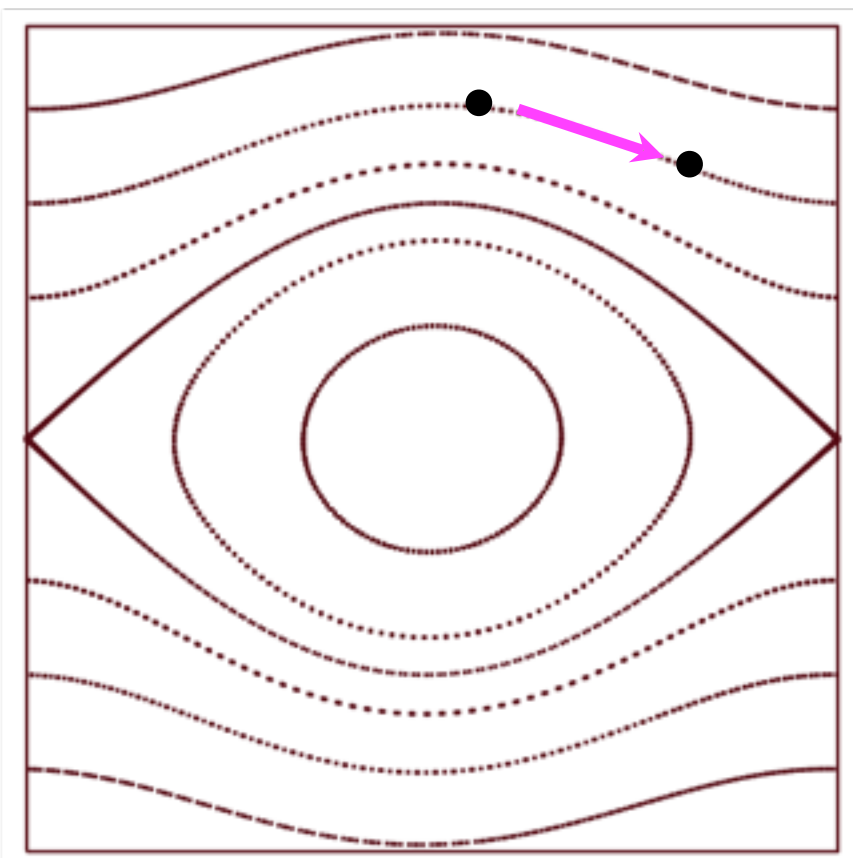
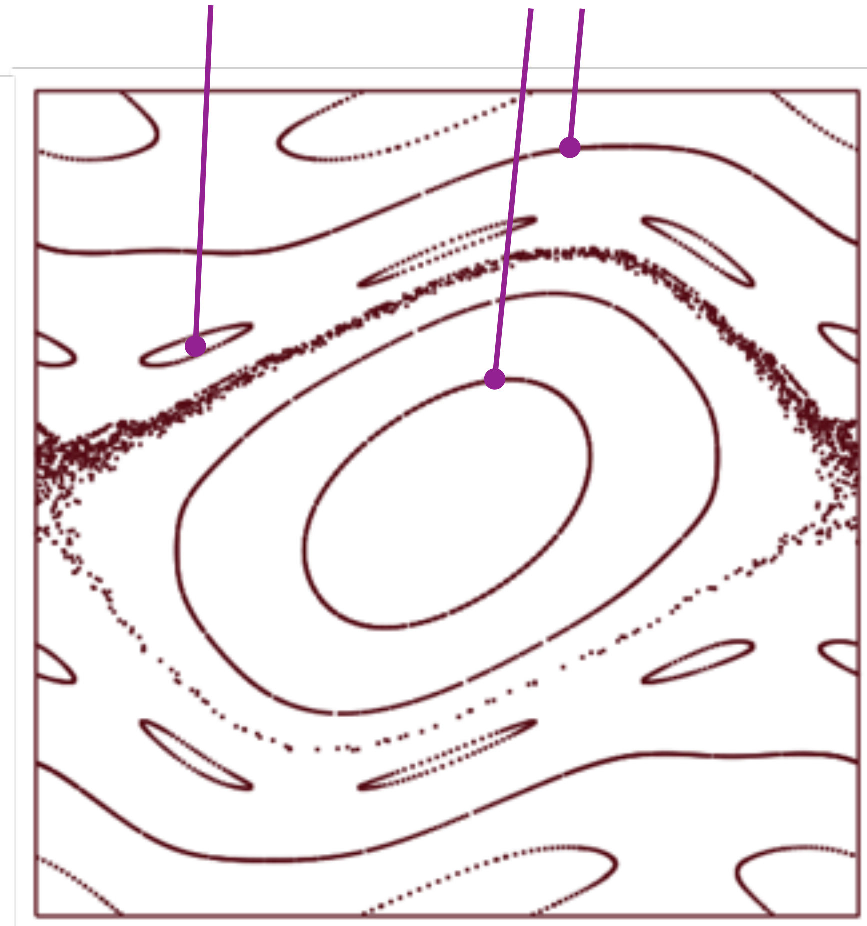


integrable

island chains

KAM curves



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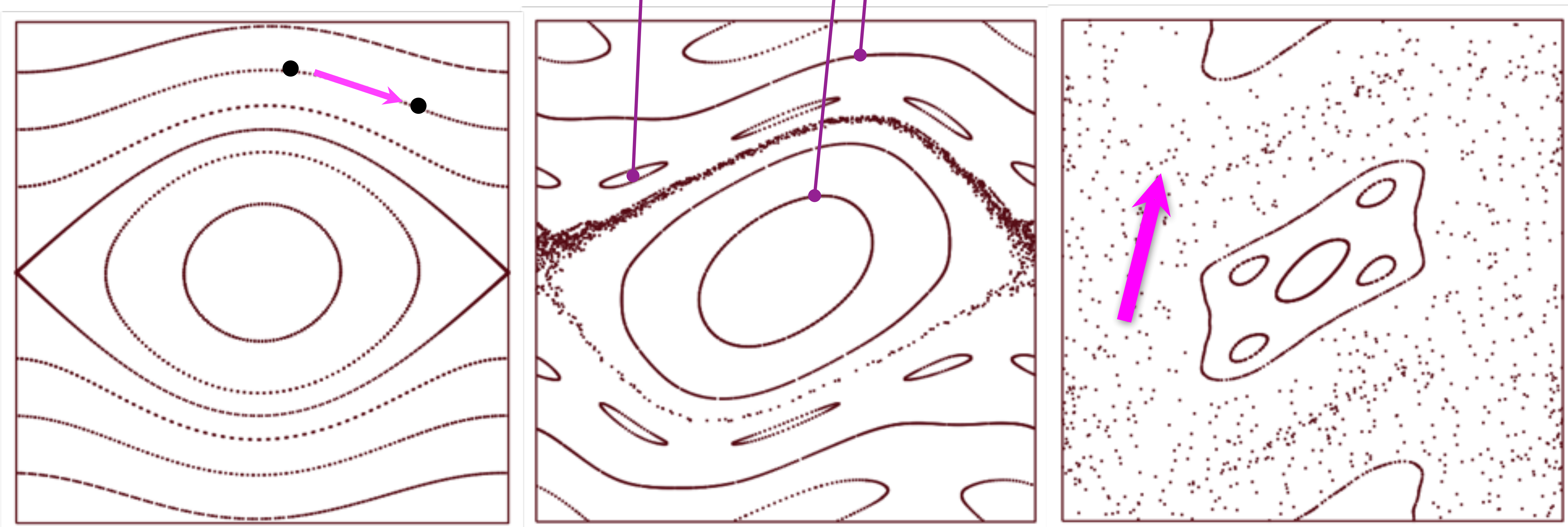

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near-integrable: stable

strong perturbation:
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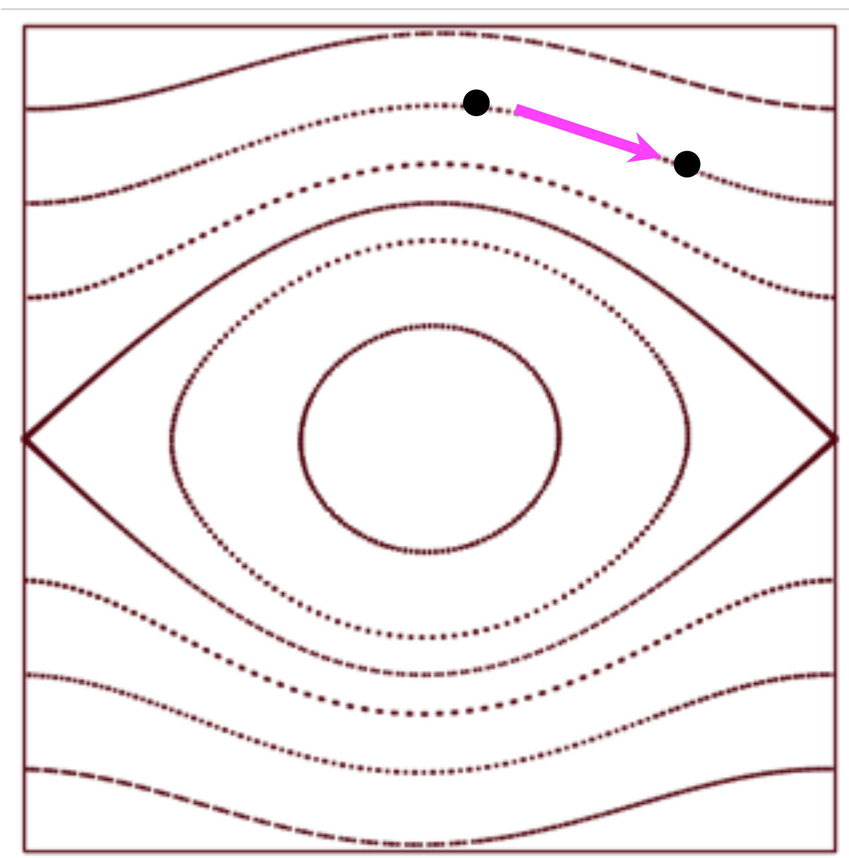

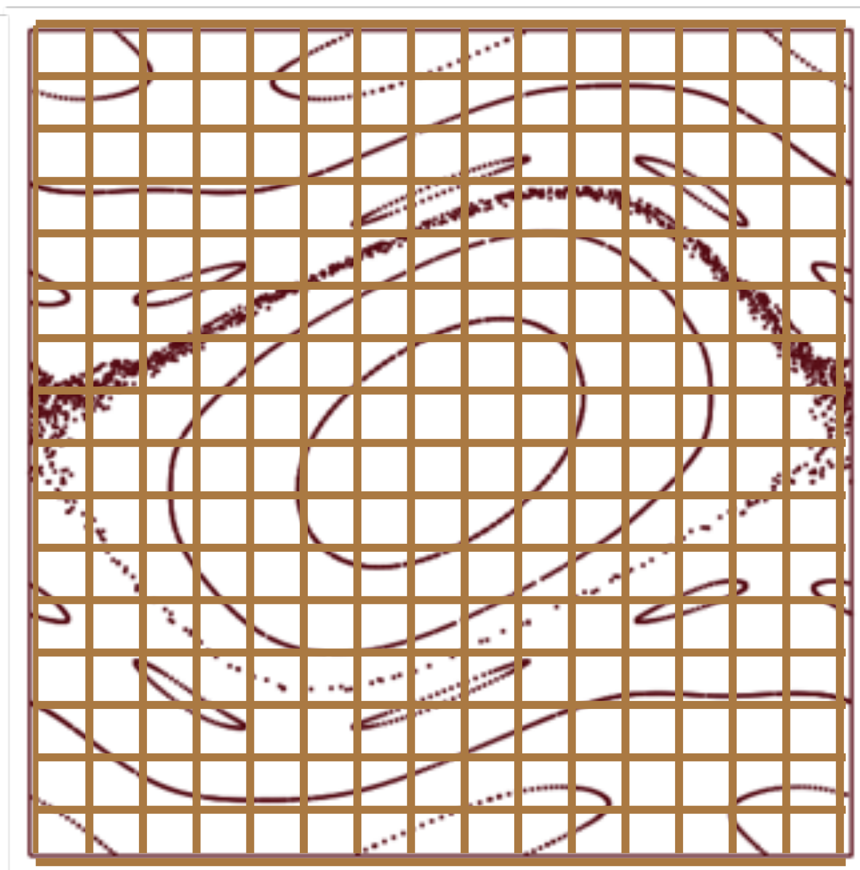

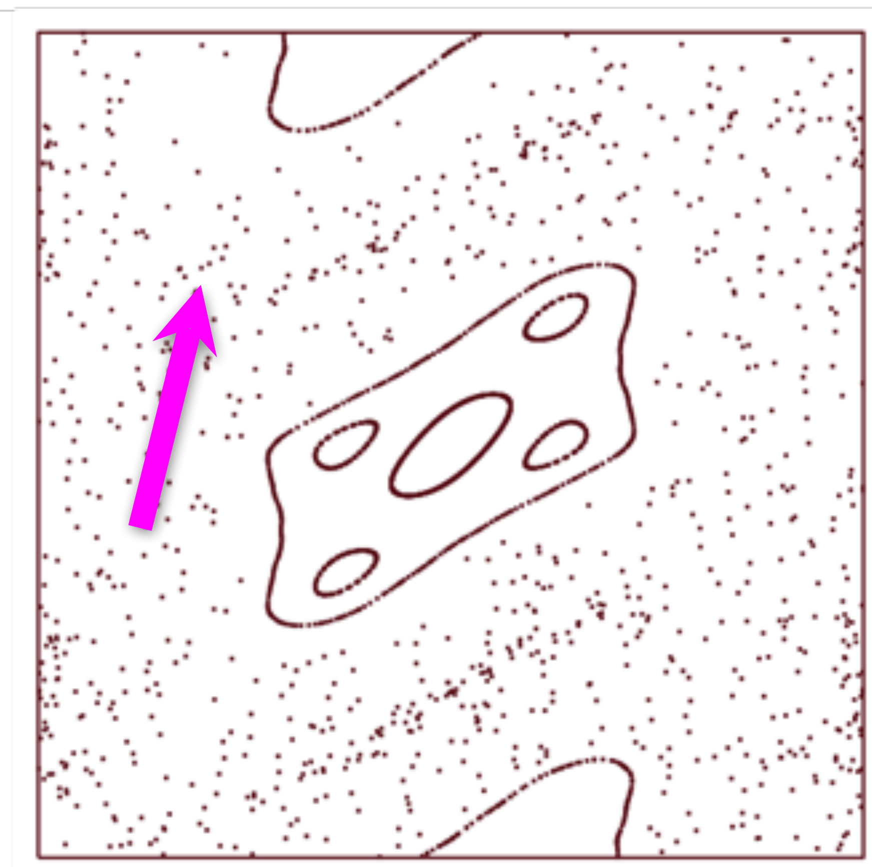

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What happens if the space is a lattice?

Smooth area-preserving maps

foliation by invariant curves

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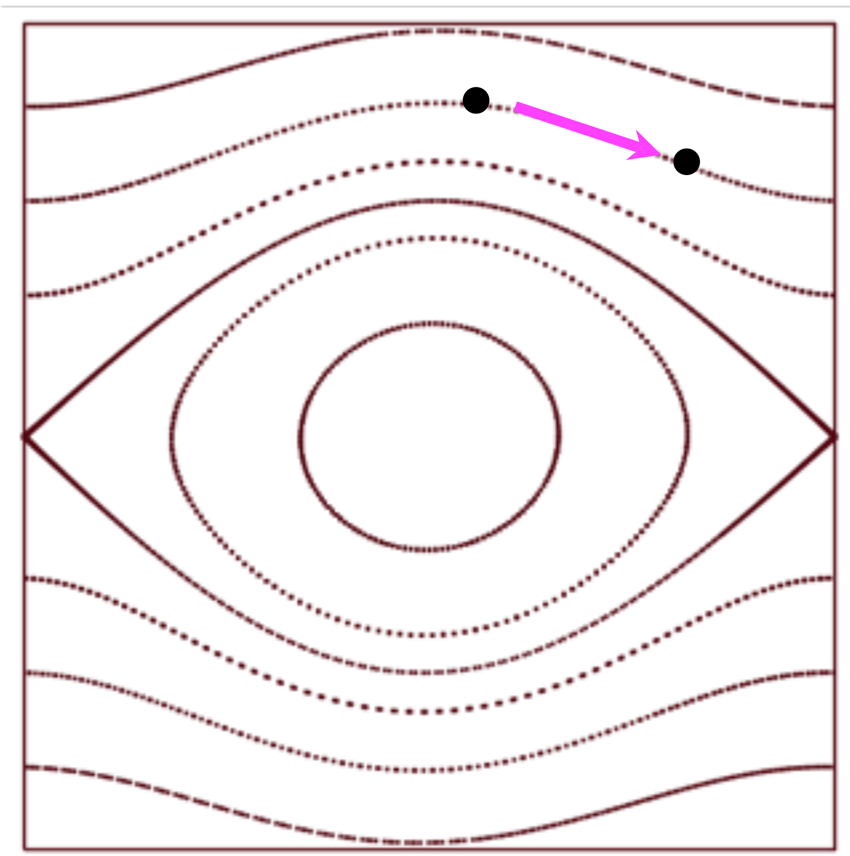

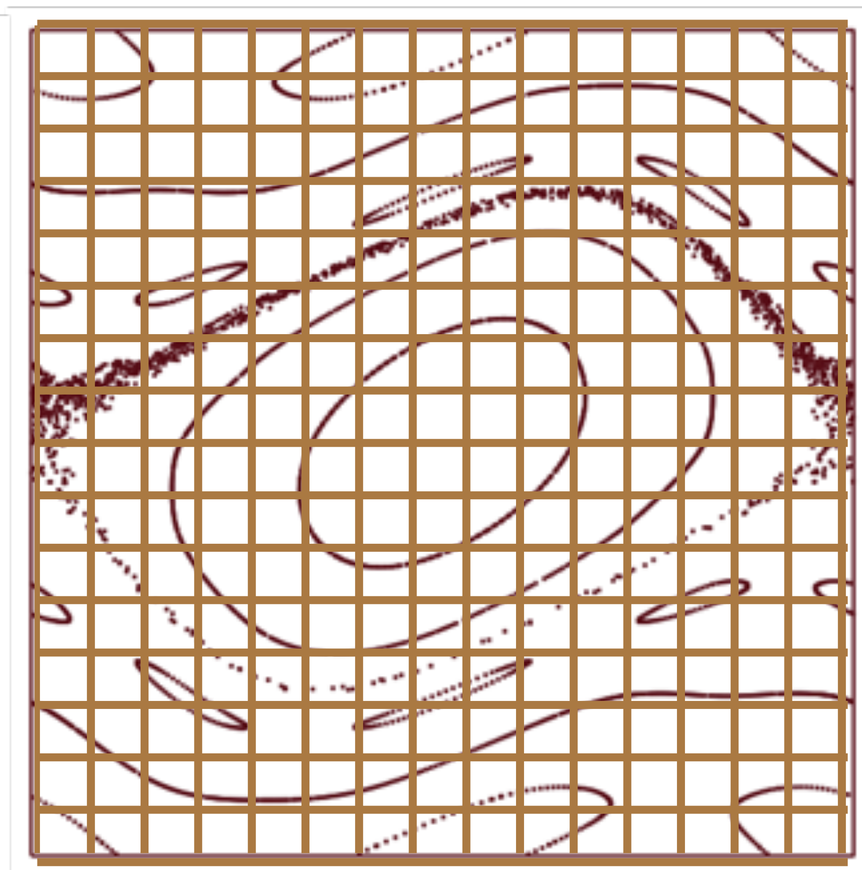

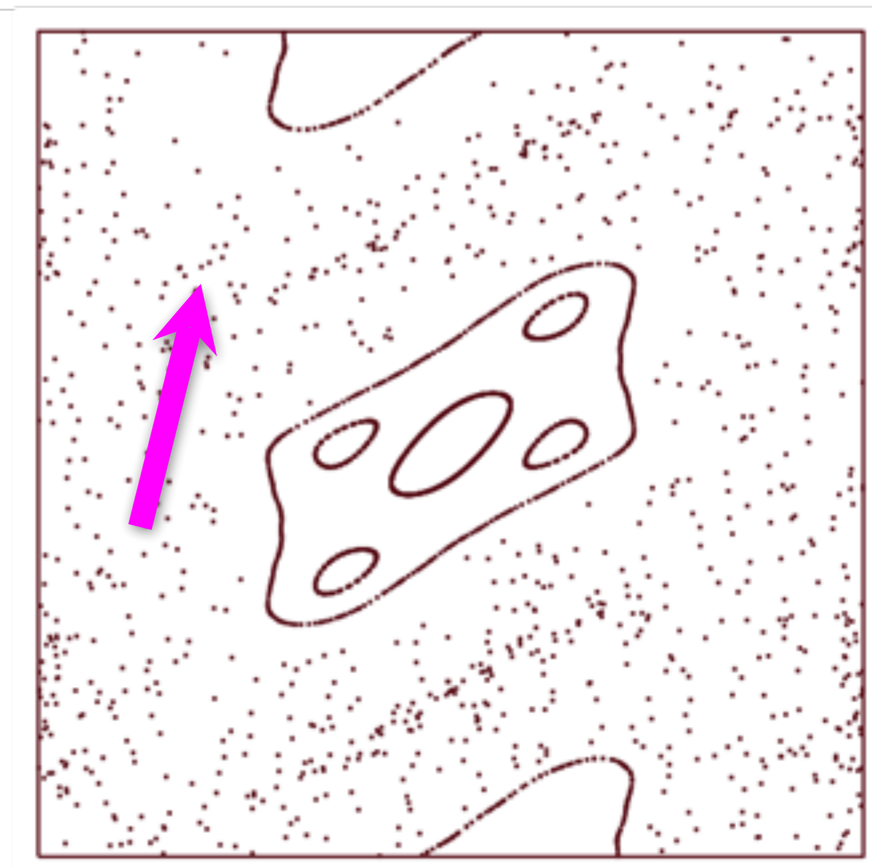

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Astron. & Astrophys. 31, 289 – 301 (1974)

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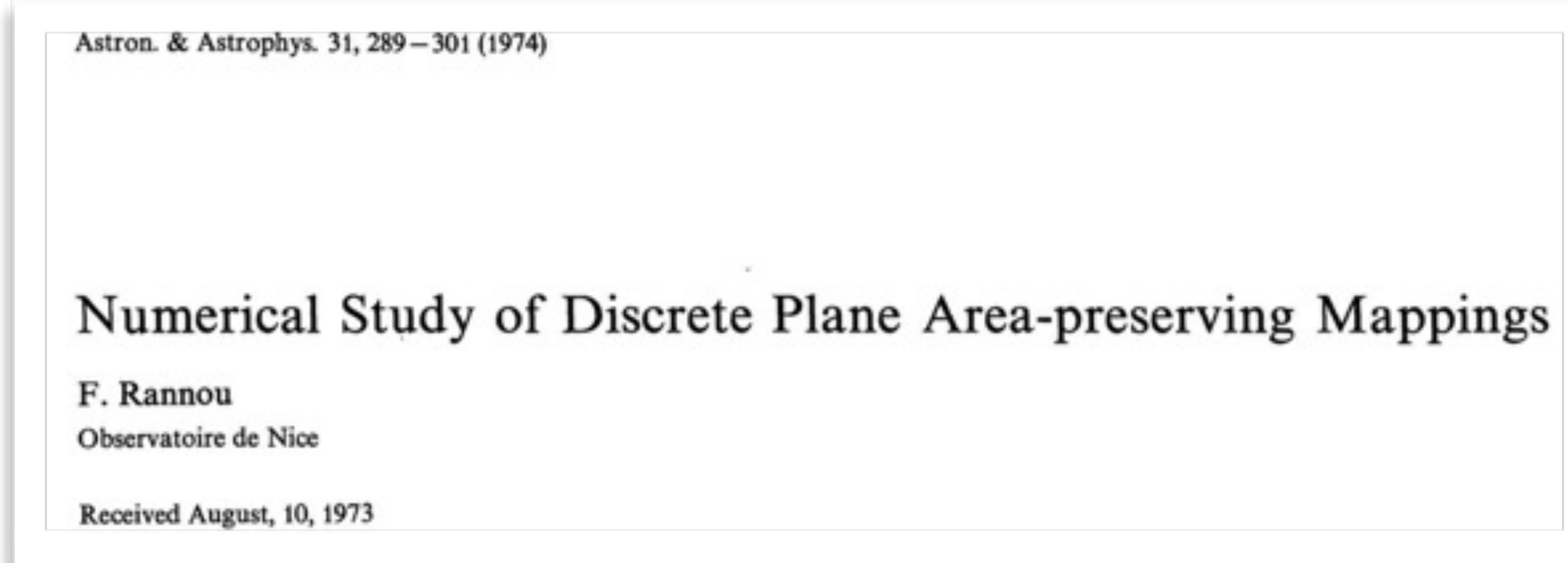
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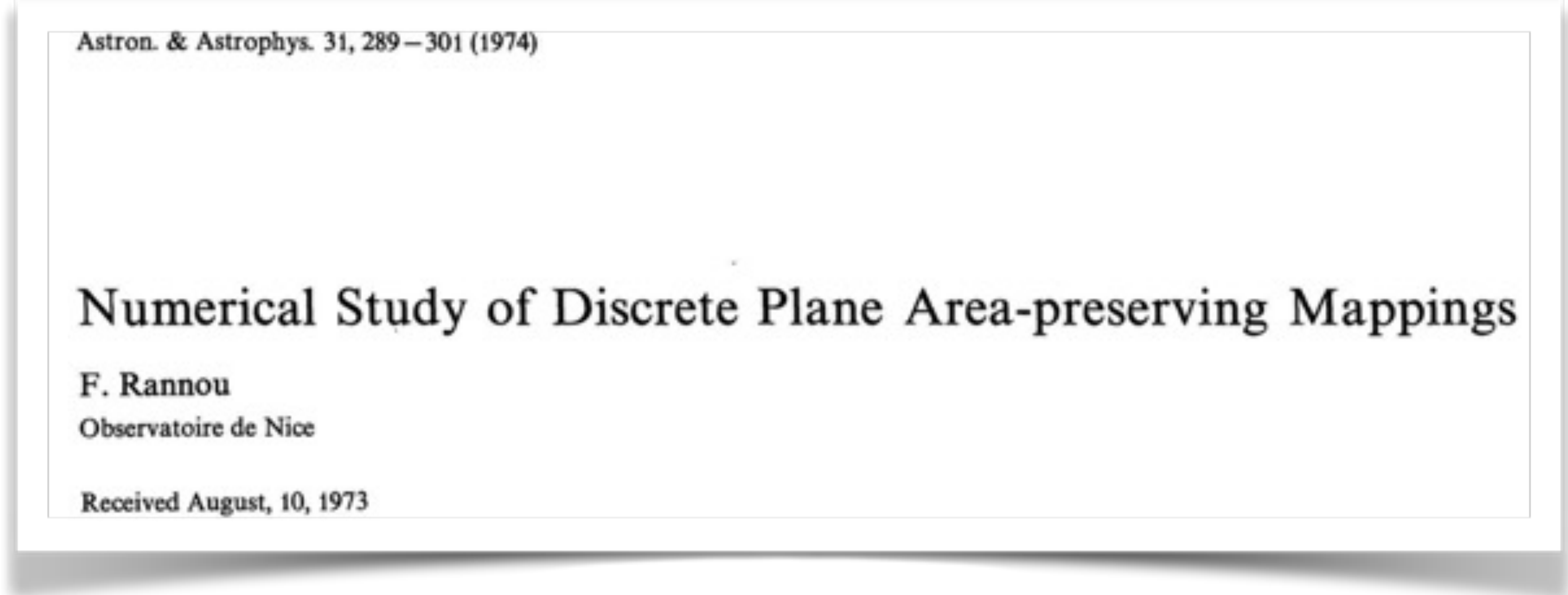

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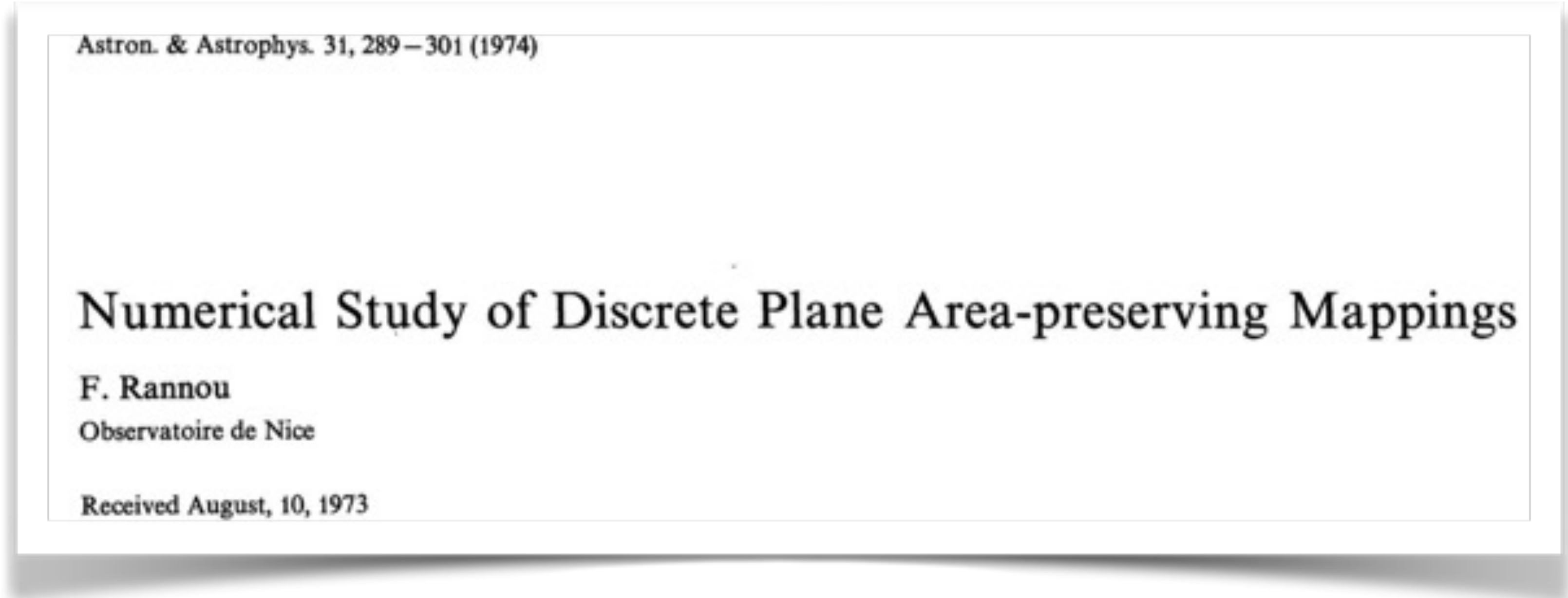

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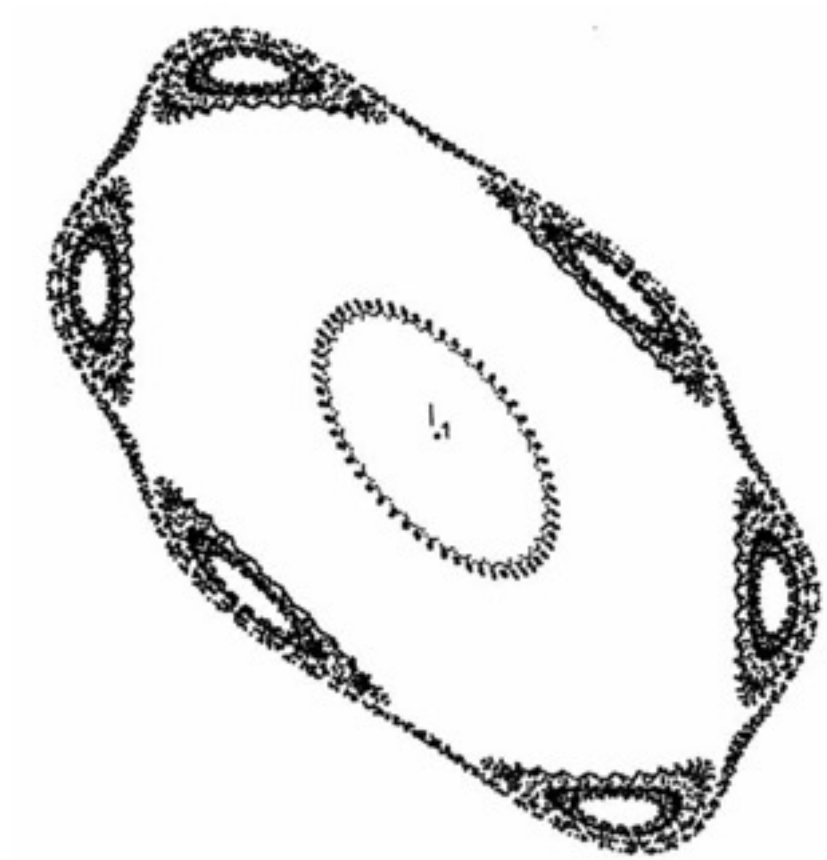

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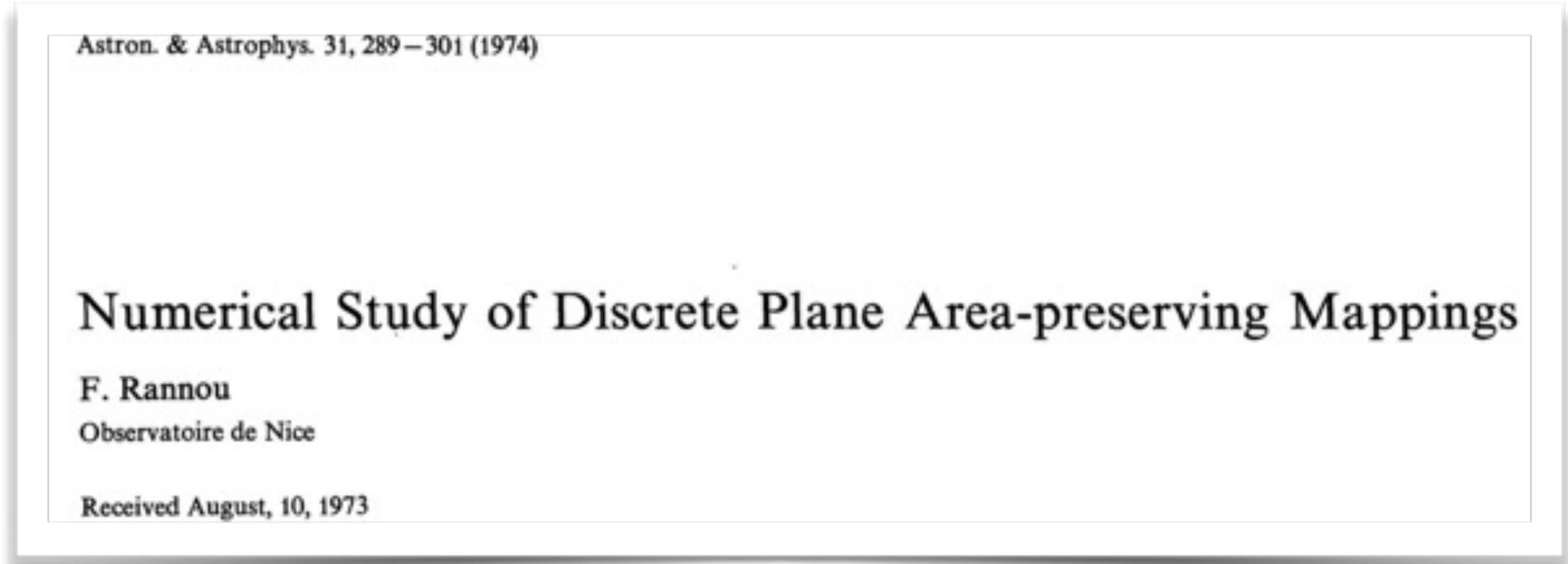
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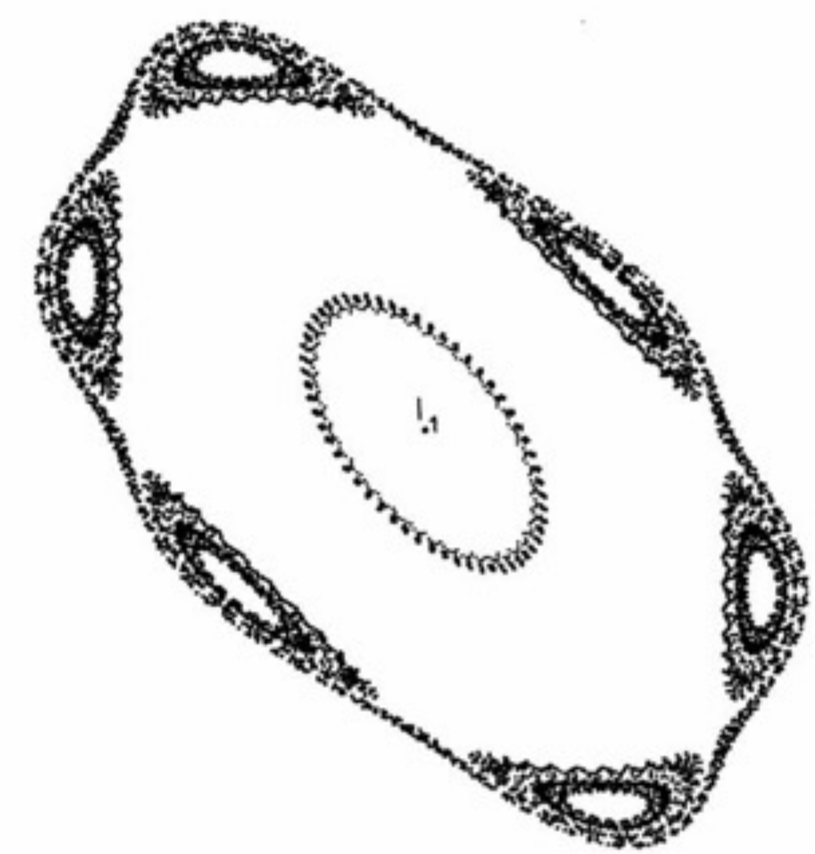

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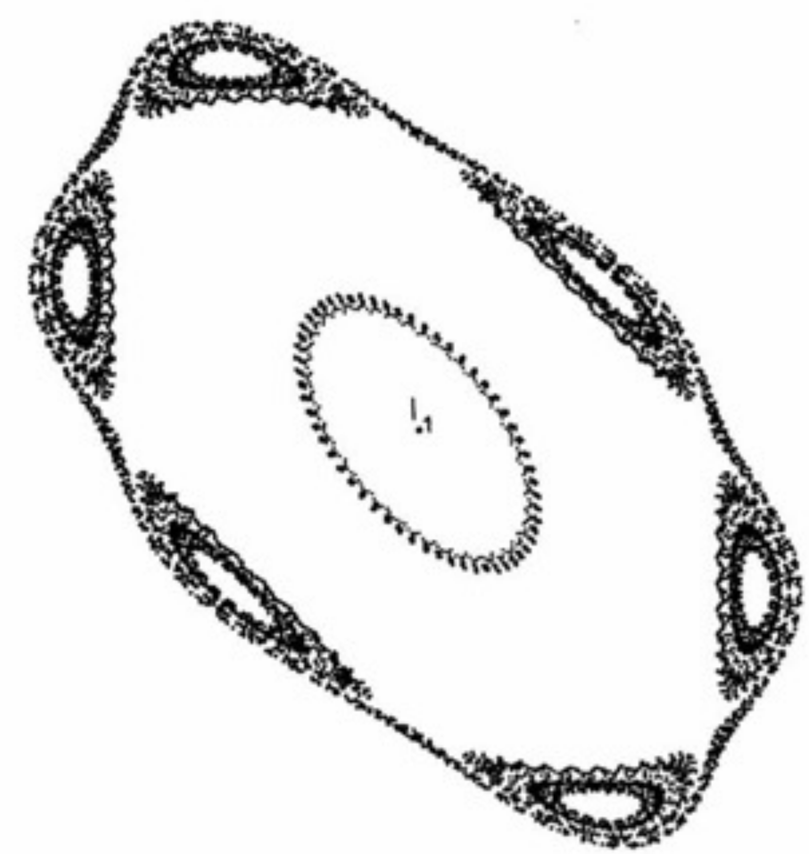

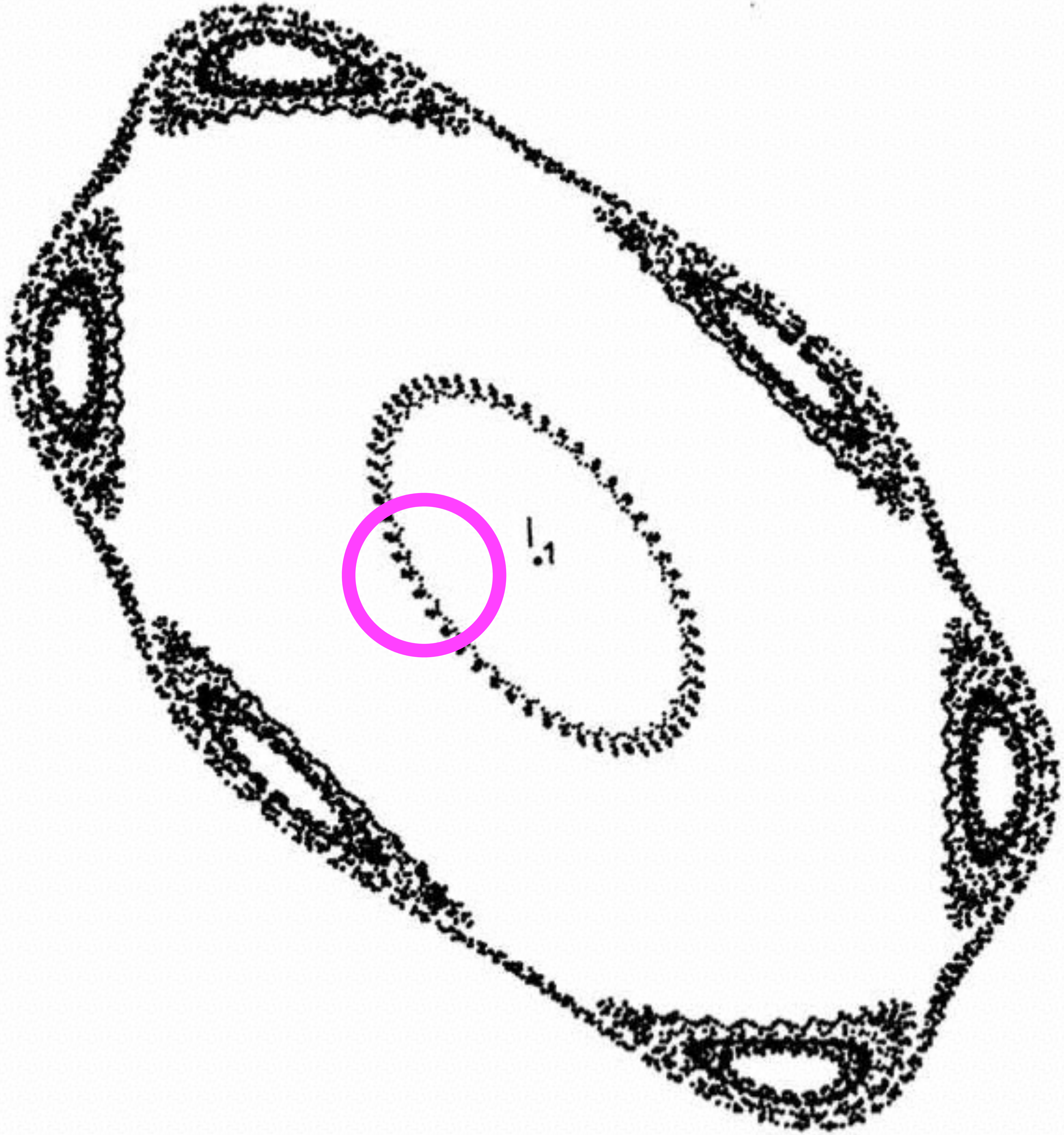
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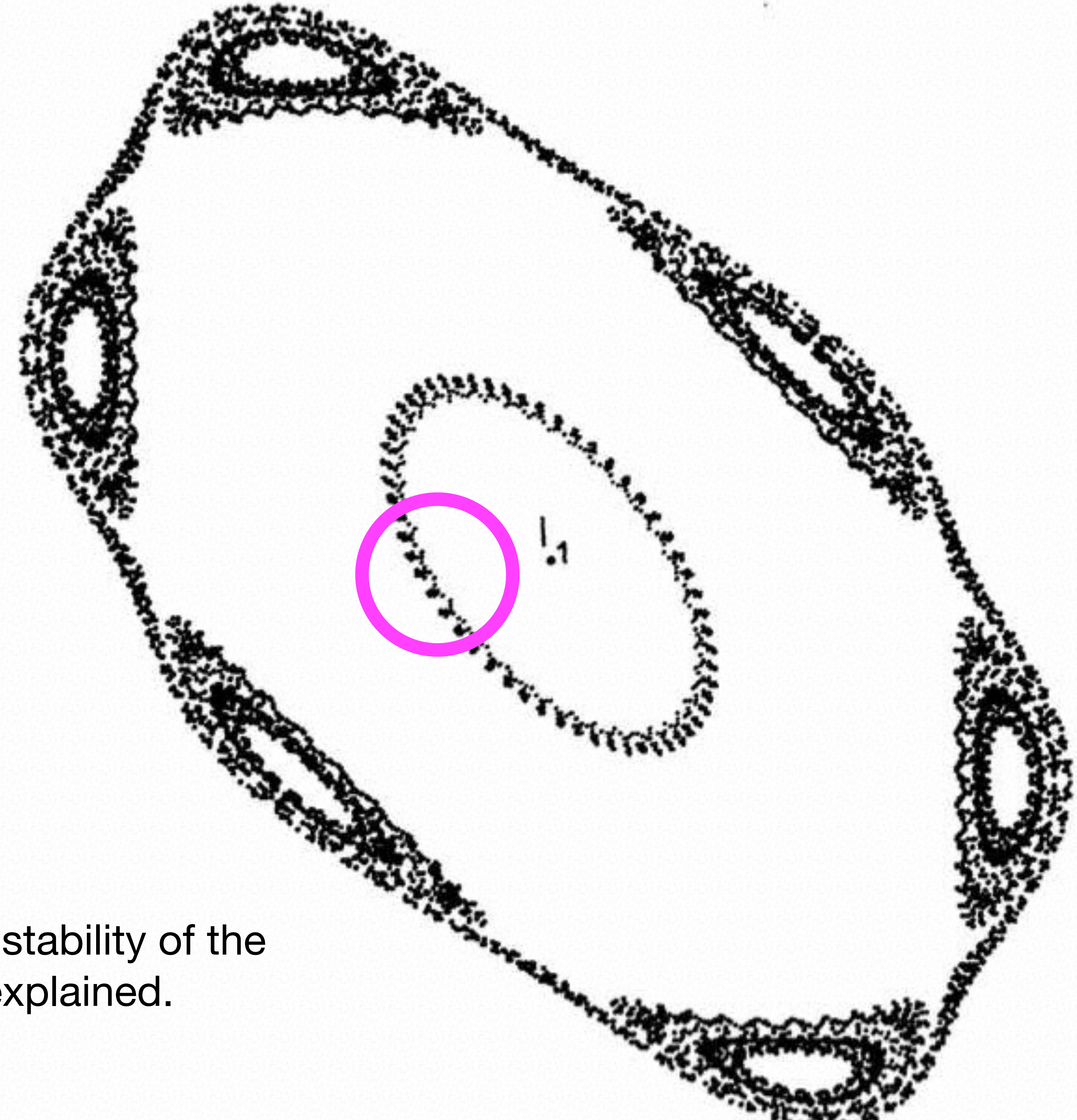

- Features and stability of the orbits are unexplained.

Stability of rotational orbits on lattices

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Few rigorous results; no general theory/framework.

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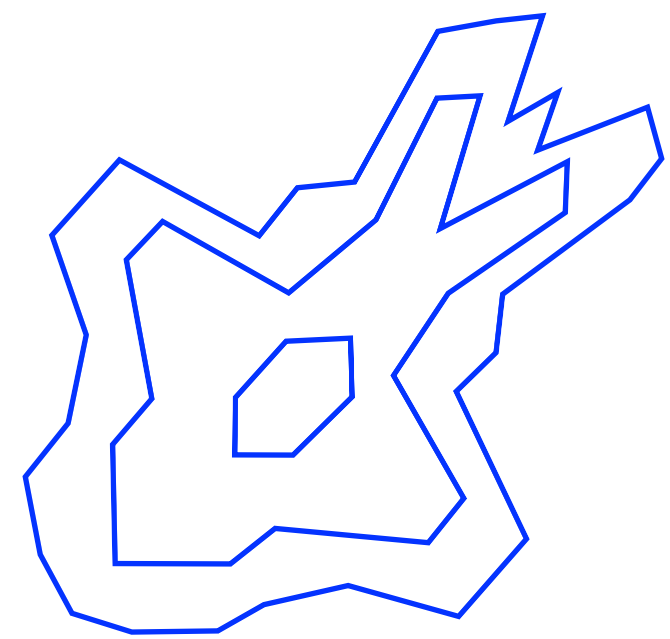
Invariant polygons of Devaney's map: $(x,y) \mapsto (1-y+|x|,x)$



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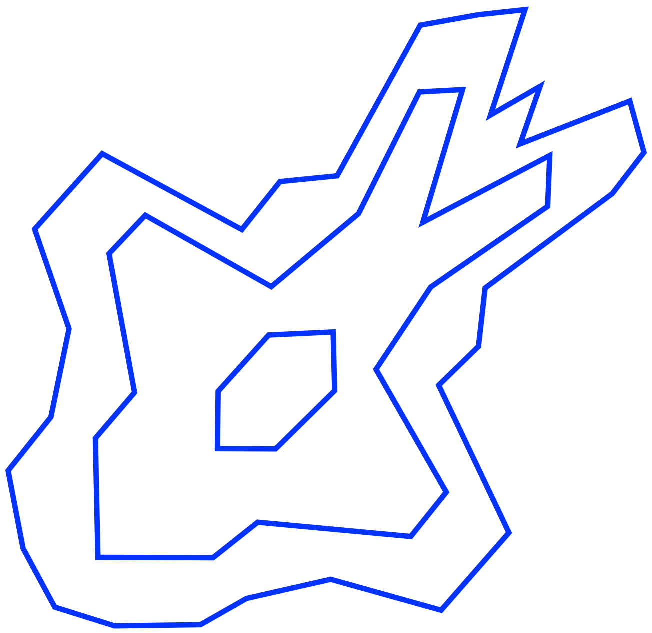

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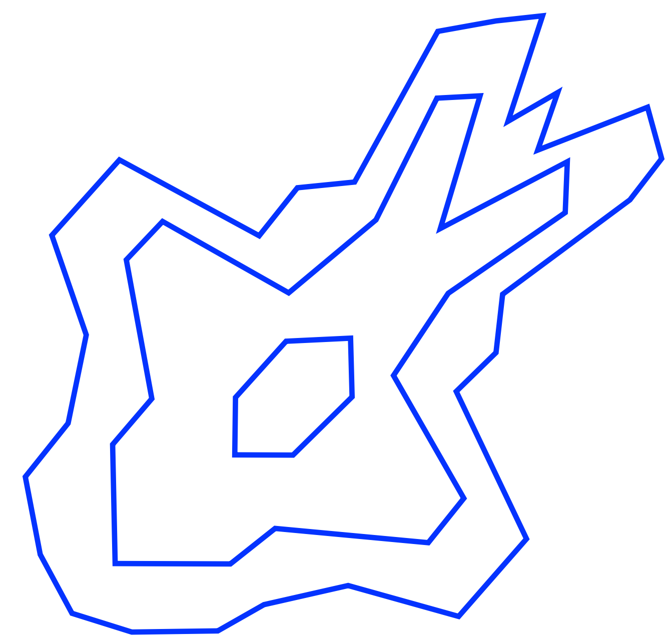

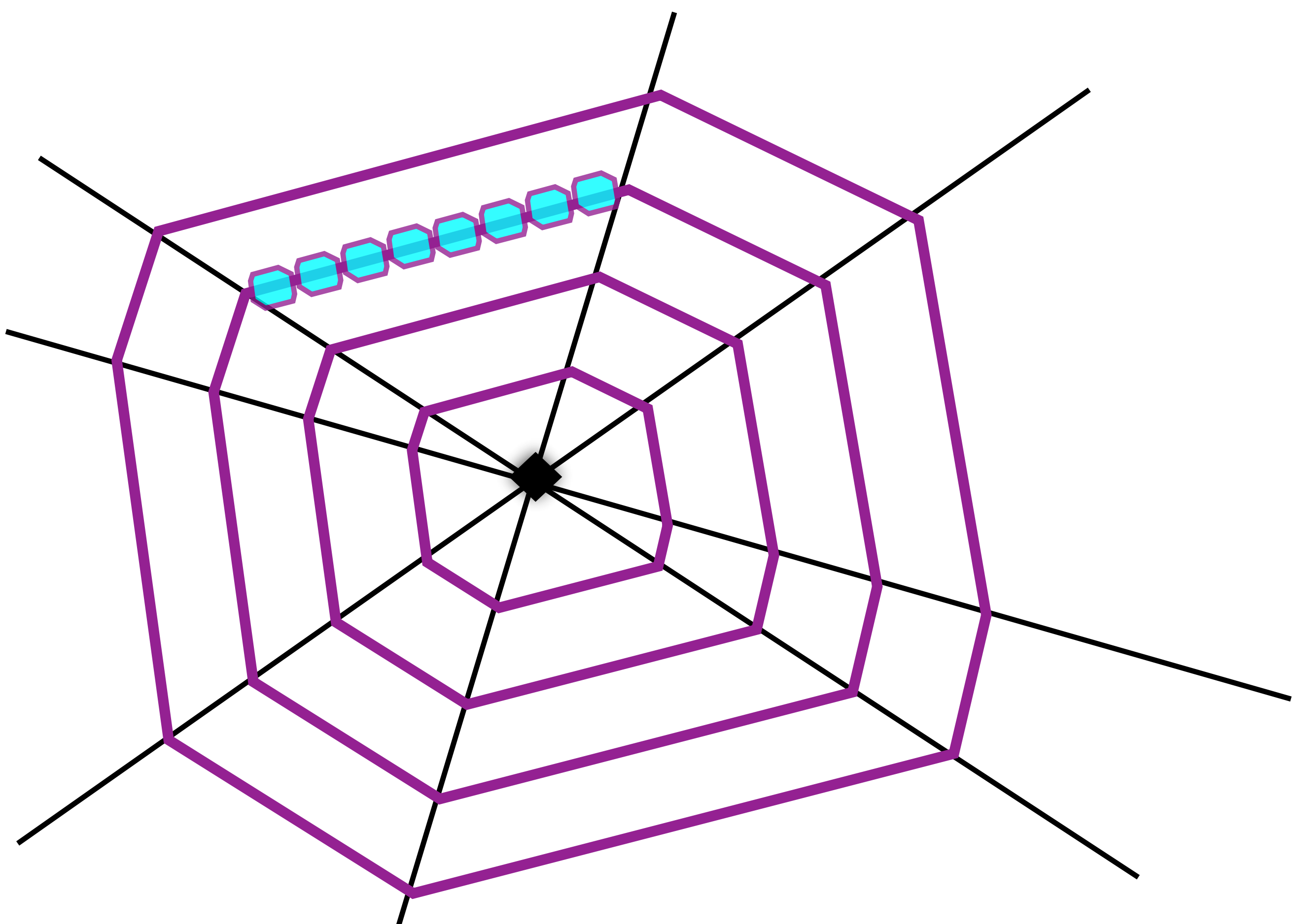

Invariant necklaces of outer billiards of rational polygons.

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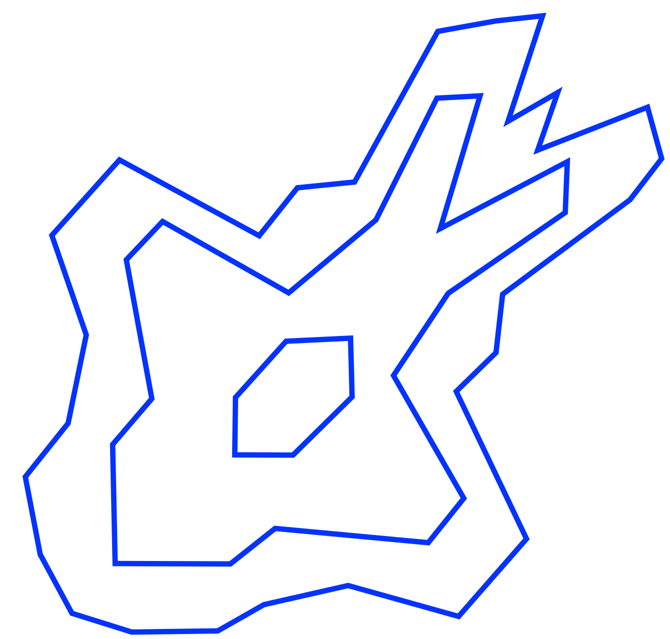

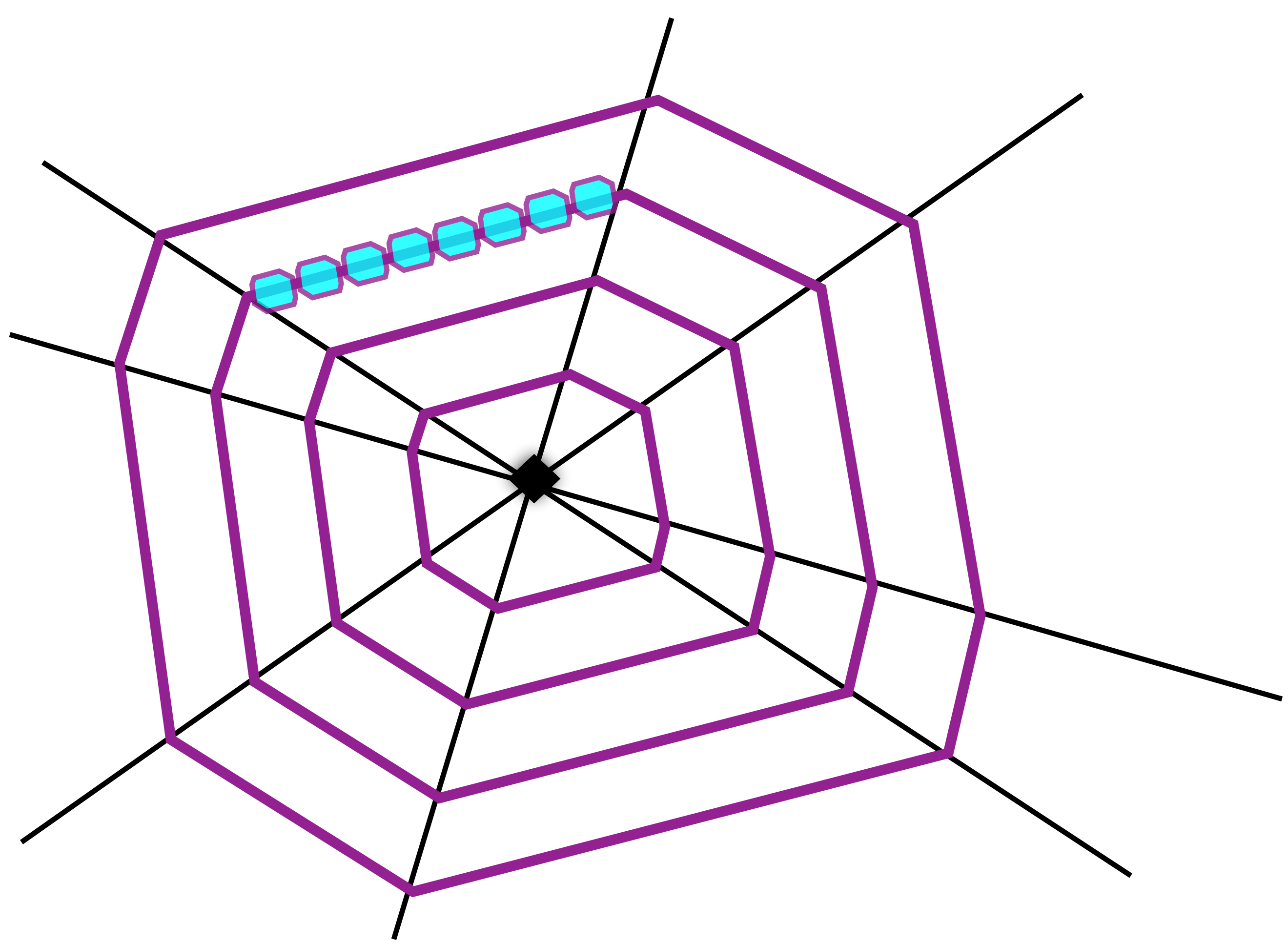

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Infinitely many invariant lattices.

Linear rotations on lattices

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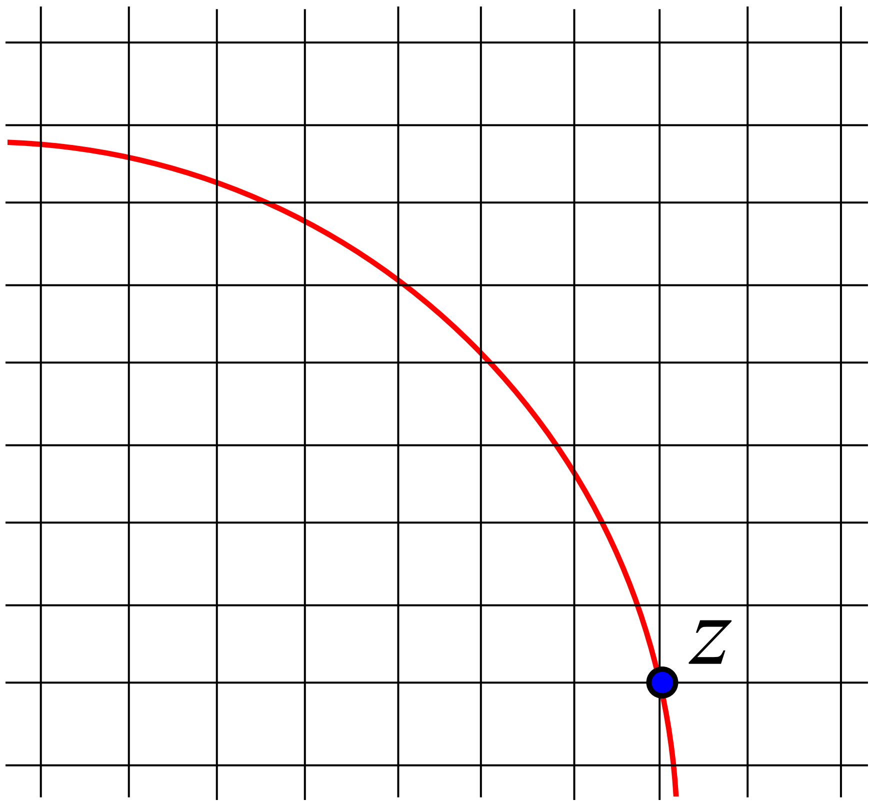

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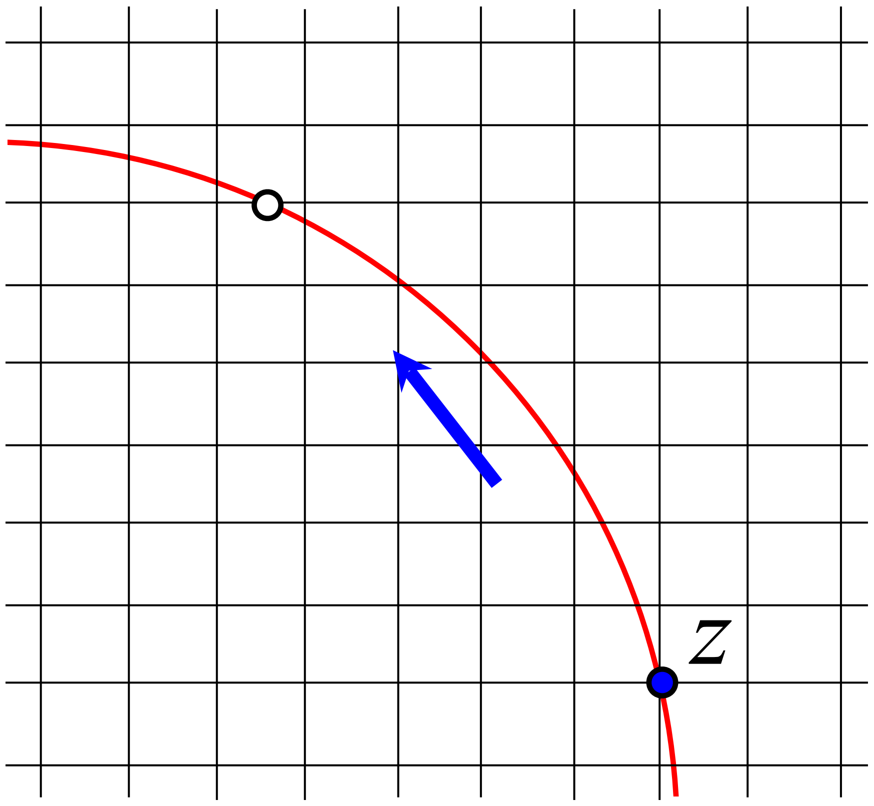

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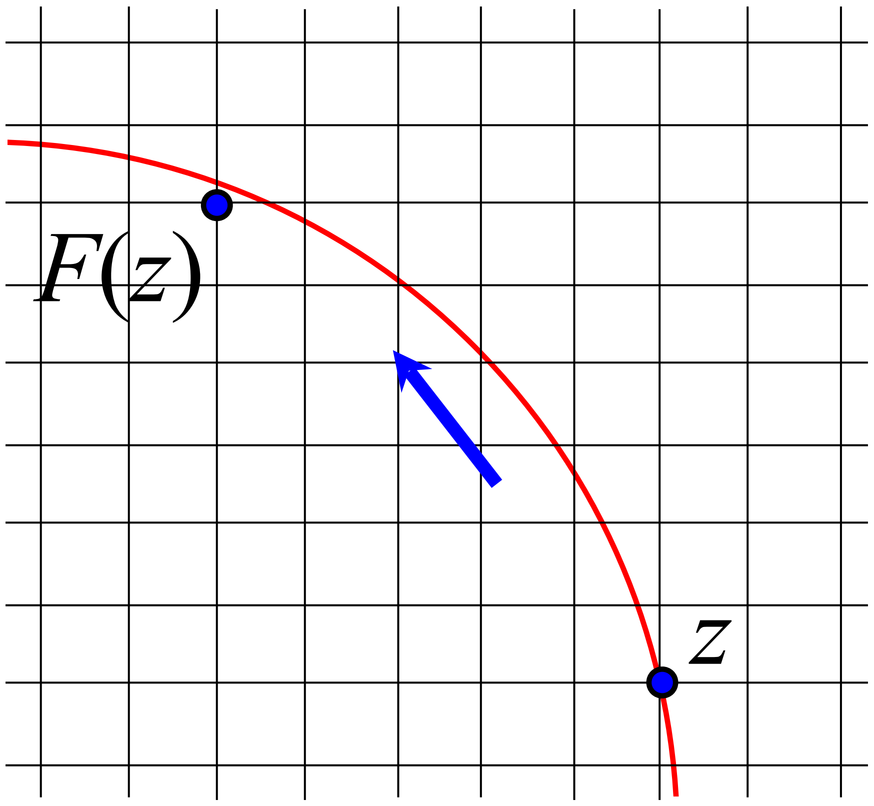

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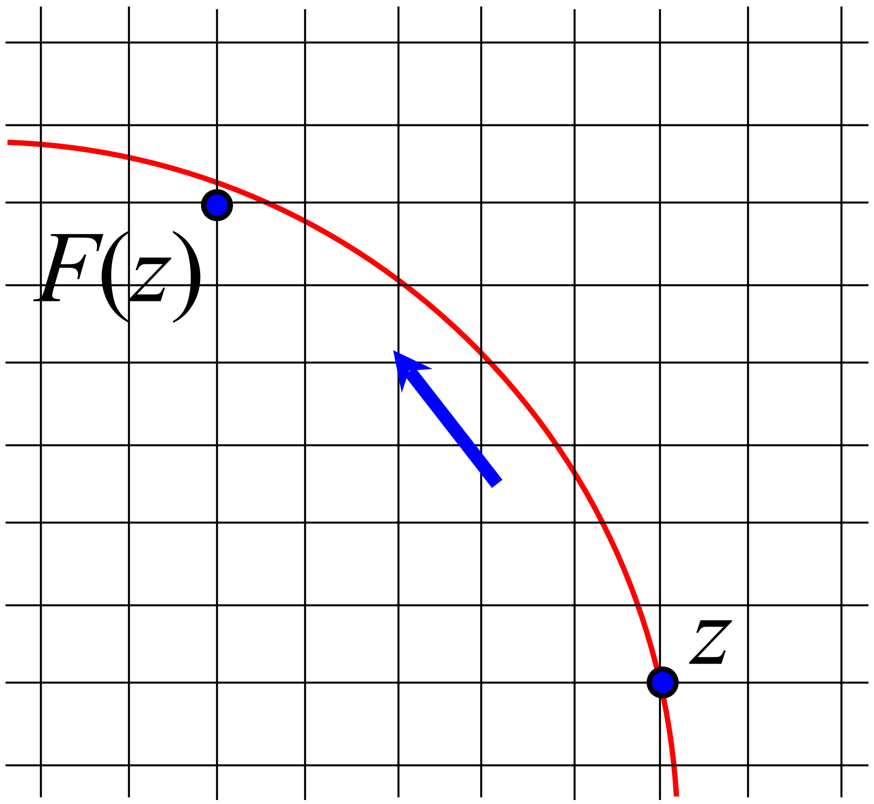

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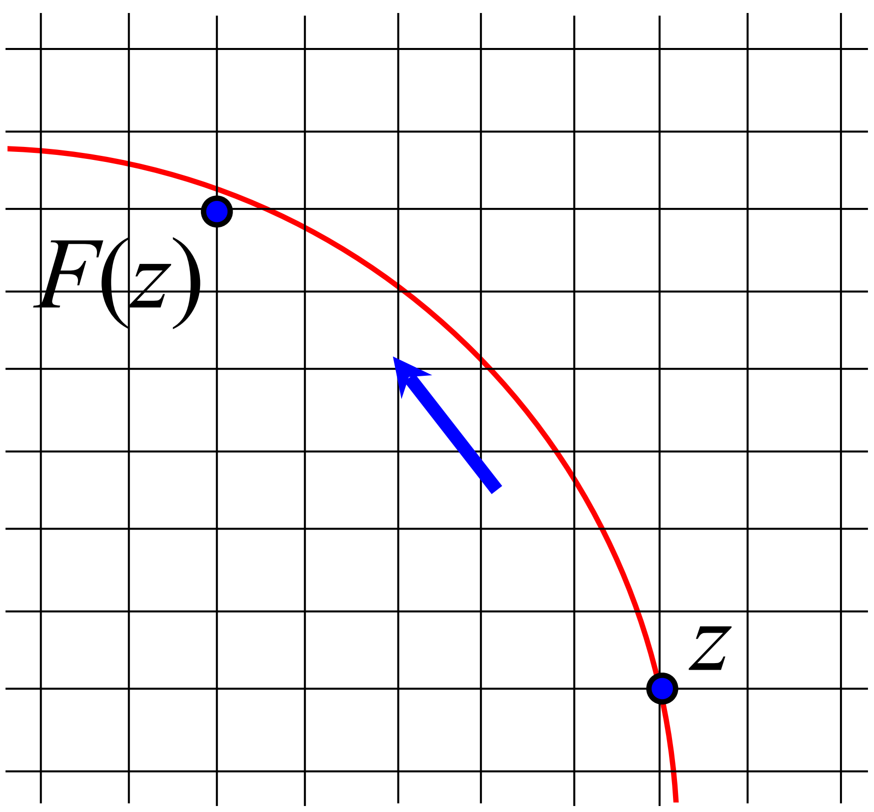

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rational (irrational) rotation number:

without round-off, all orbits are periodic (quasi-periodic).

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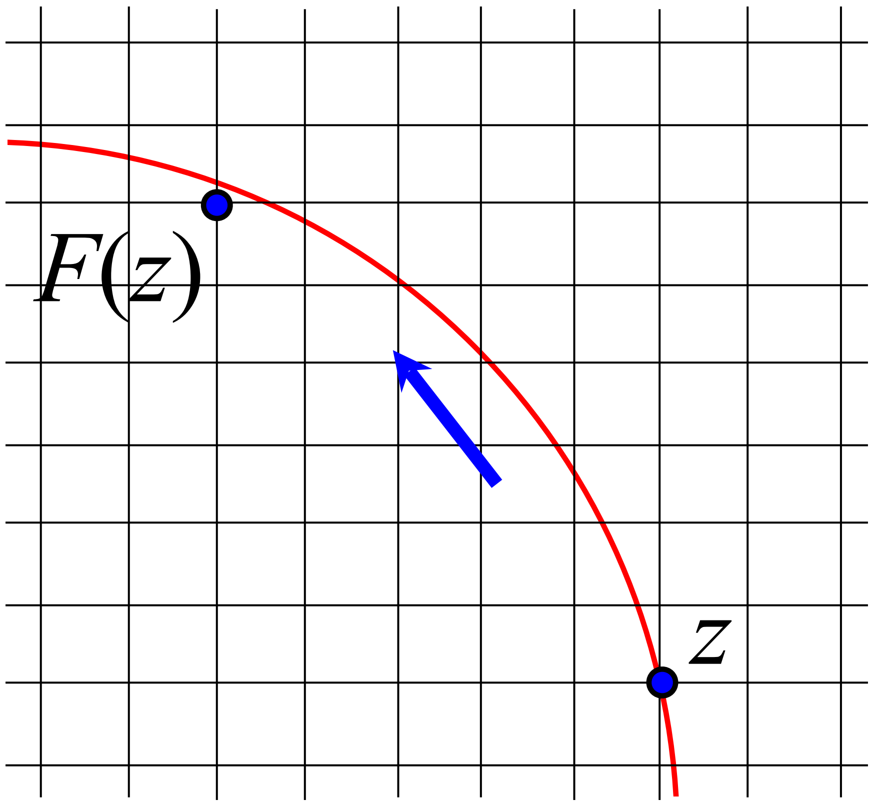

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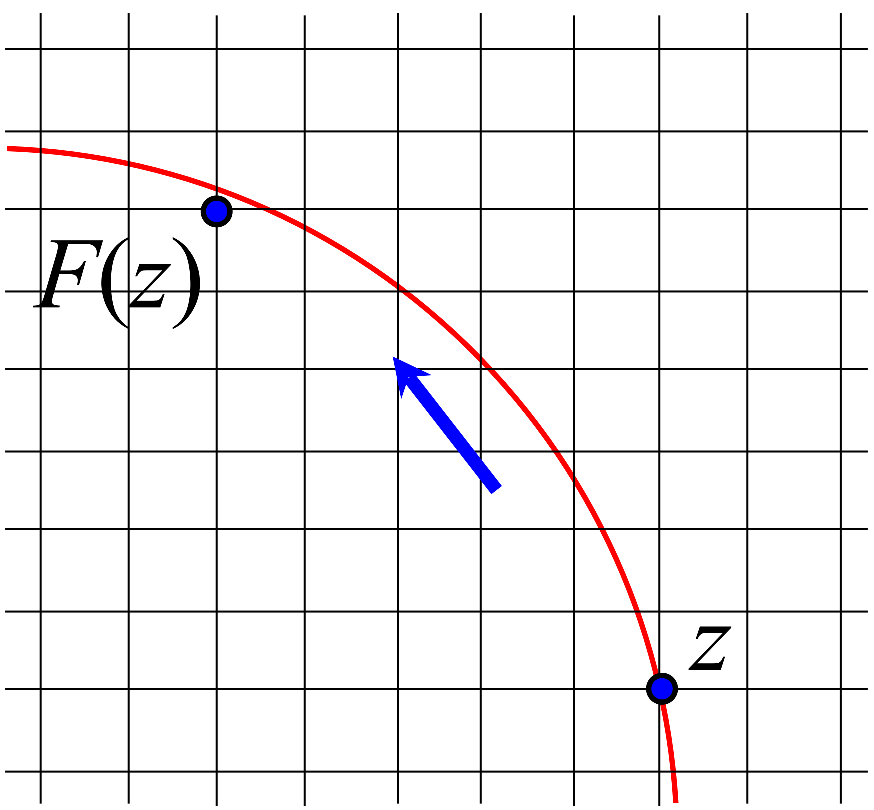

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Proved for only eight (non-trivial) values of the parameter.

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If $\nu = p/q$, then the map F^q is close to the identity on $\mathbb{Z}^2$.

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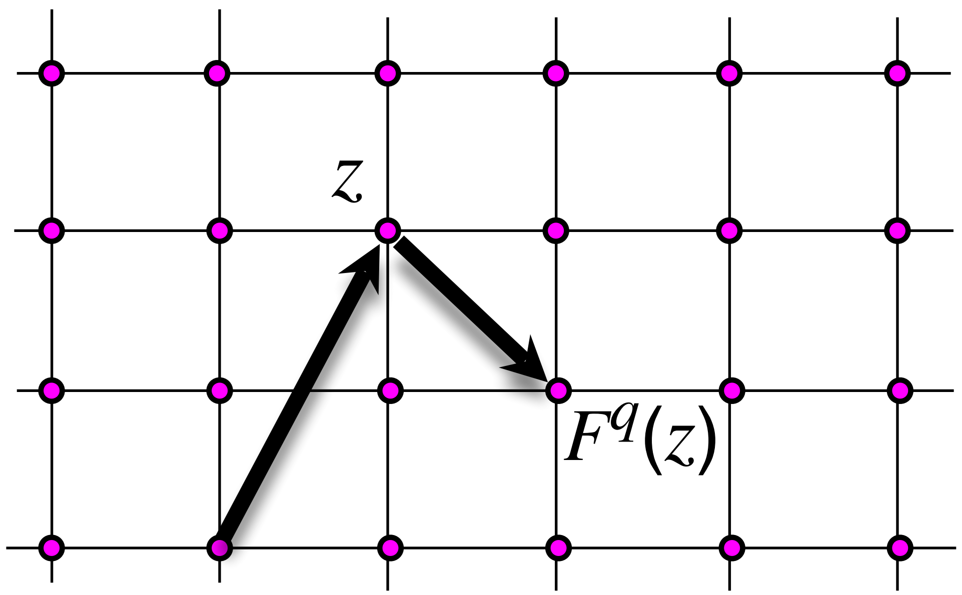

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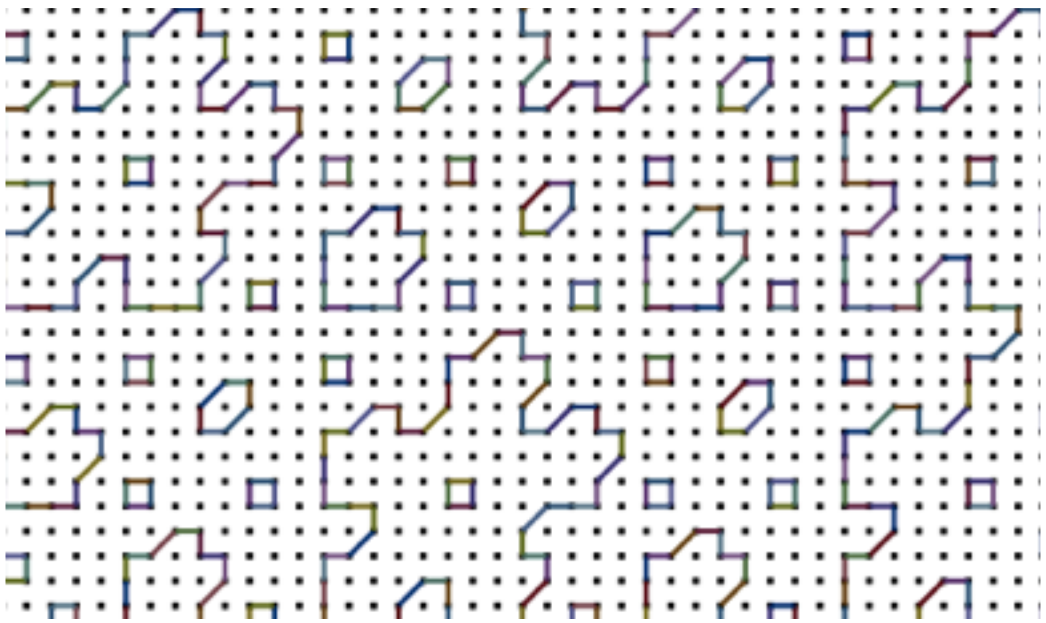

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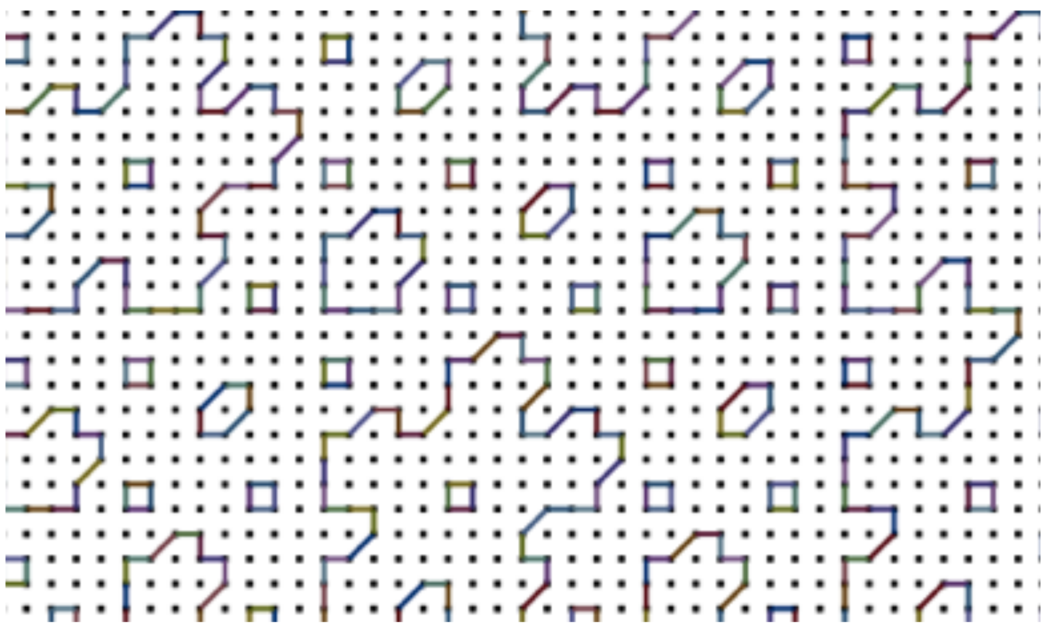

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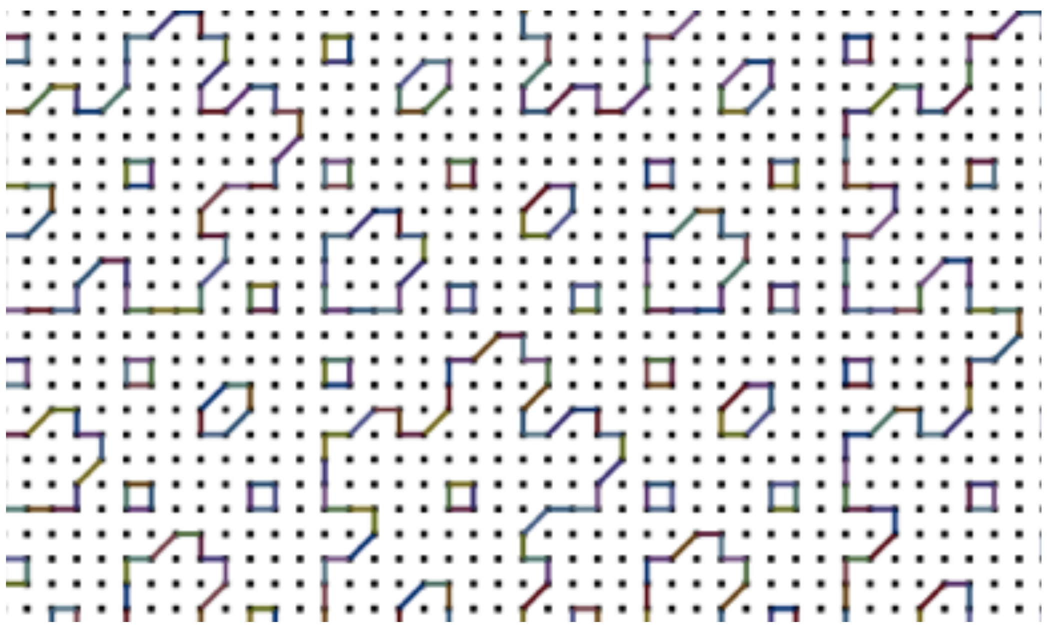

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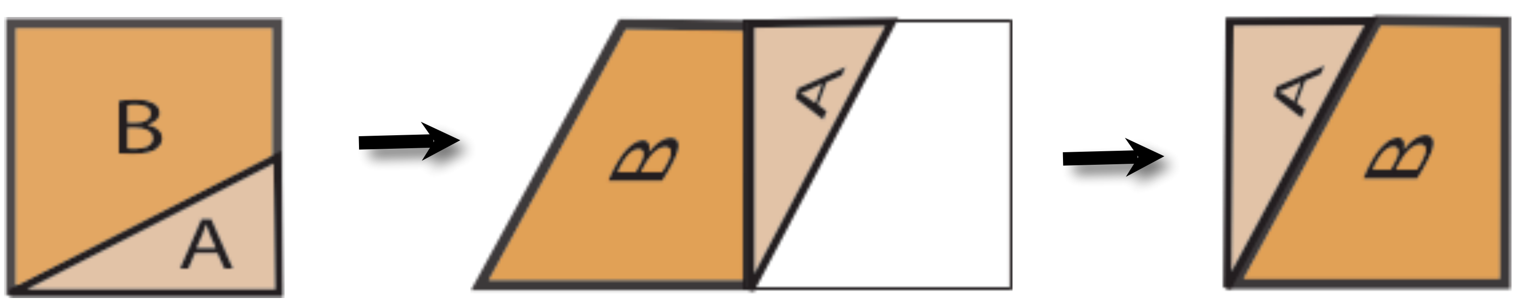

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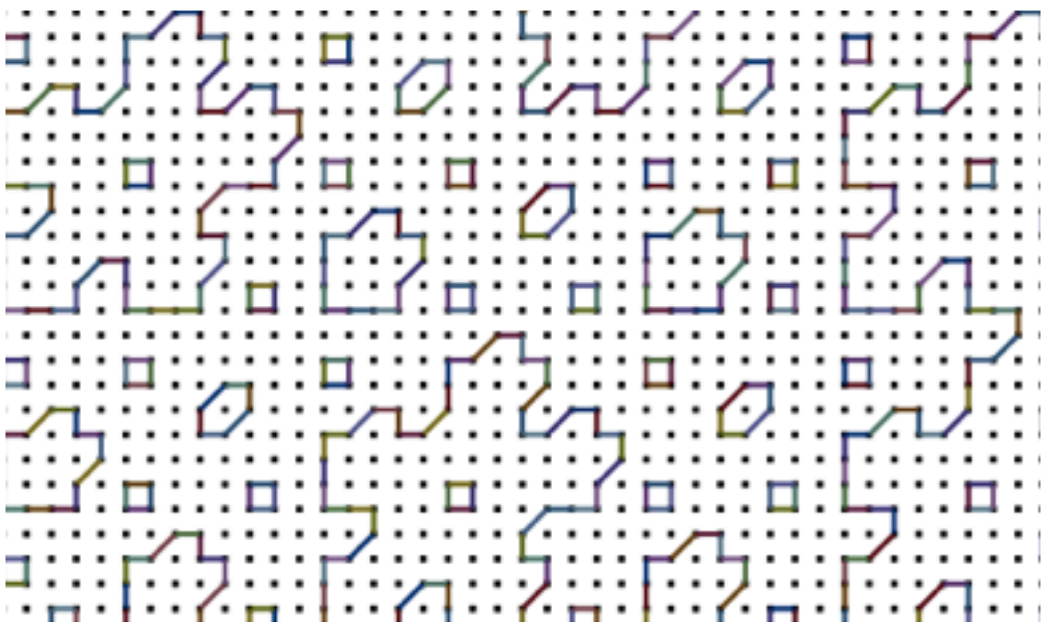

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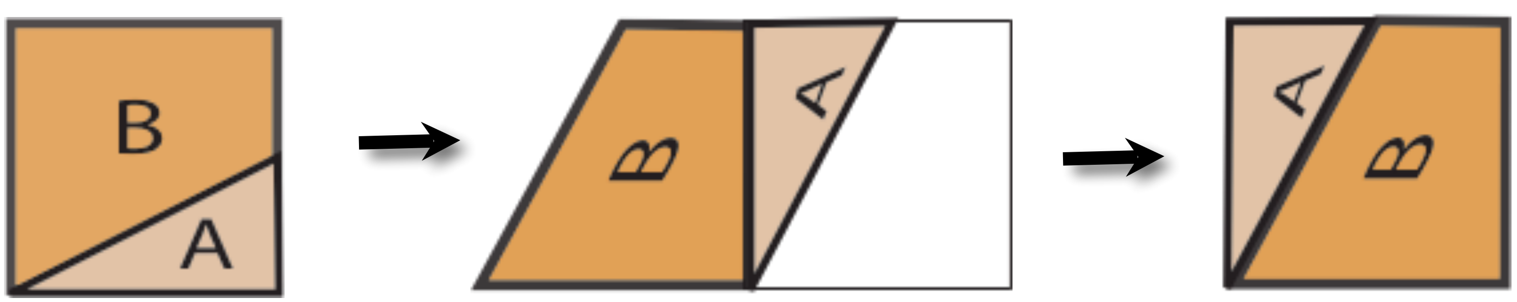

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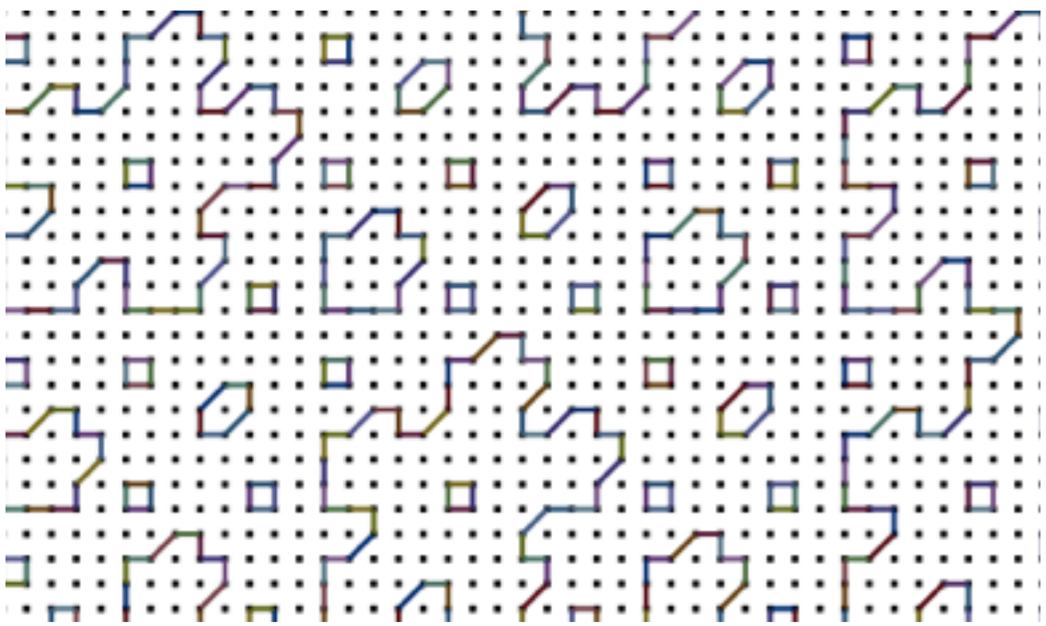

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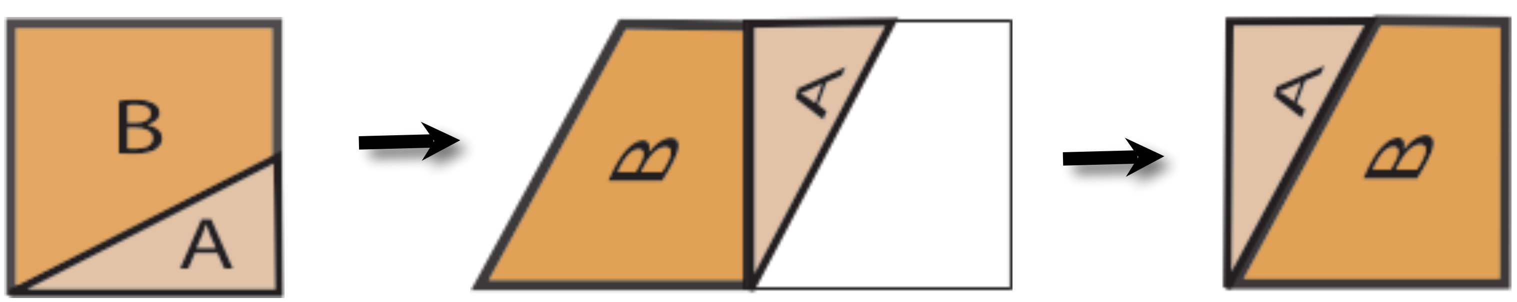

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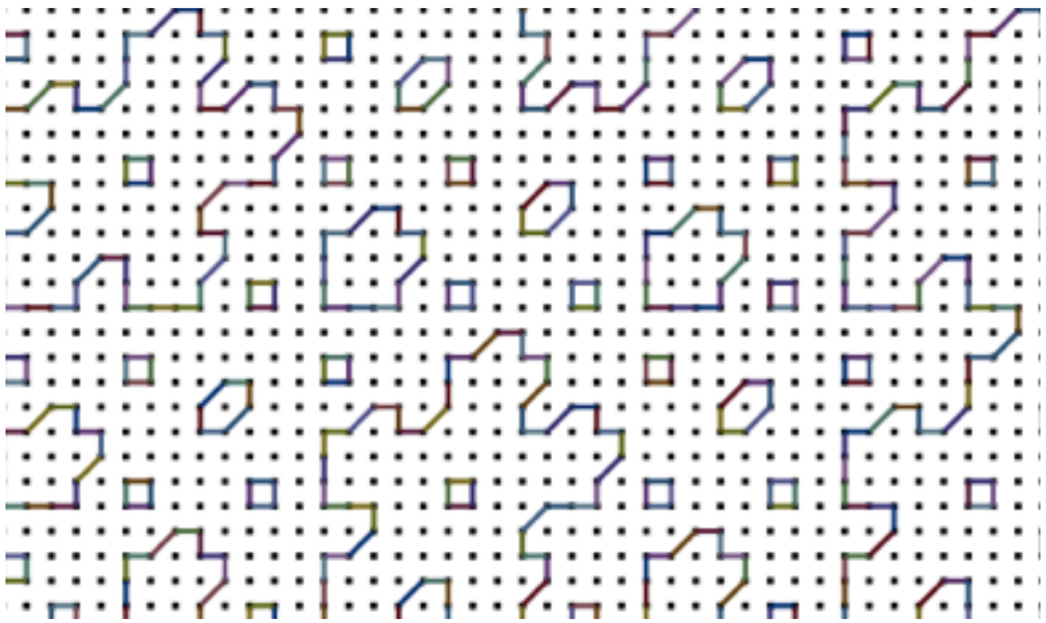

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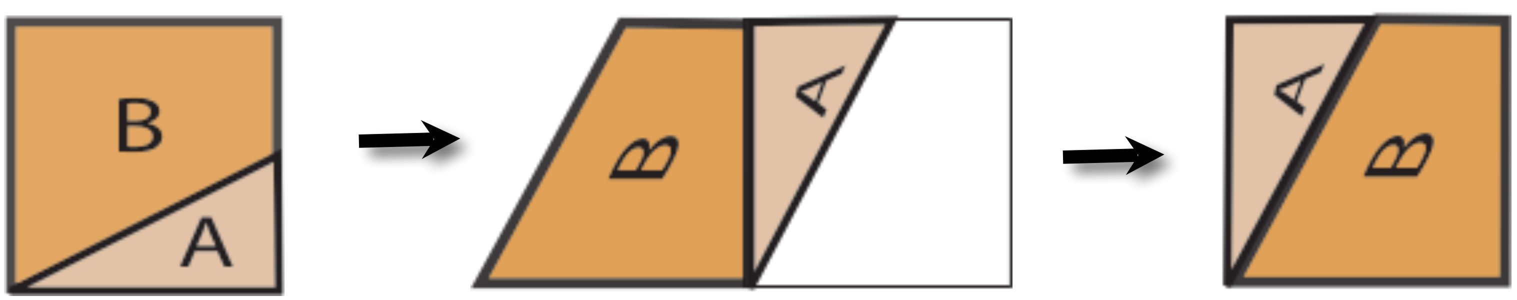

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PERIODICITY PROOFS:
Lowenstein, Hatjispyros & fv (1997),
Koupstov, Lowenstein & fv (2002),
Akiyama, Brunotte, Pethö & Steiner (2008).

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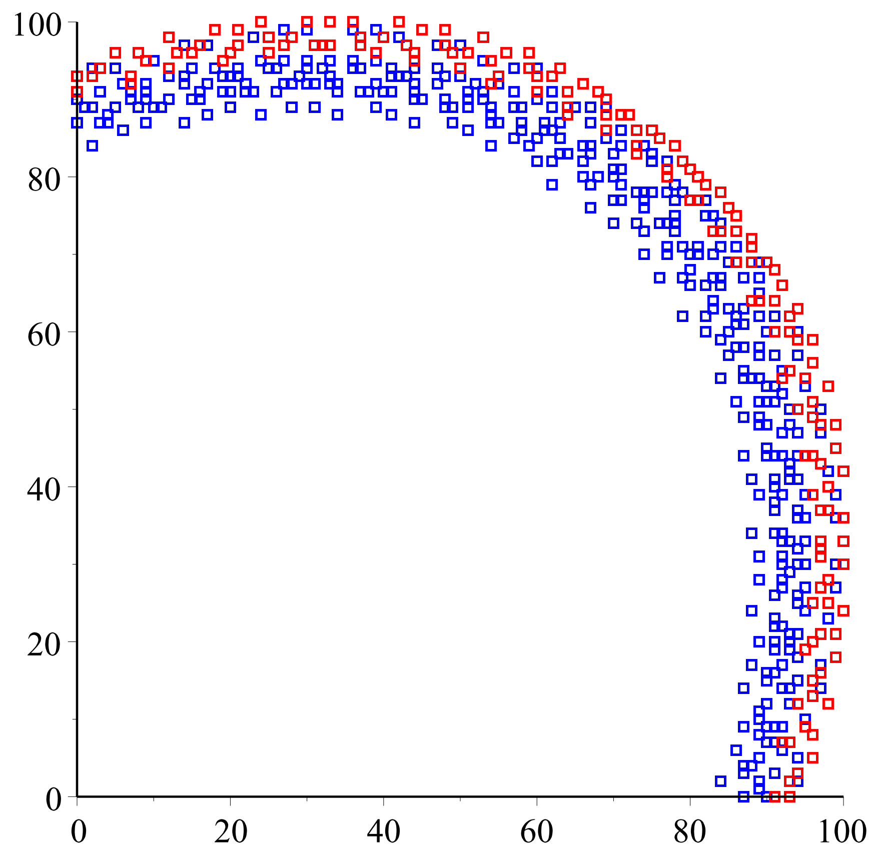

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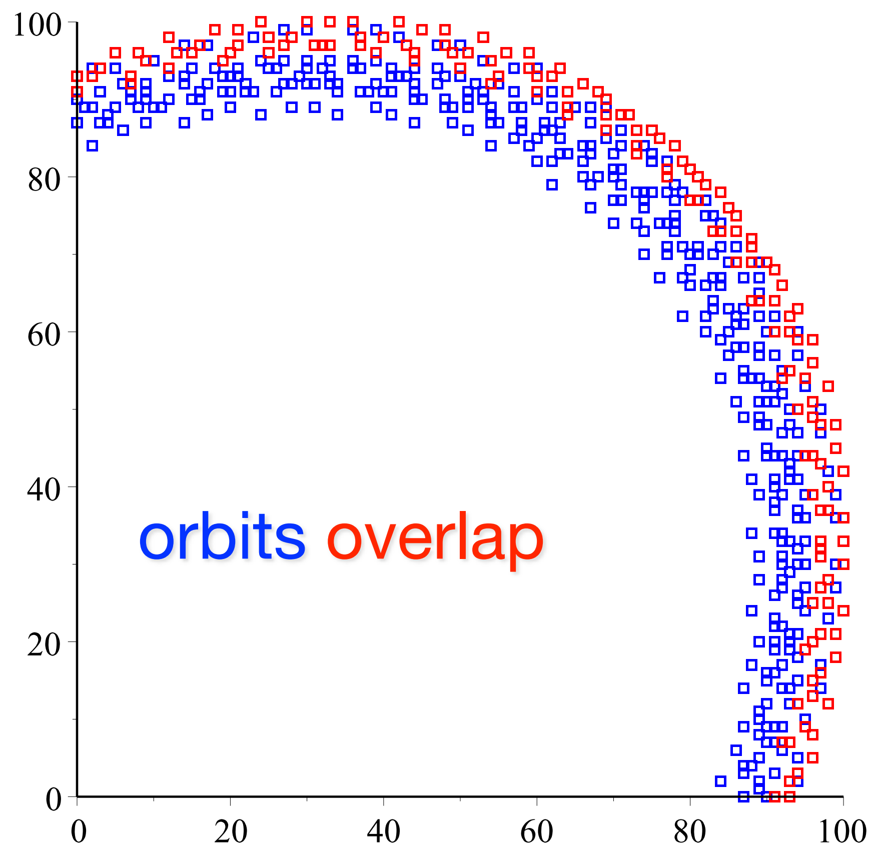

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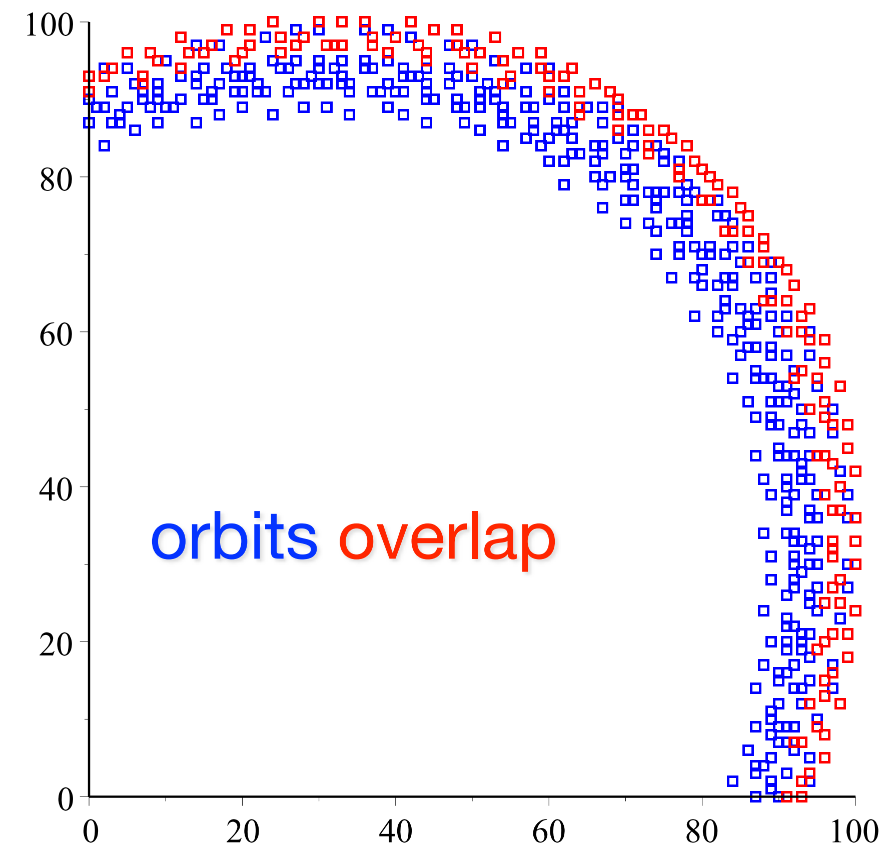

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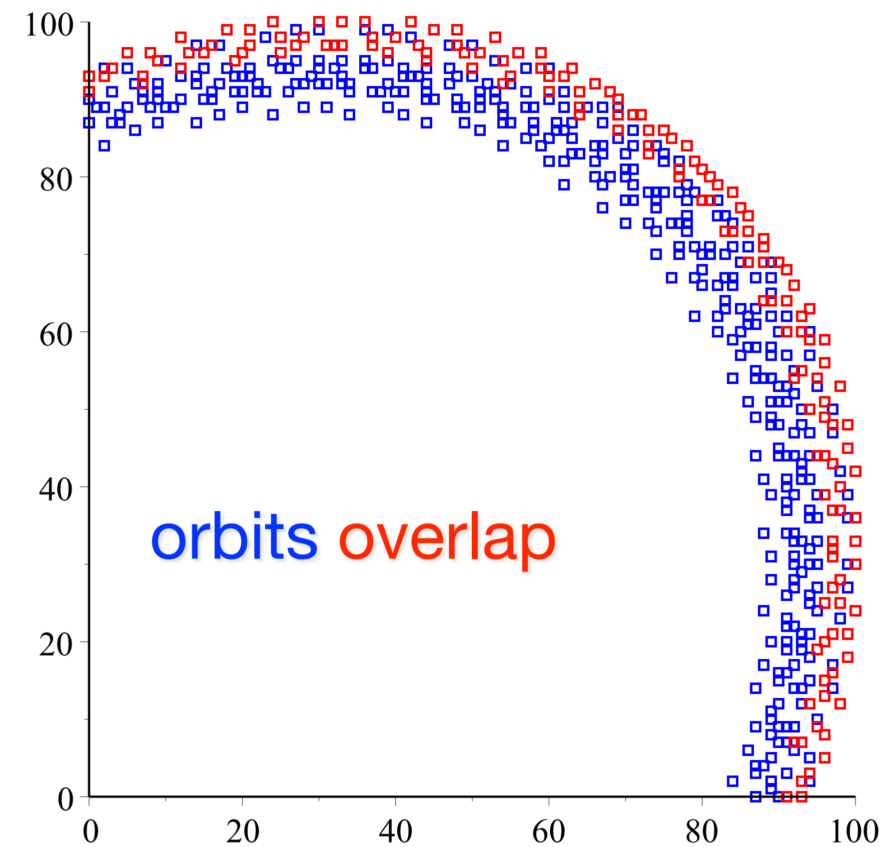
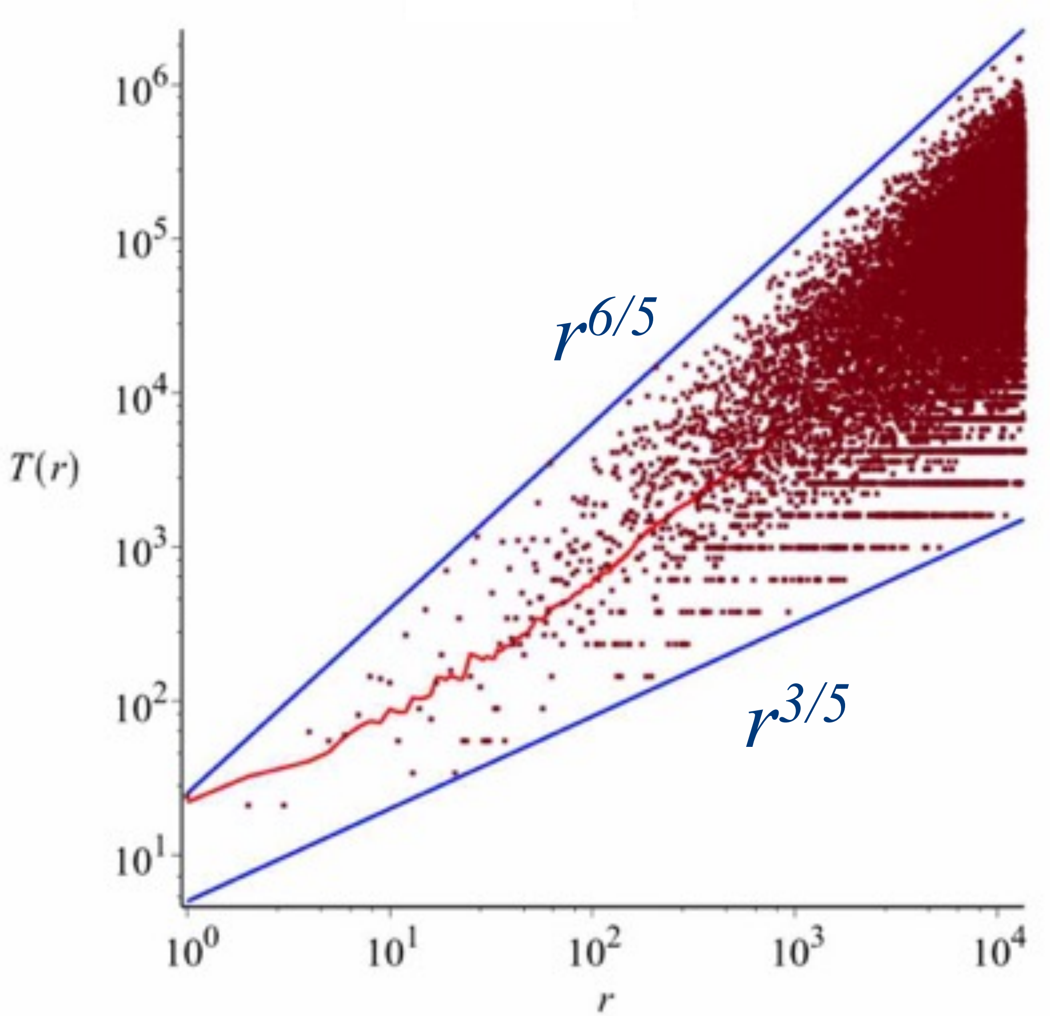


period vs radius



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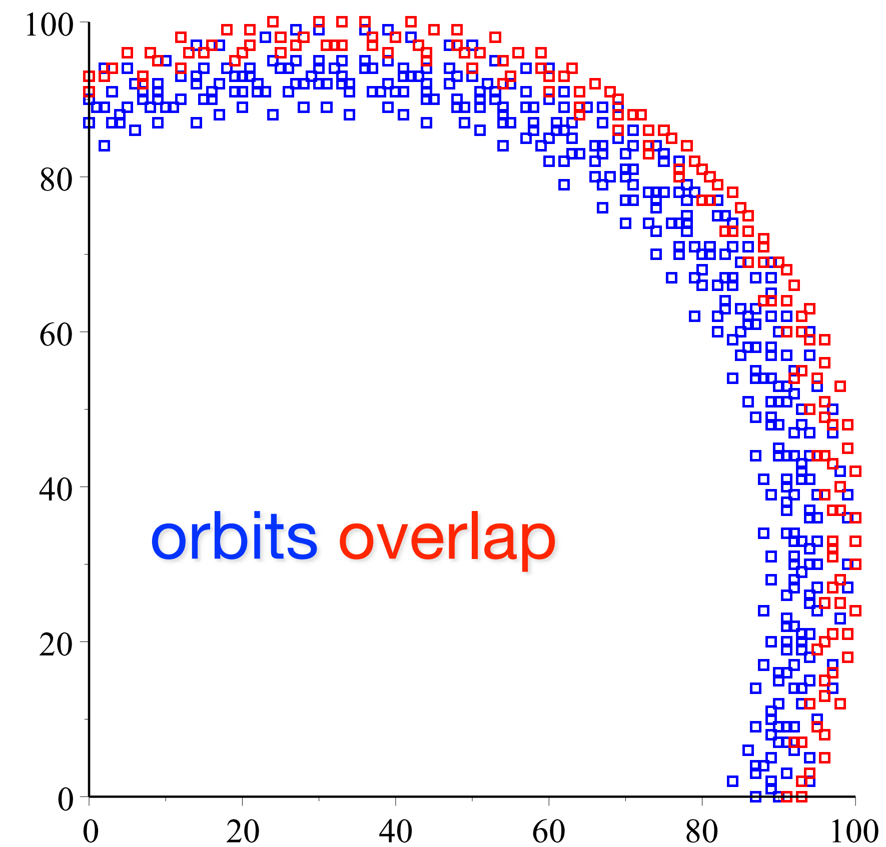

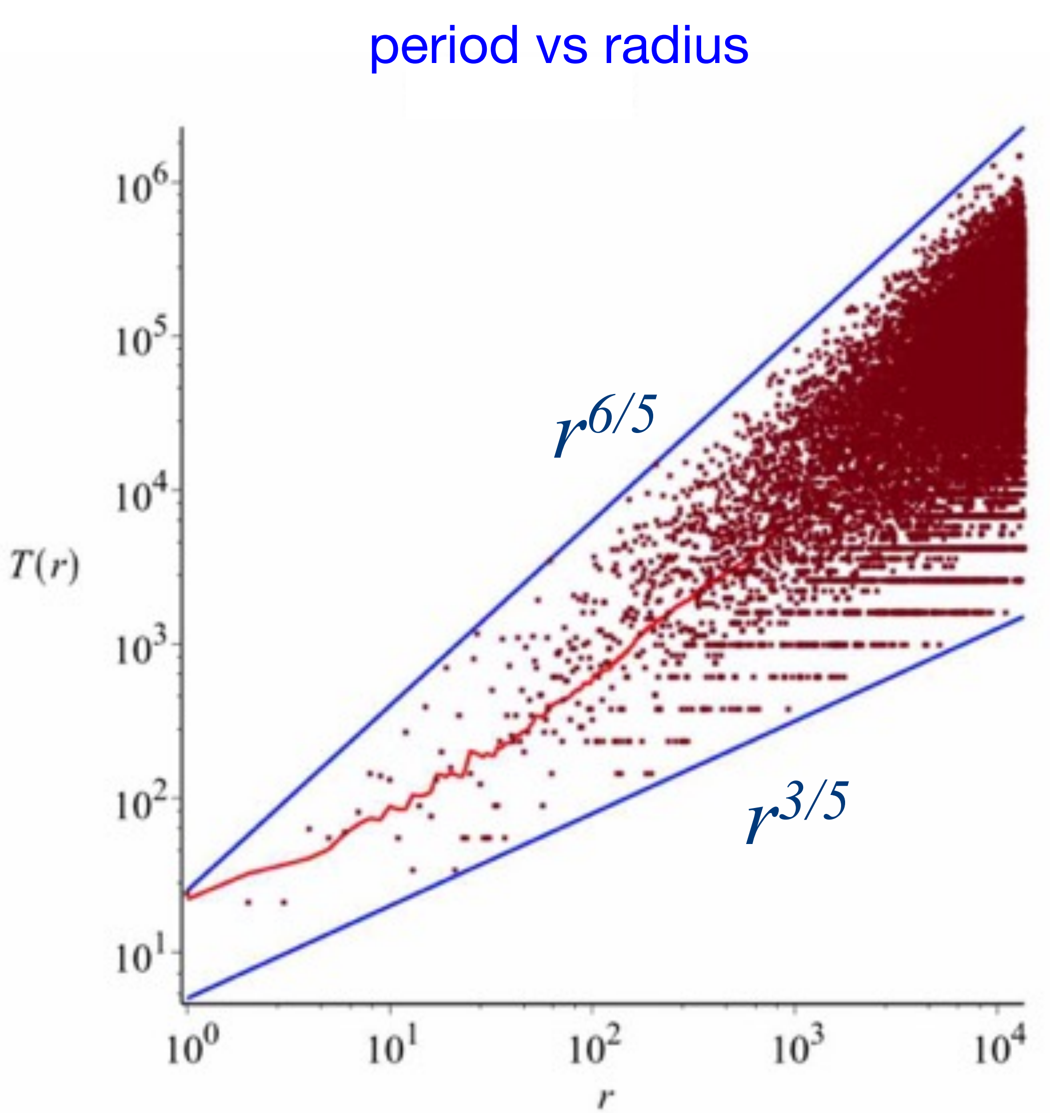

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- Very large fluctuations, with some structure.

What do we know?

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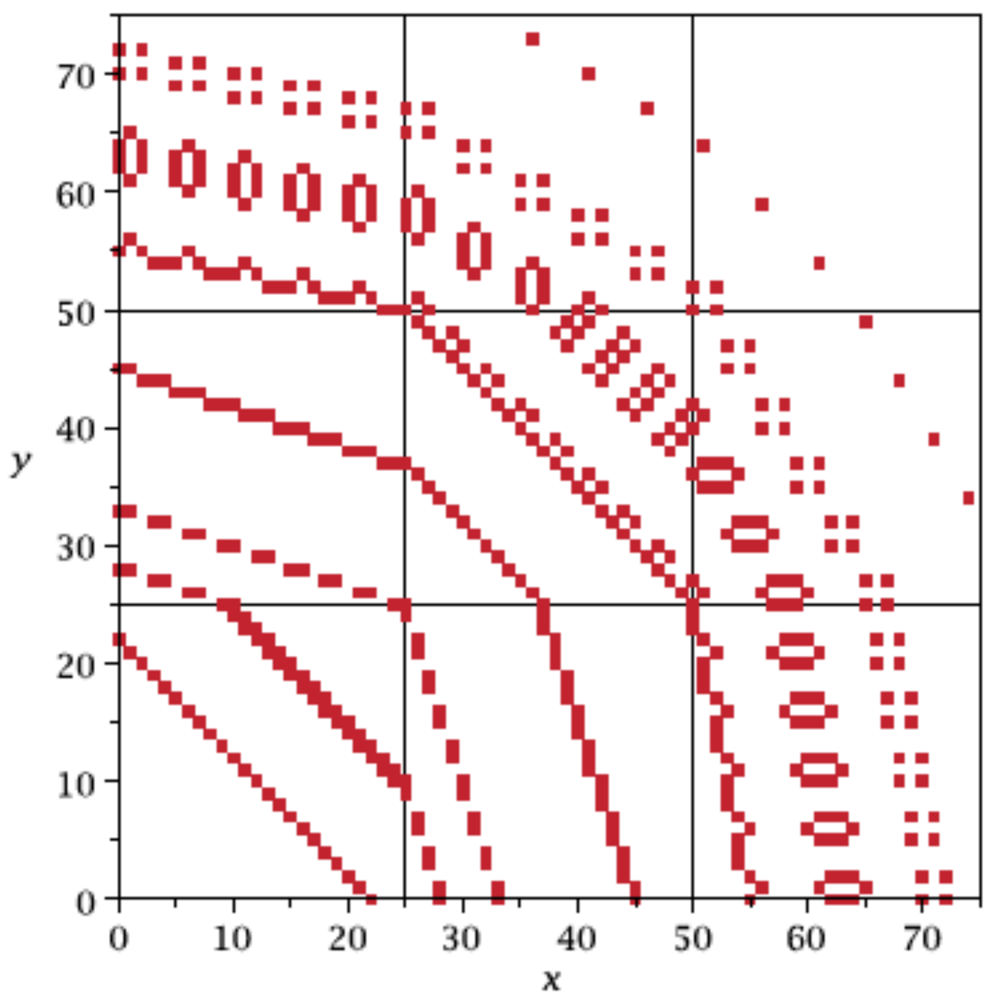

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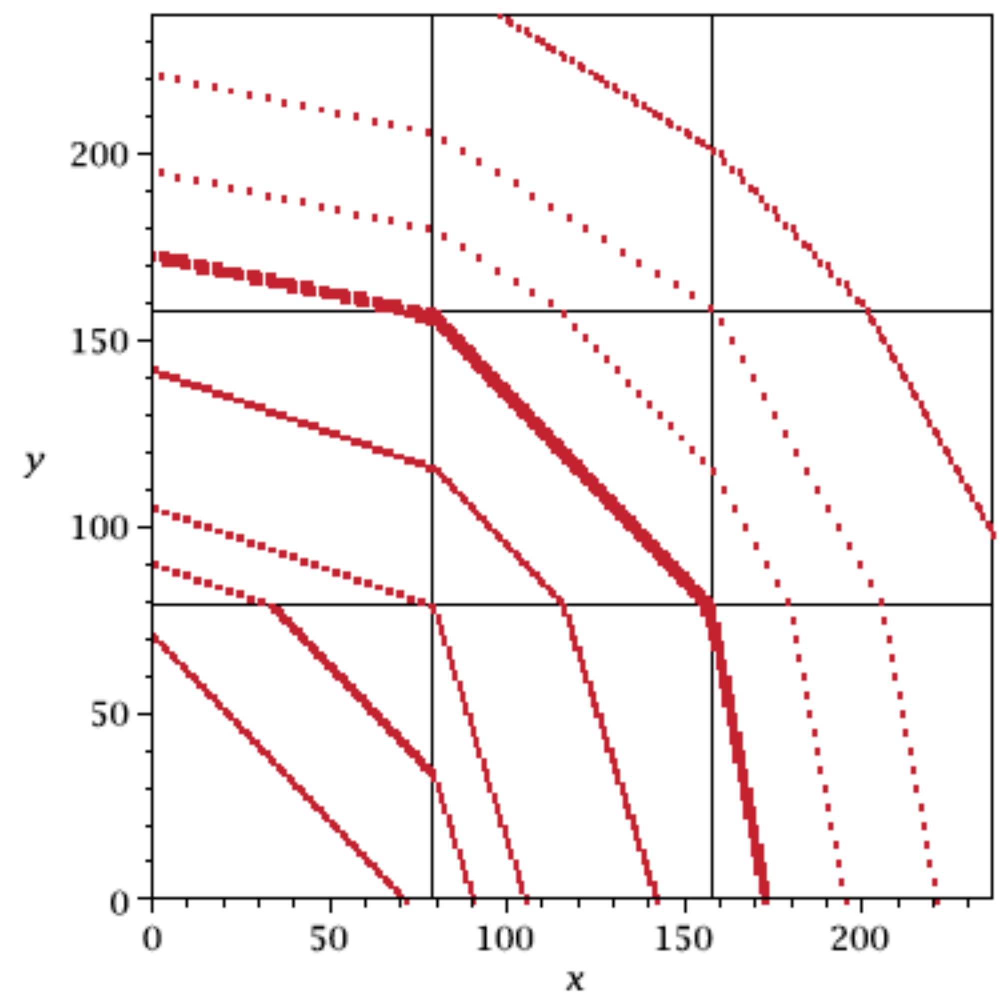

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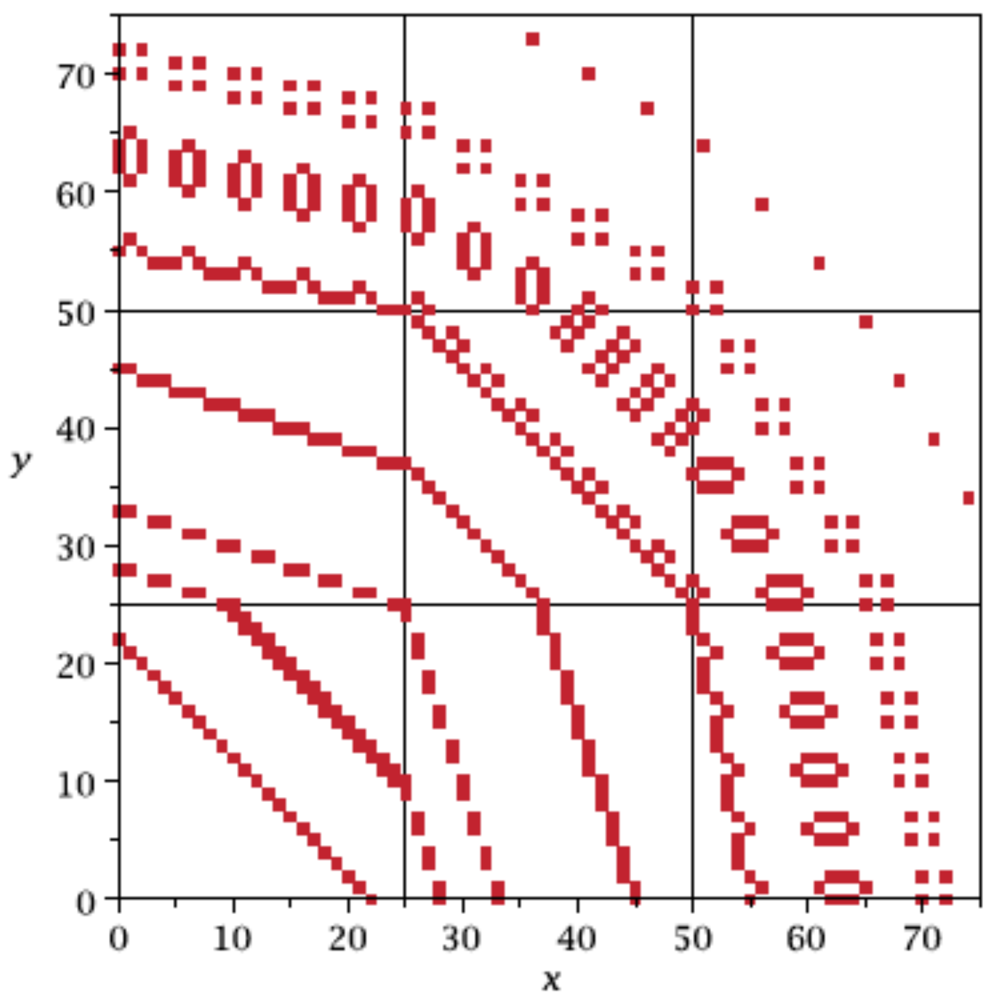

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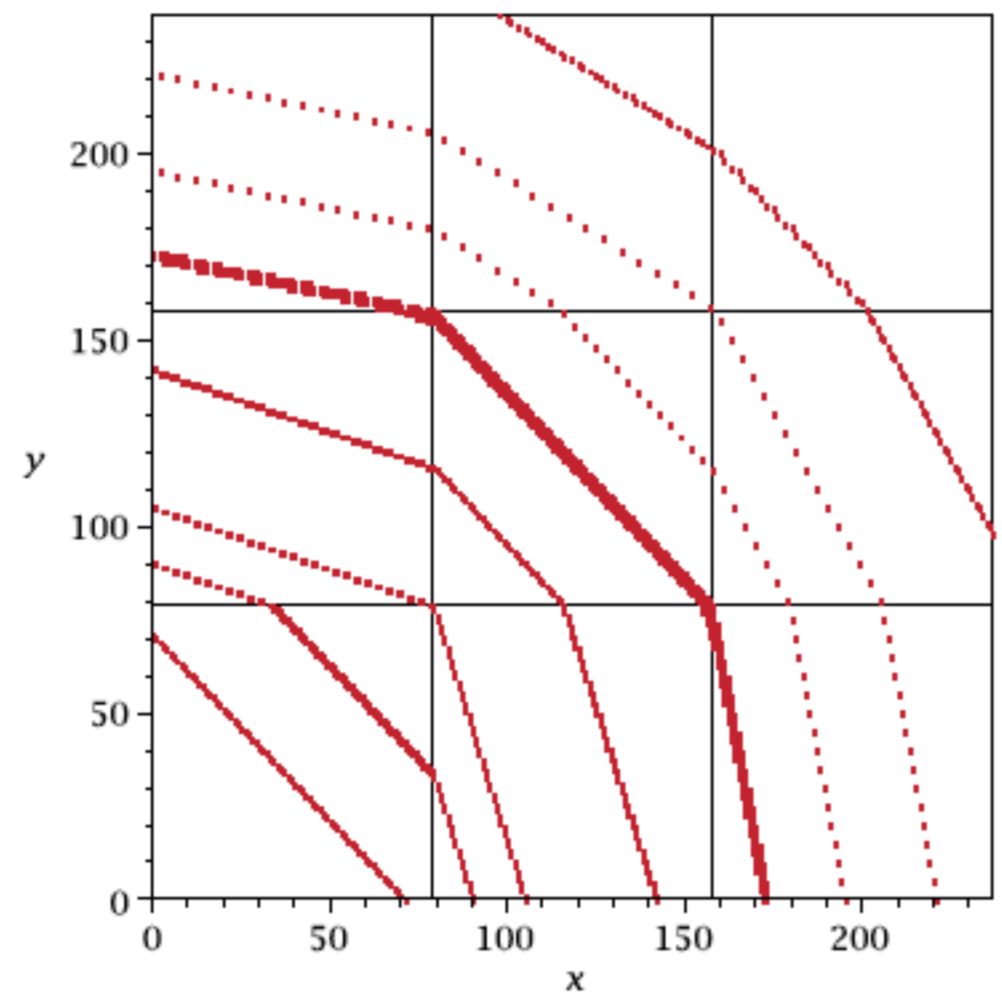

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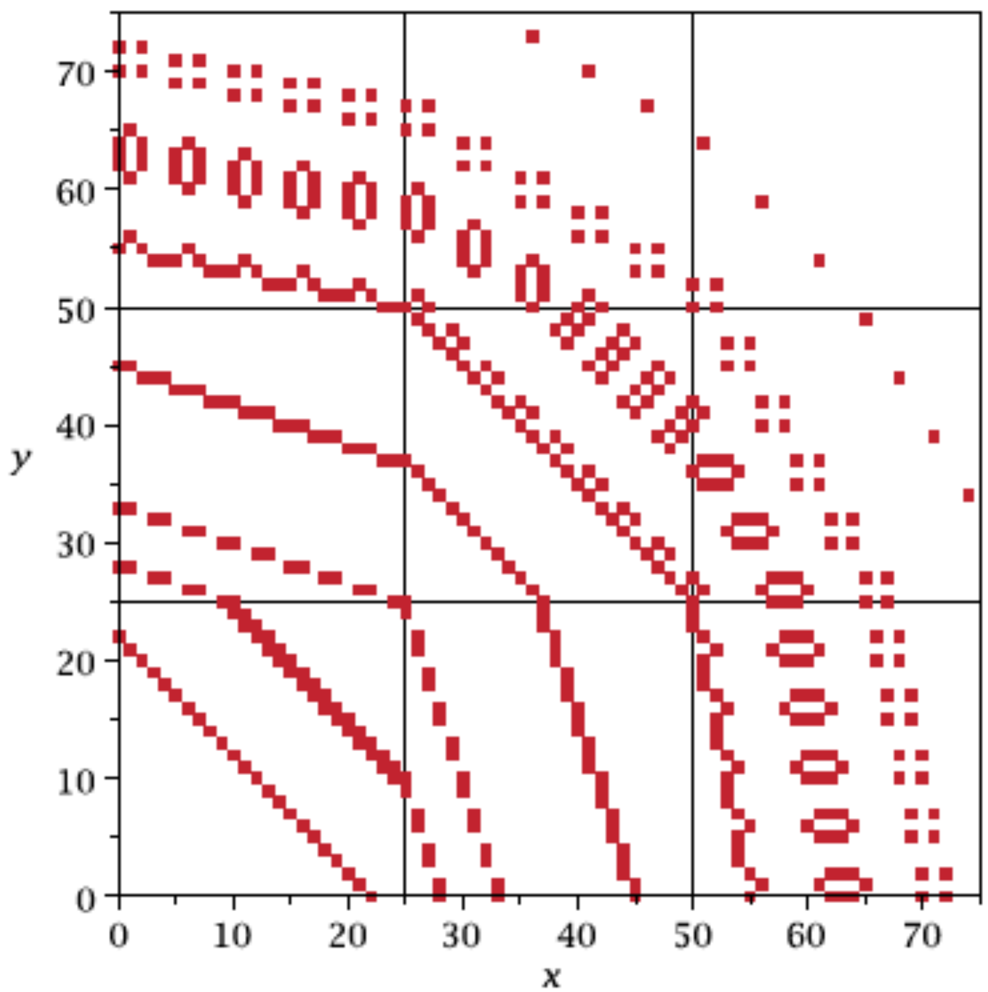

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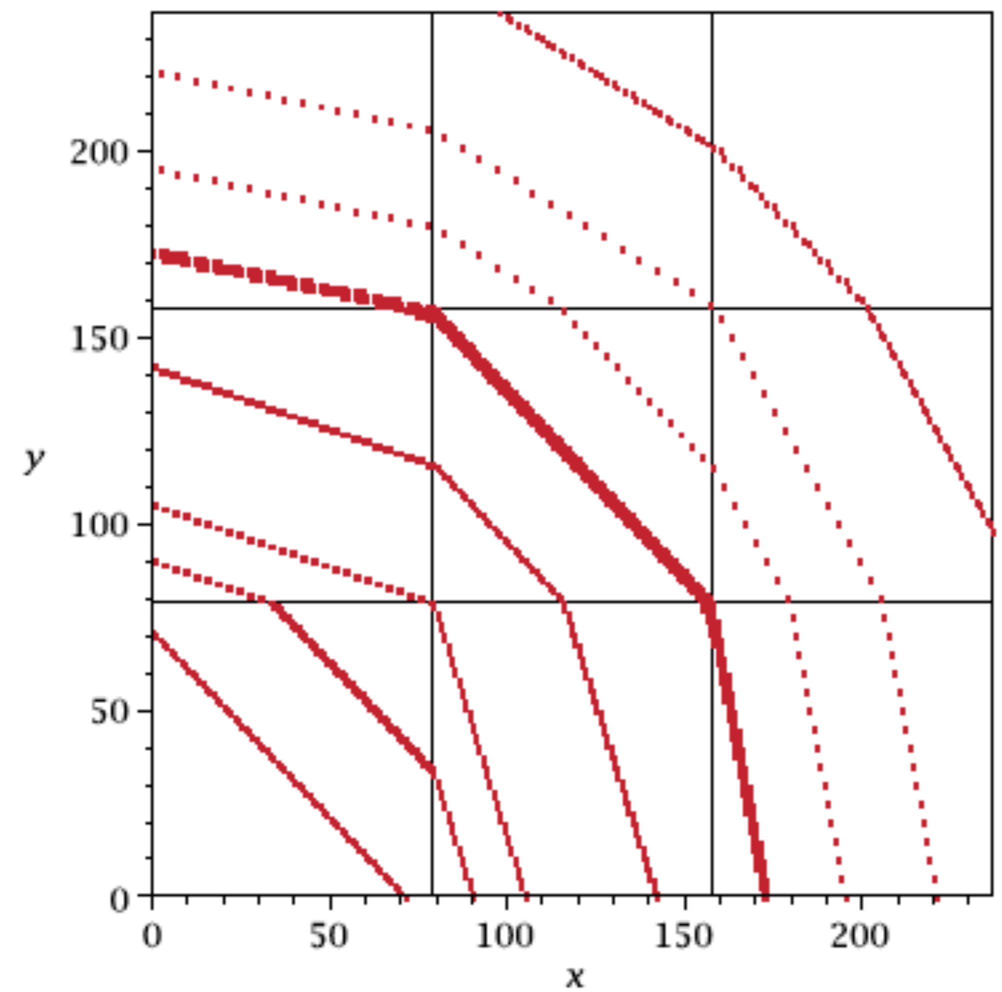

limit integrable system?

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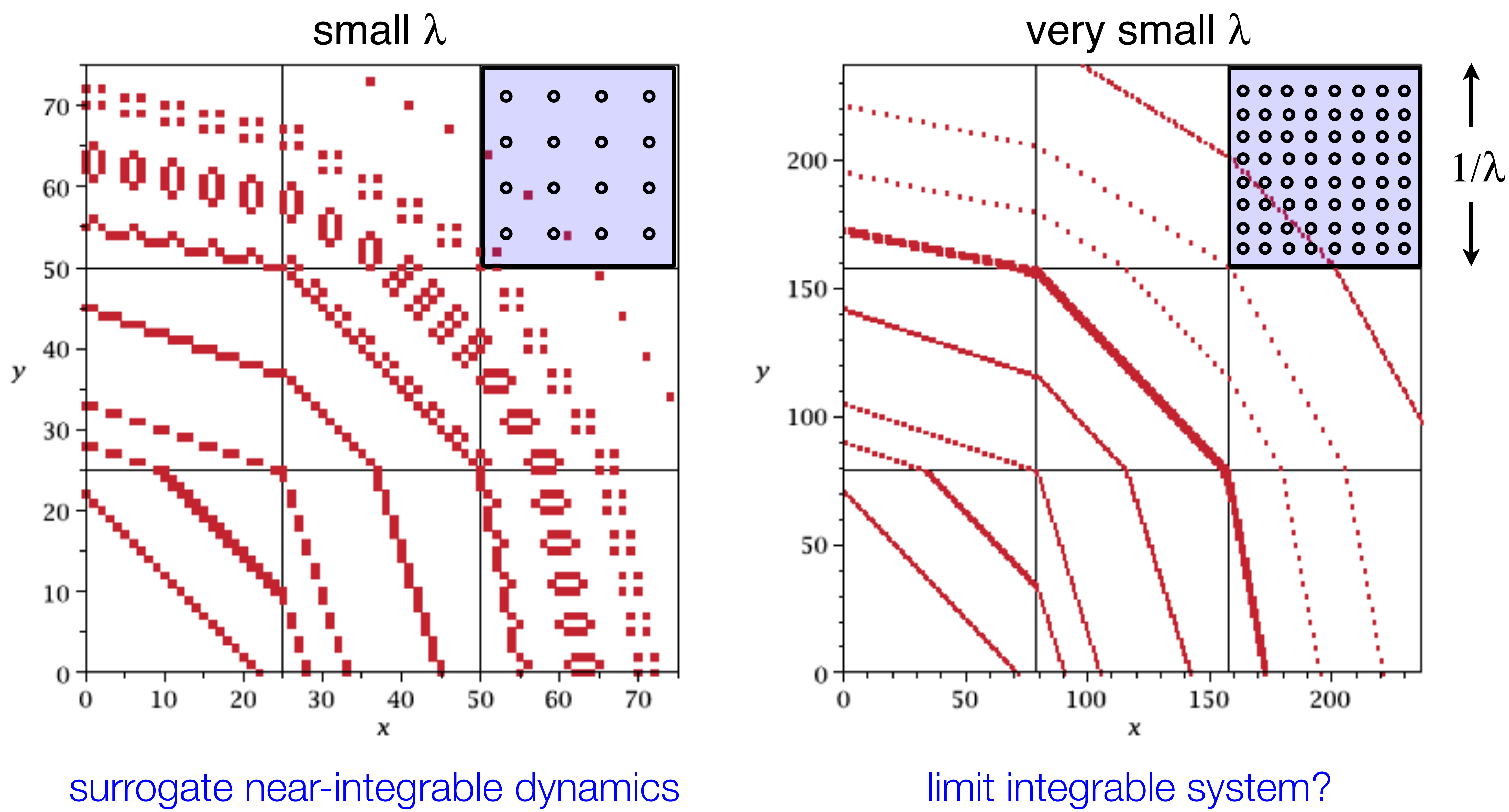
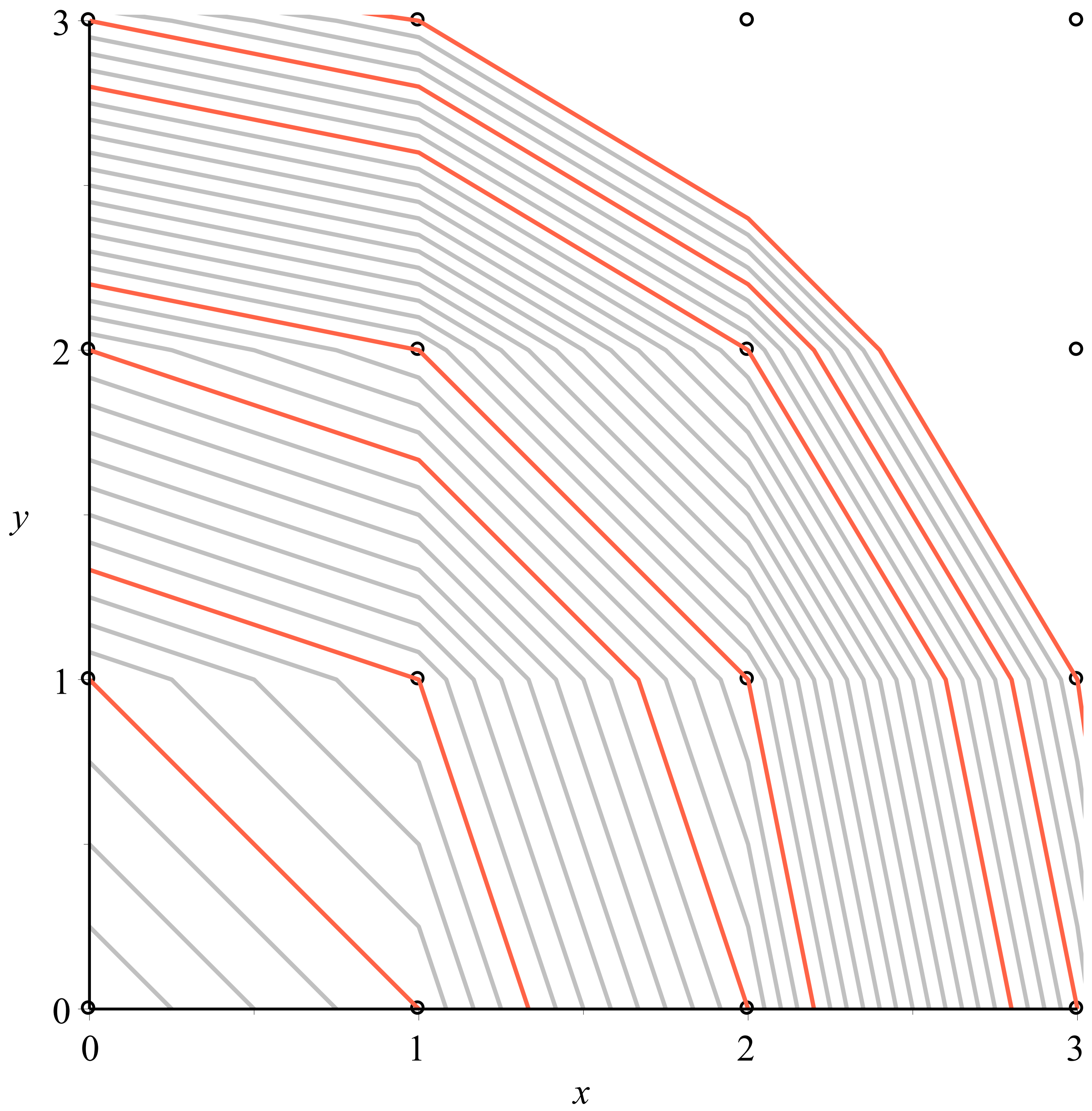


Scaling will be needed for convergence: $\mathbb{Z}^2 \to \lambda\mathbb{Z}^2$

The integrable system



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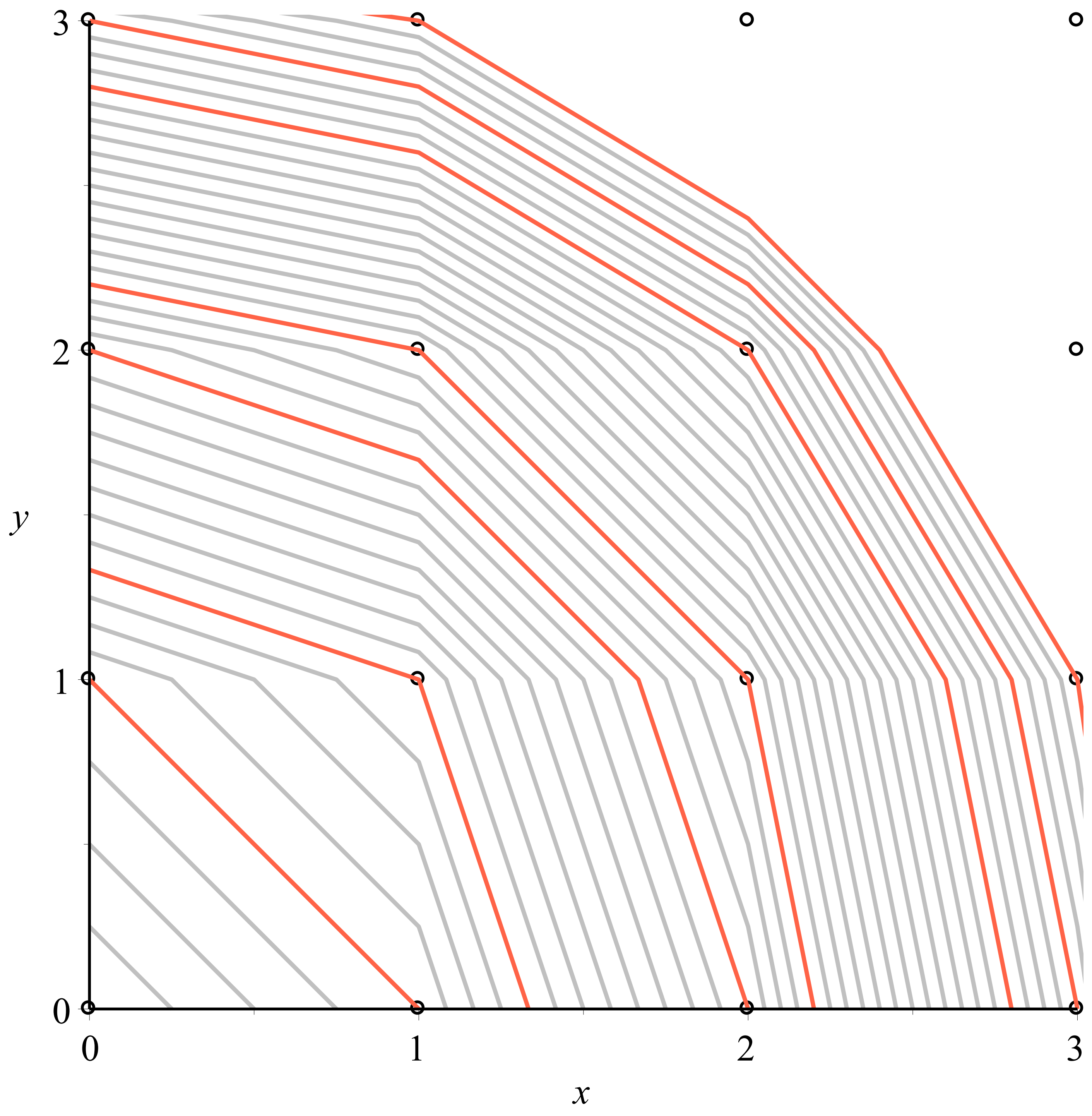

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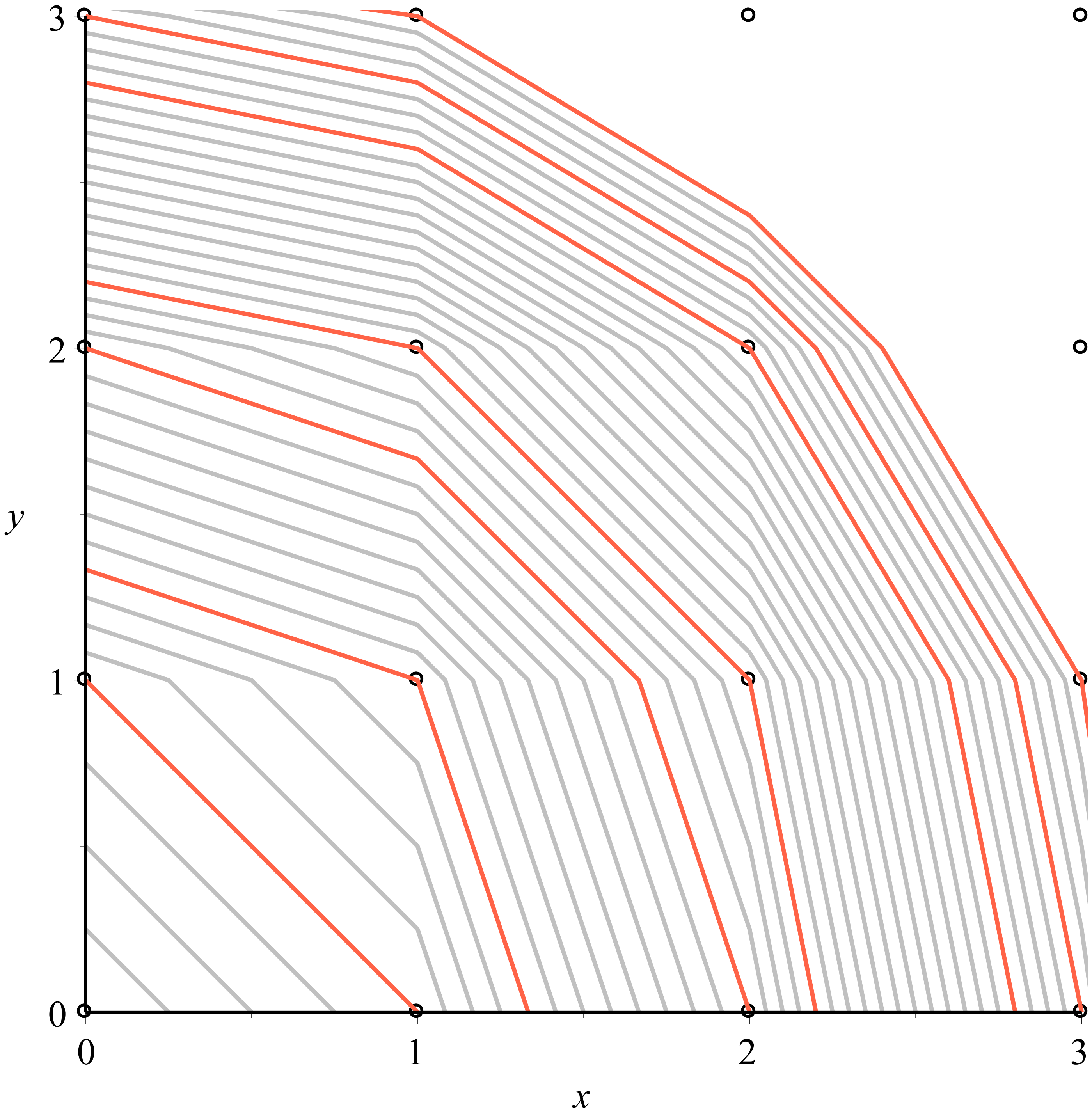

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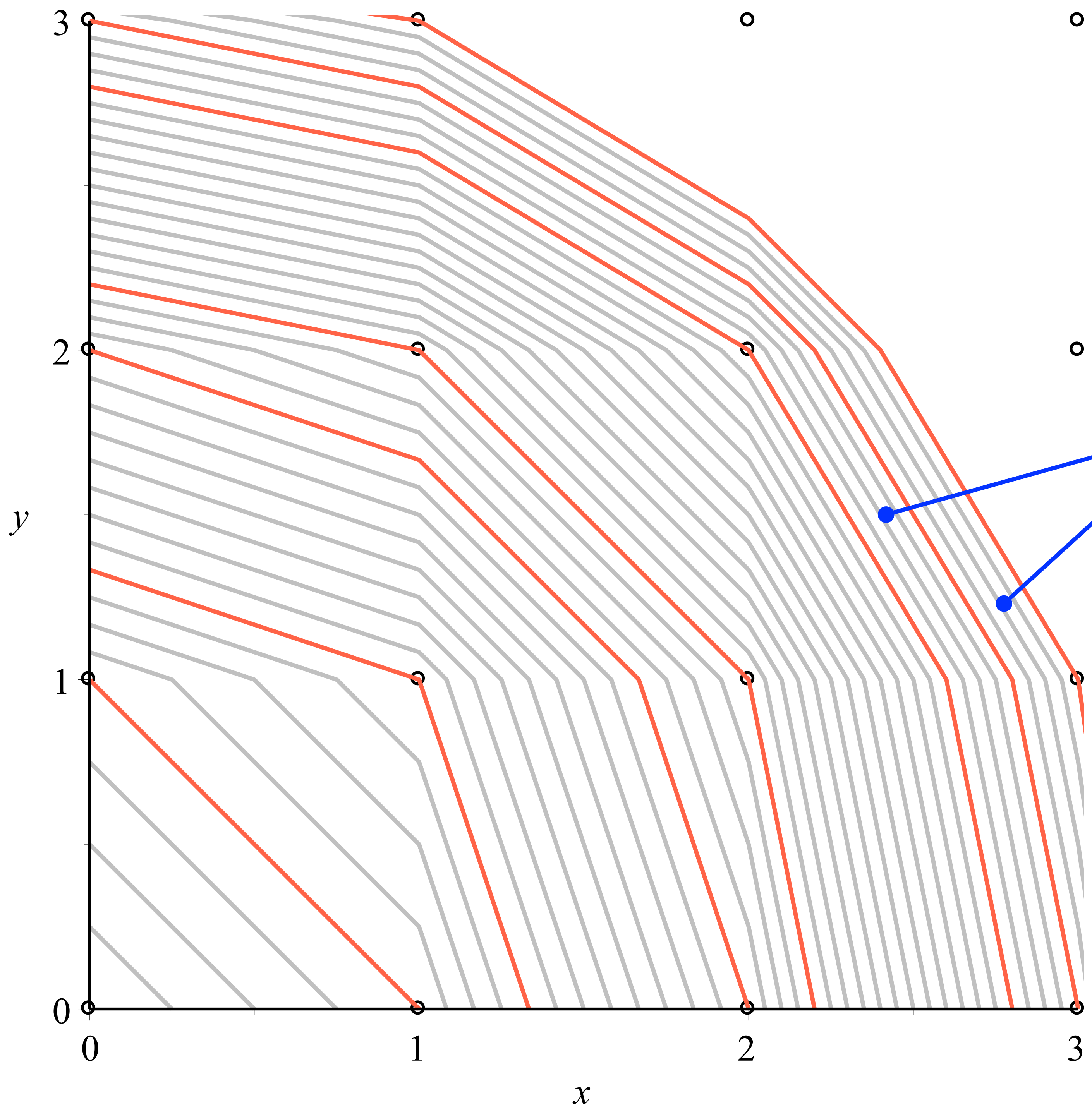

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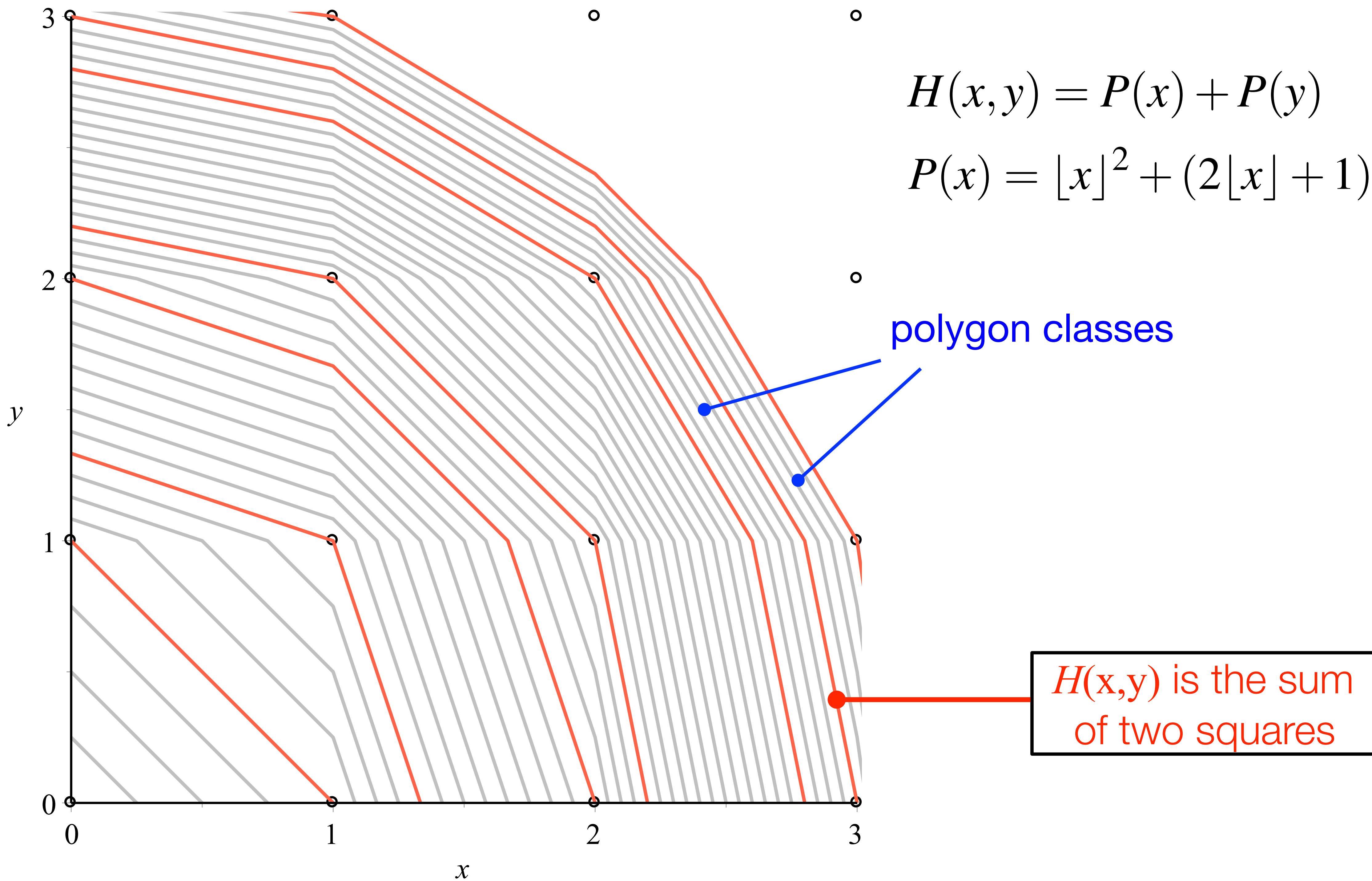

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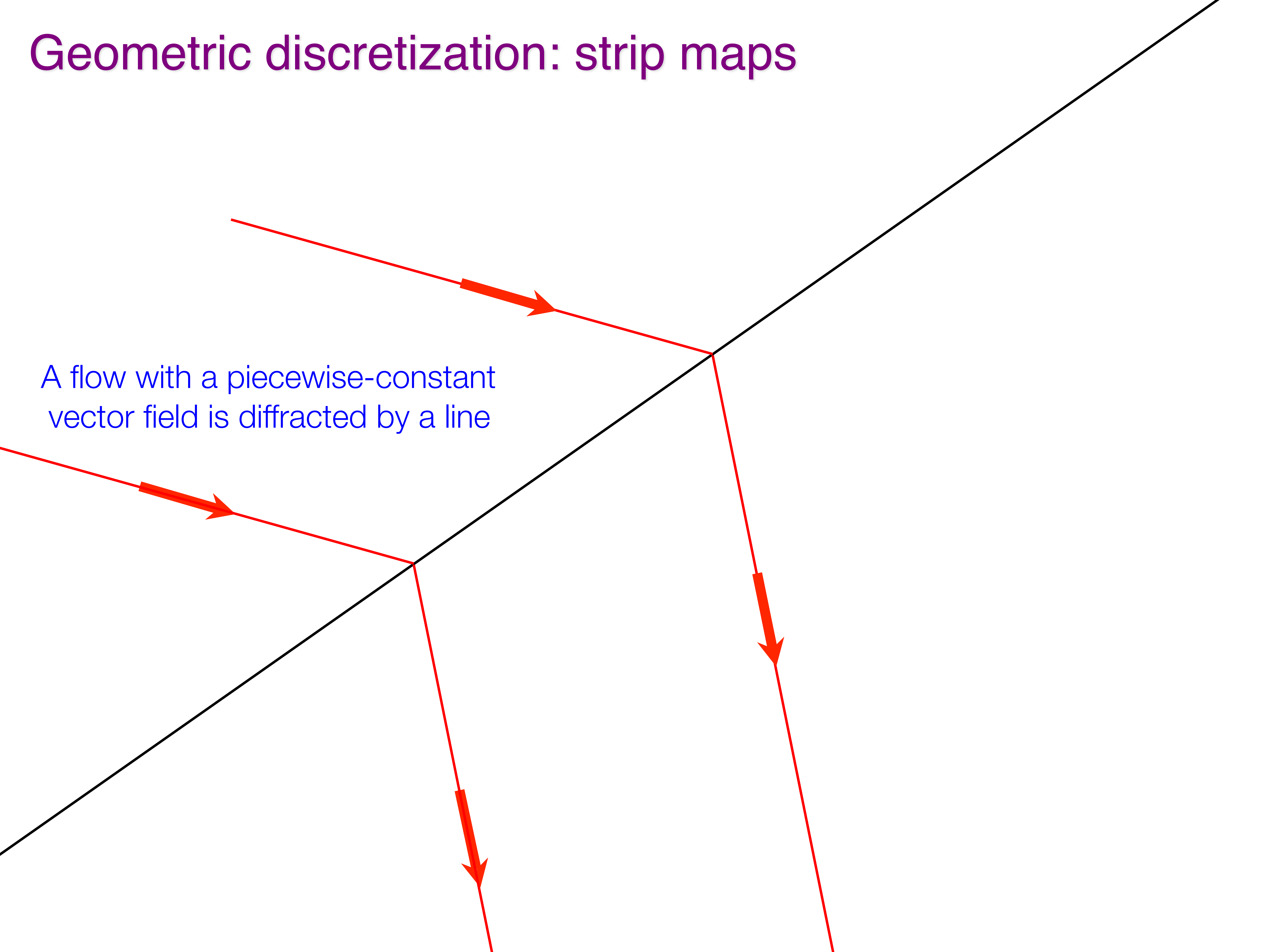
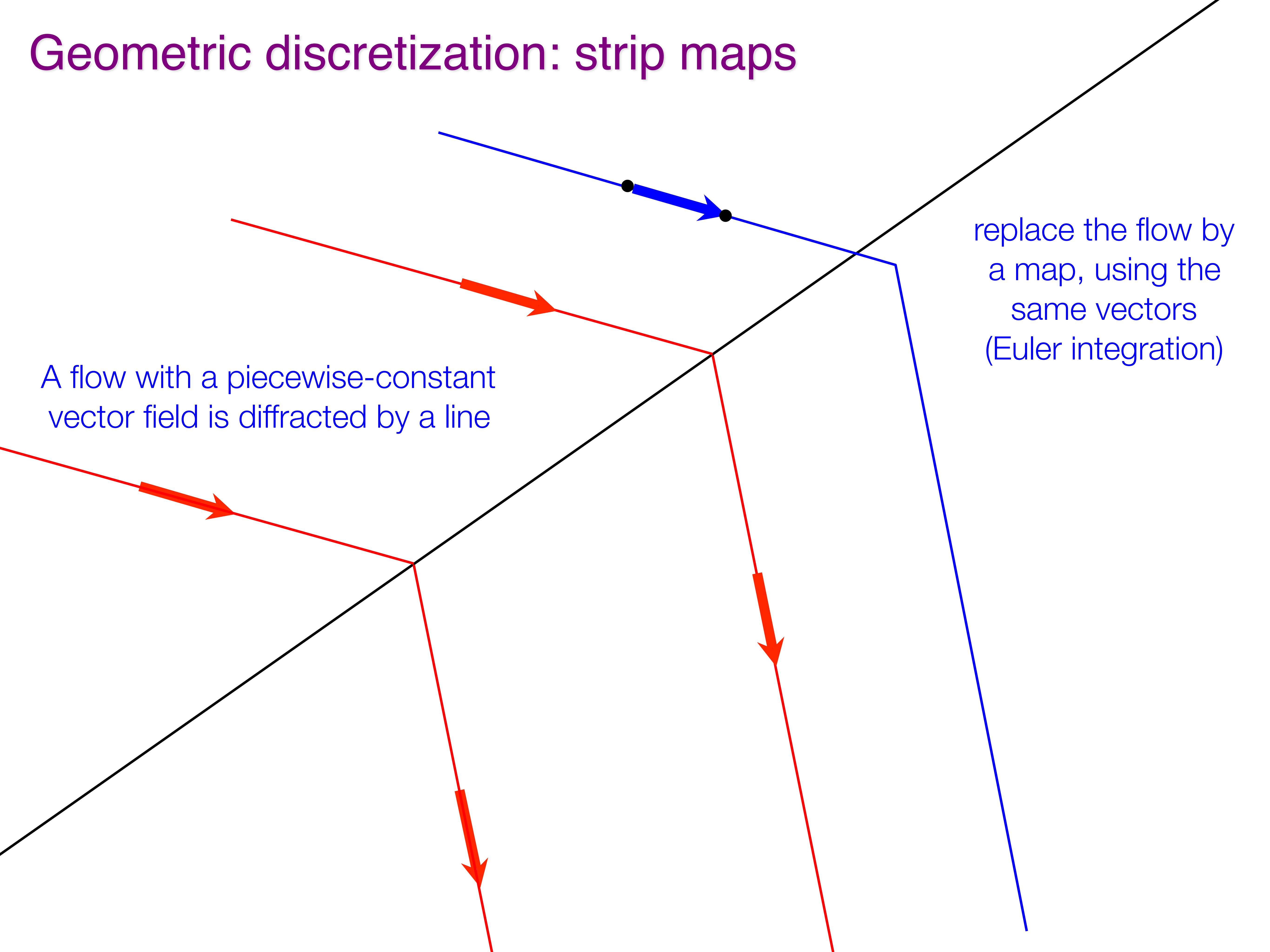


A flow with a piecewise-constant vector field is diffracted by a line

Geometric discretization: strip maps



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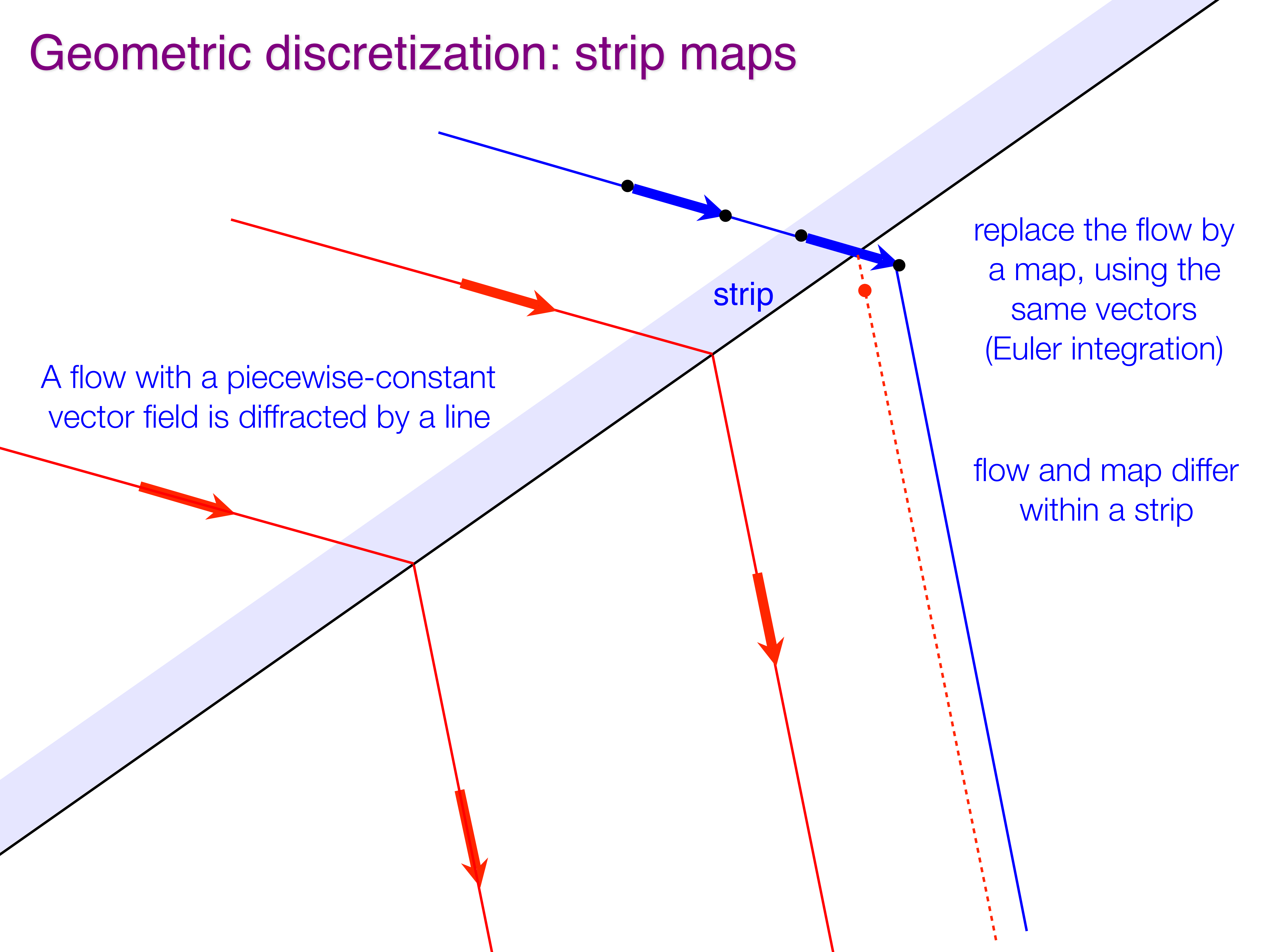

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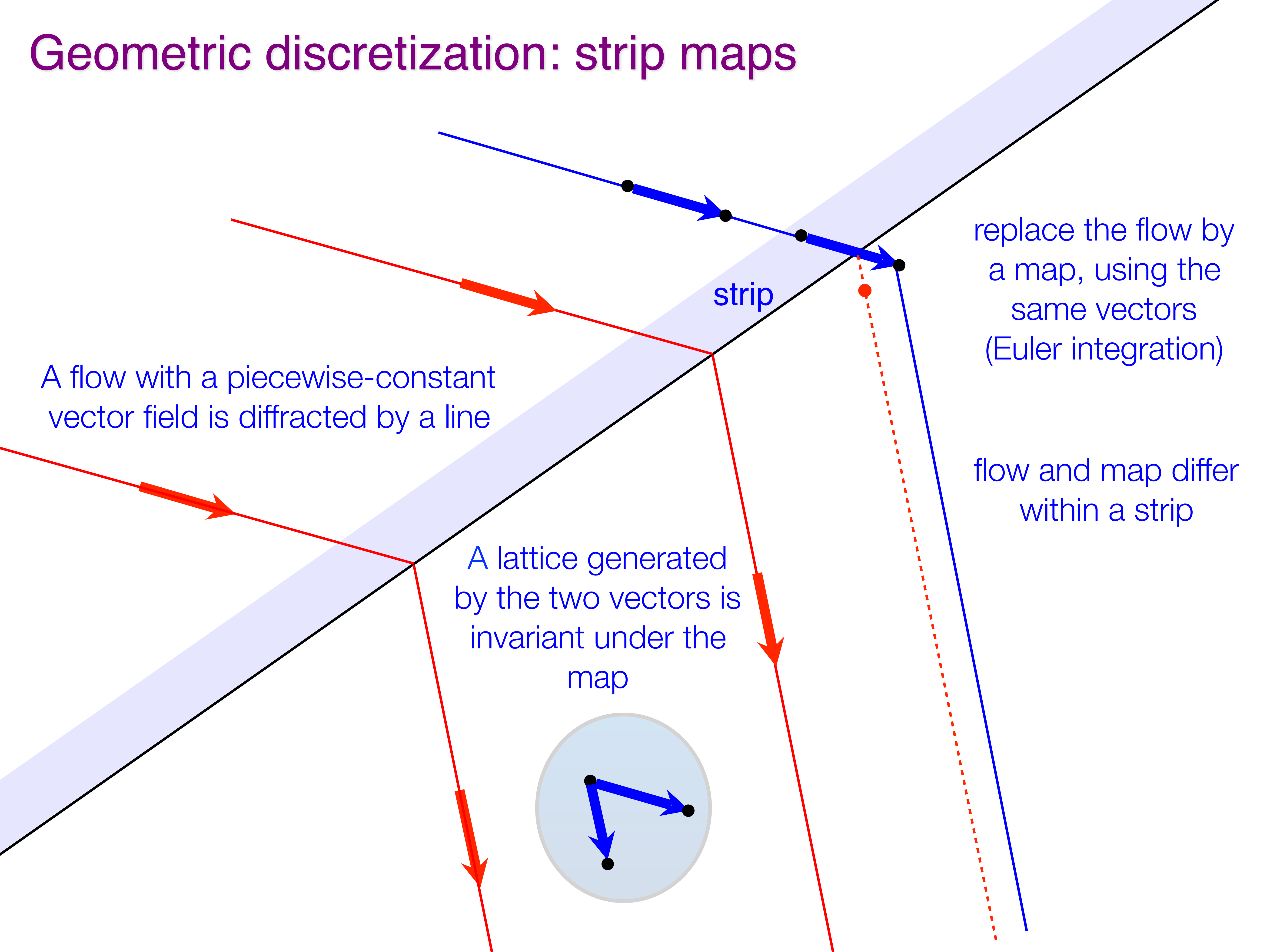
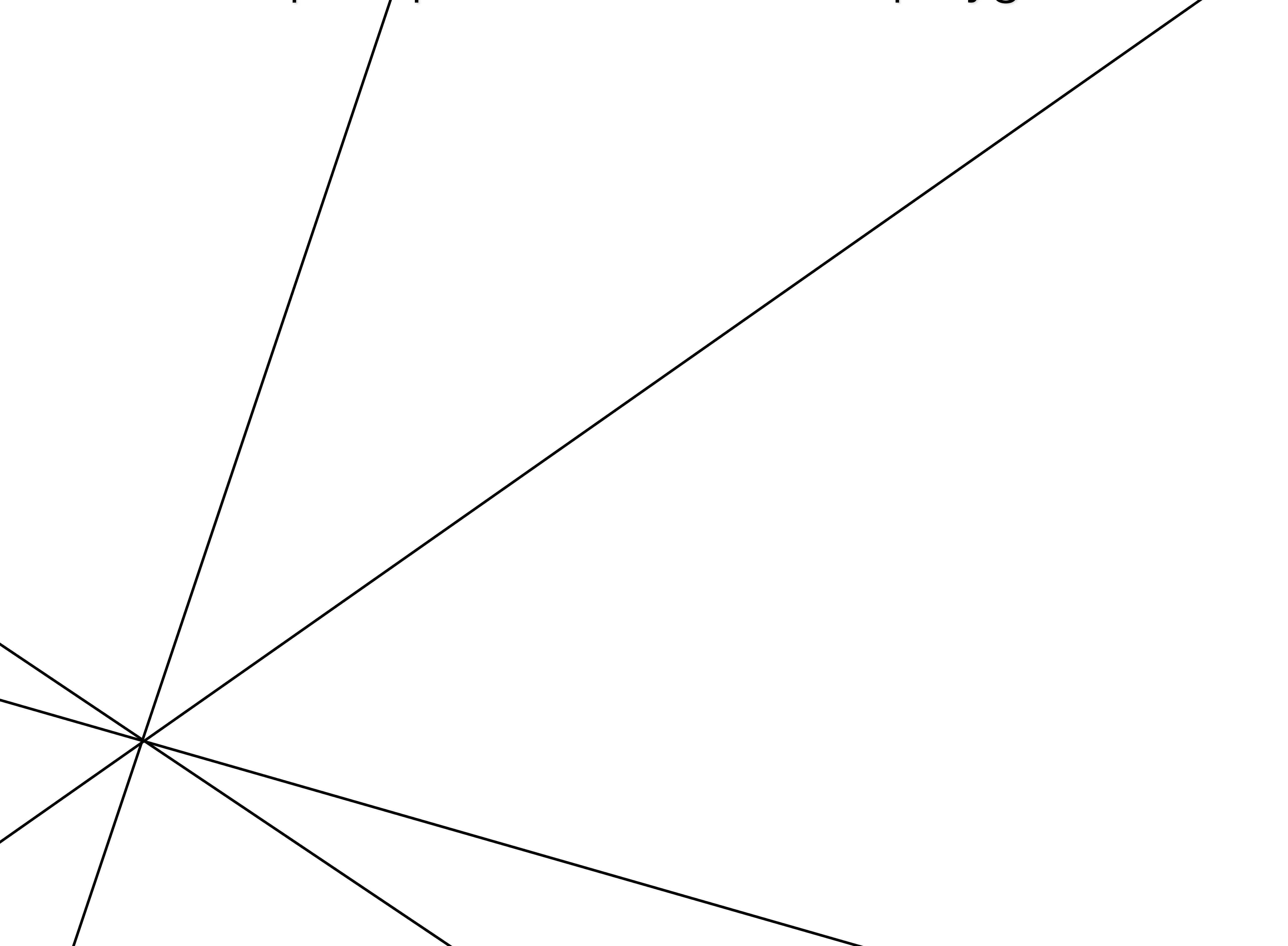


Linked strip maps: outer billiards of polygons



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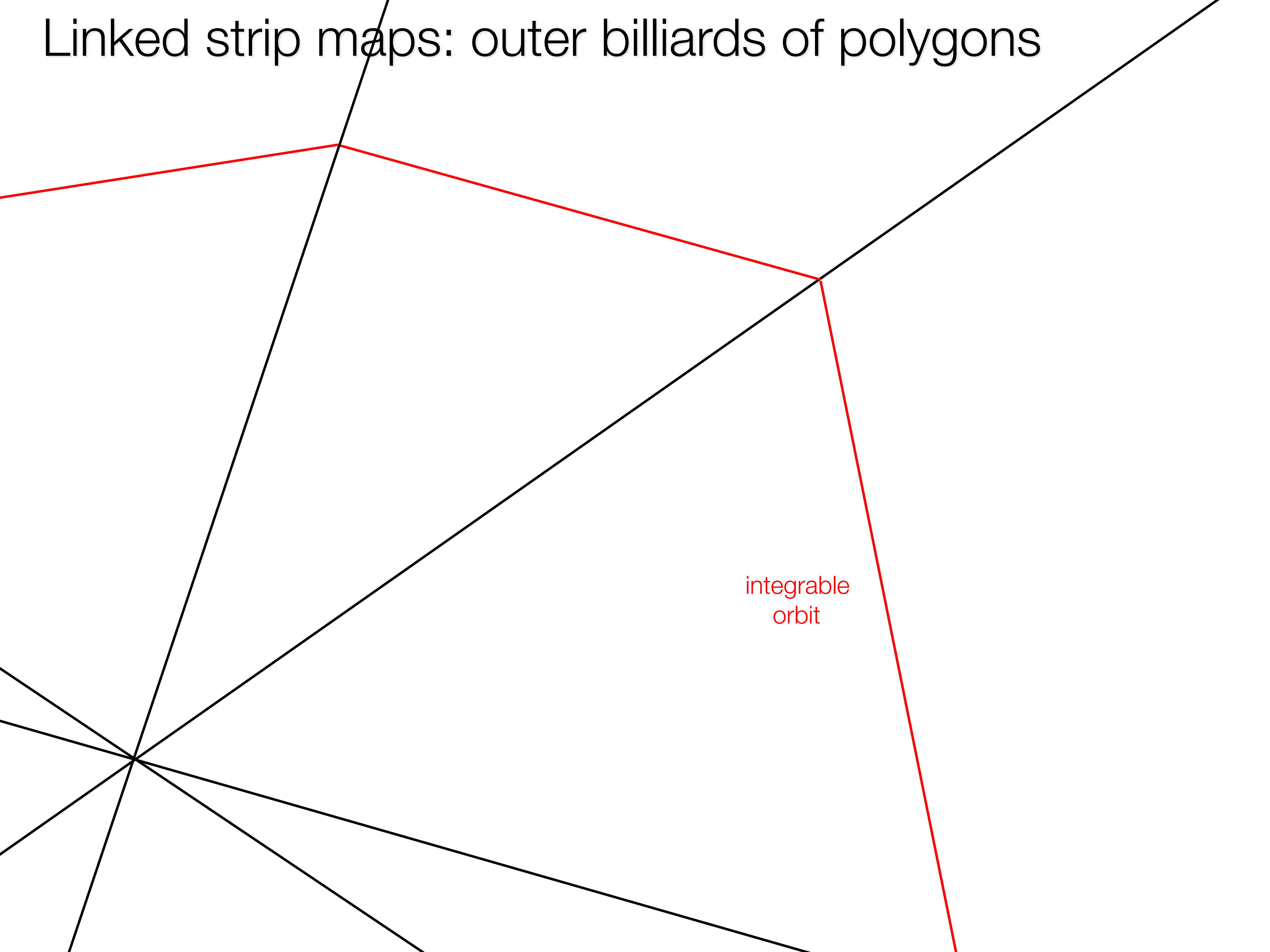

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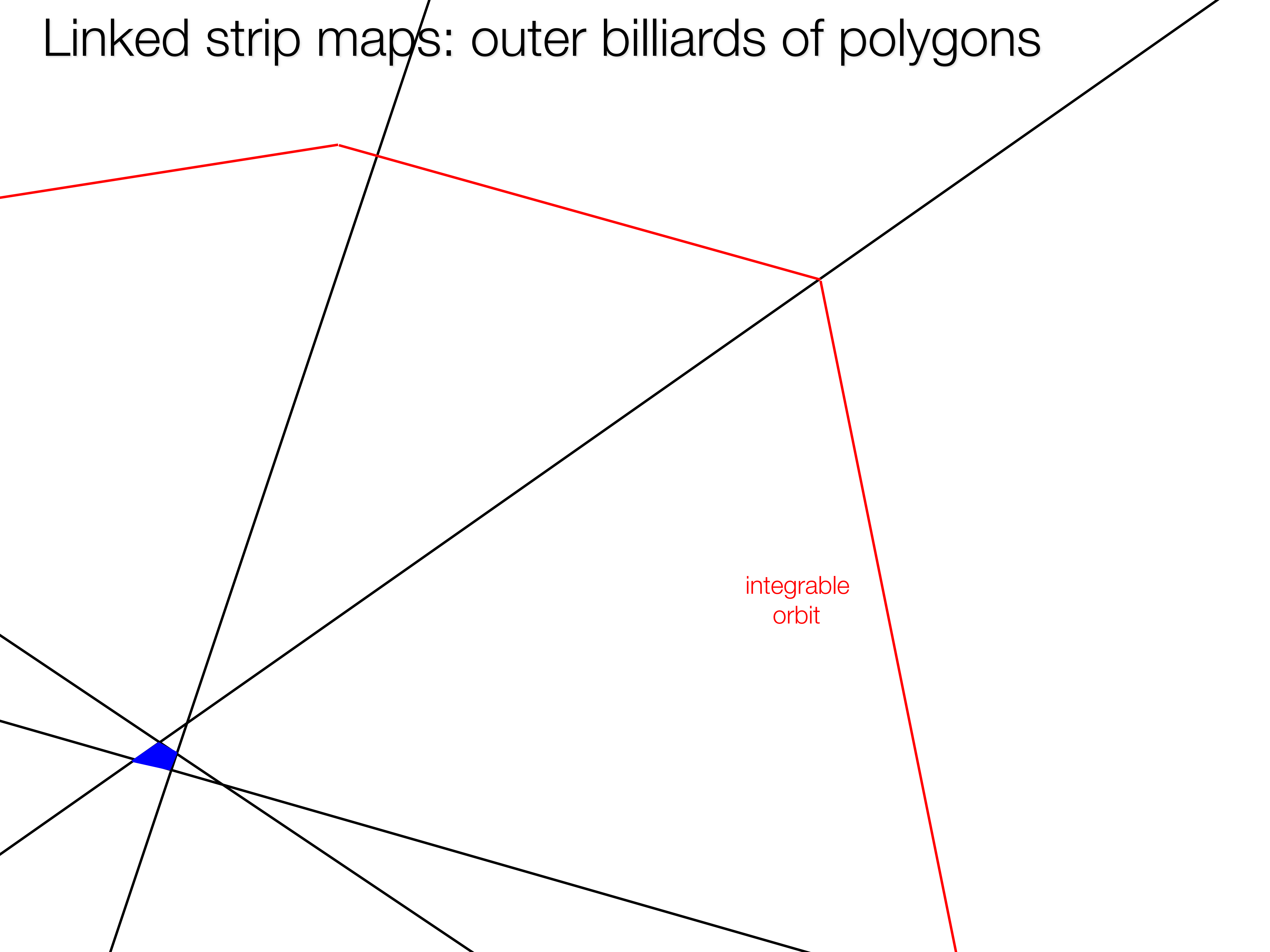

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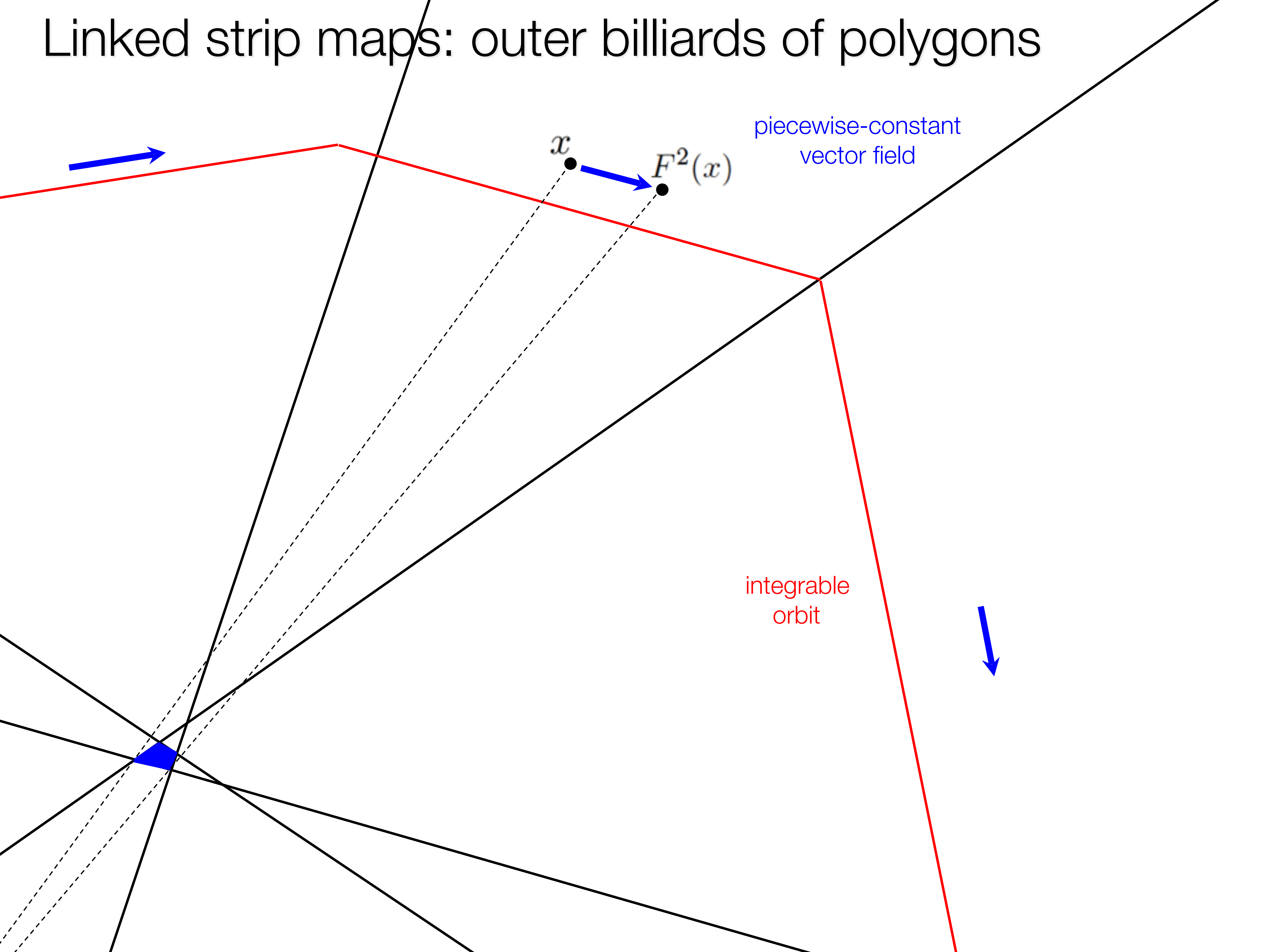

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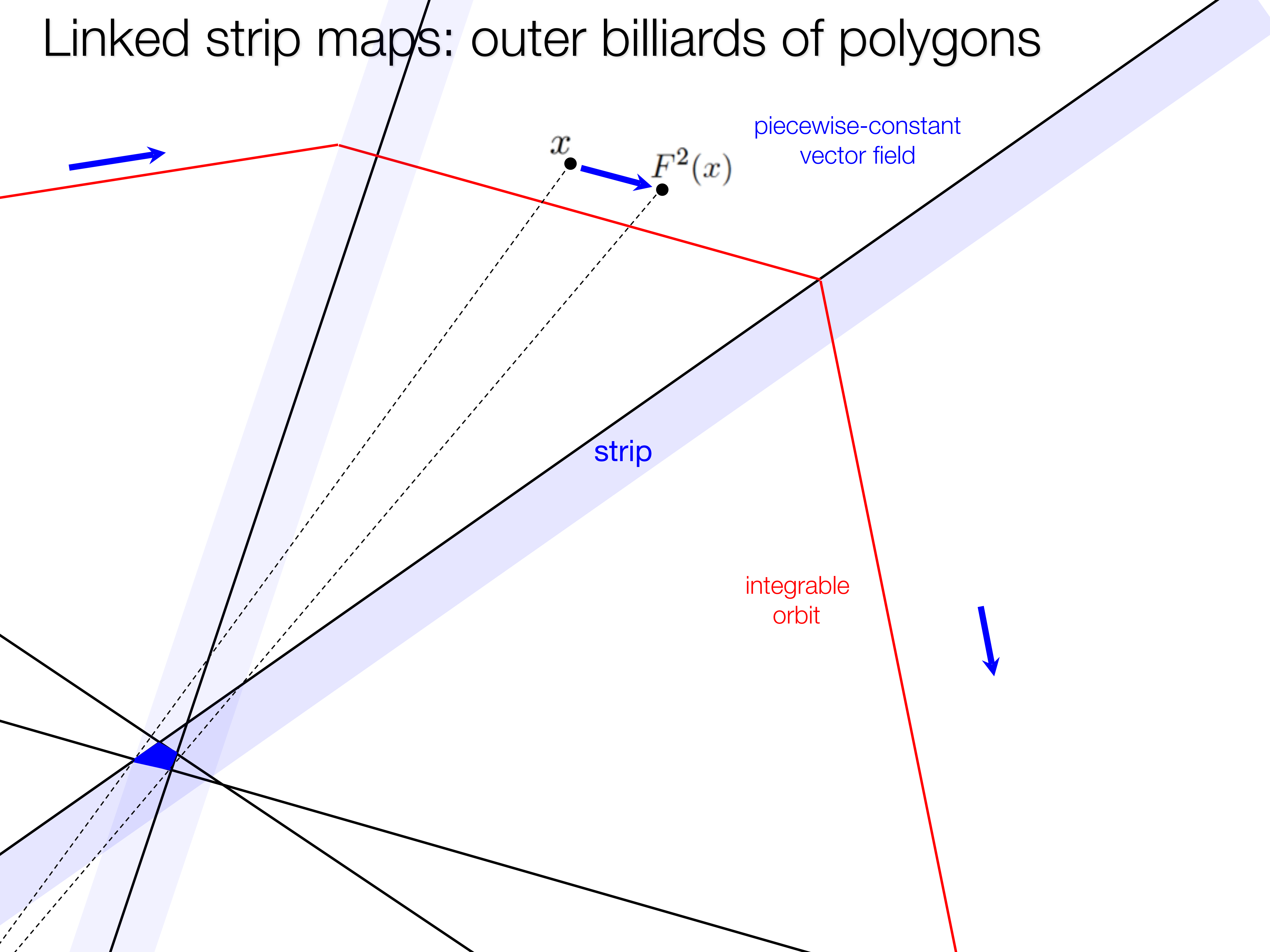

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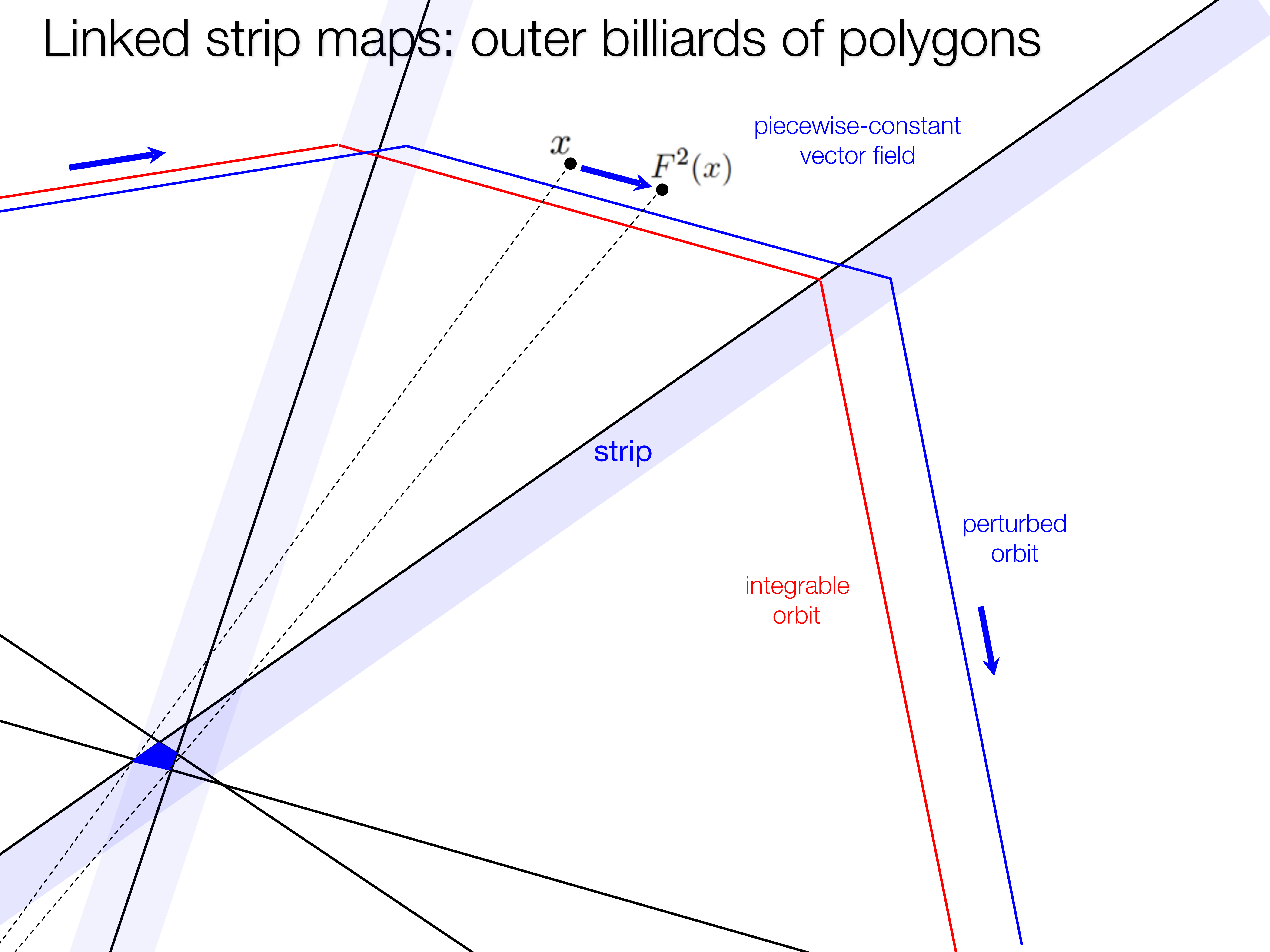
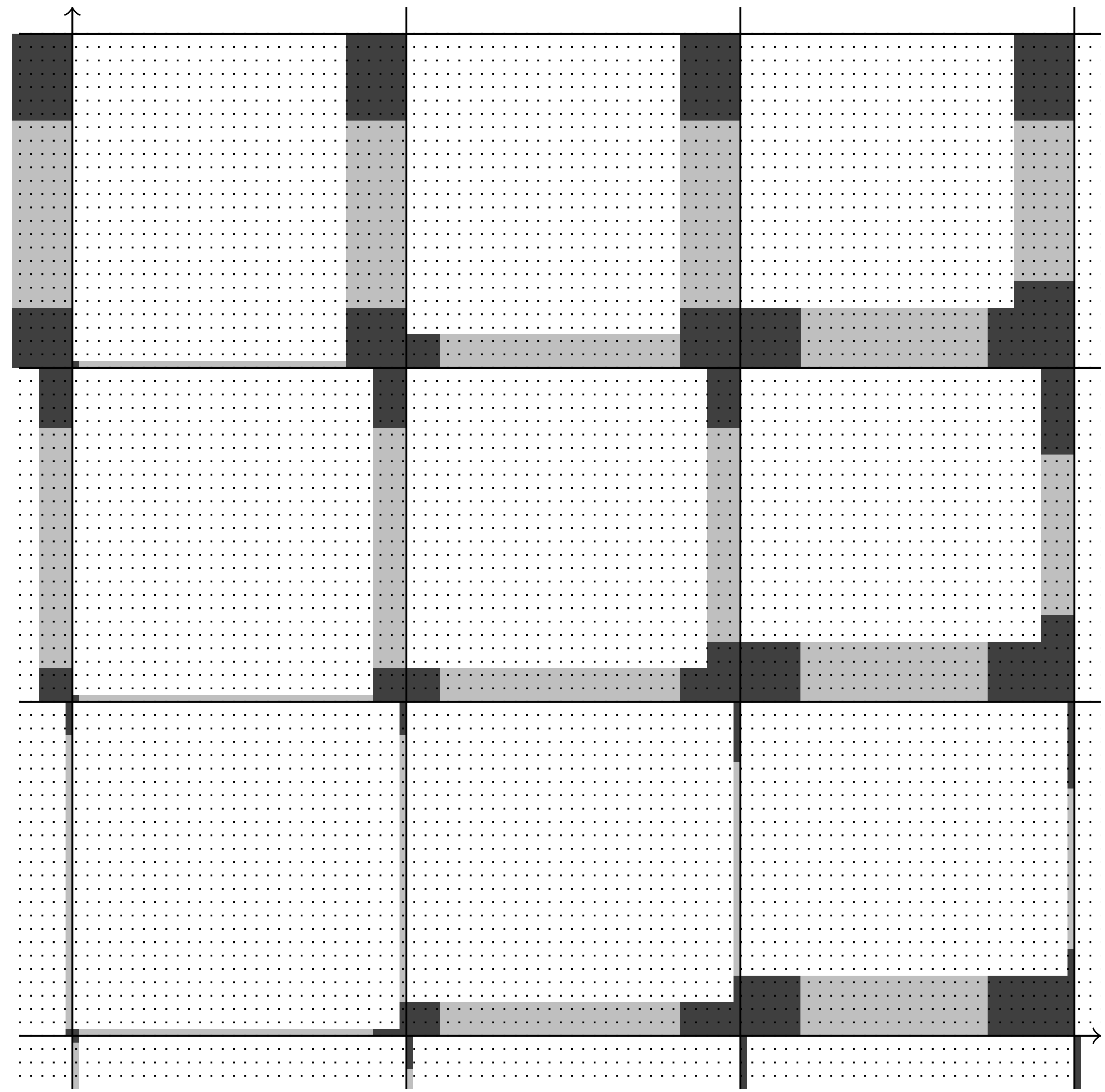


Linked strip maps: the λ→0 round-off problem



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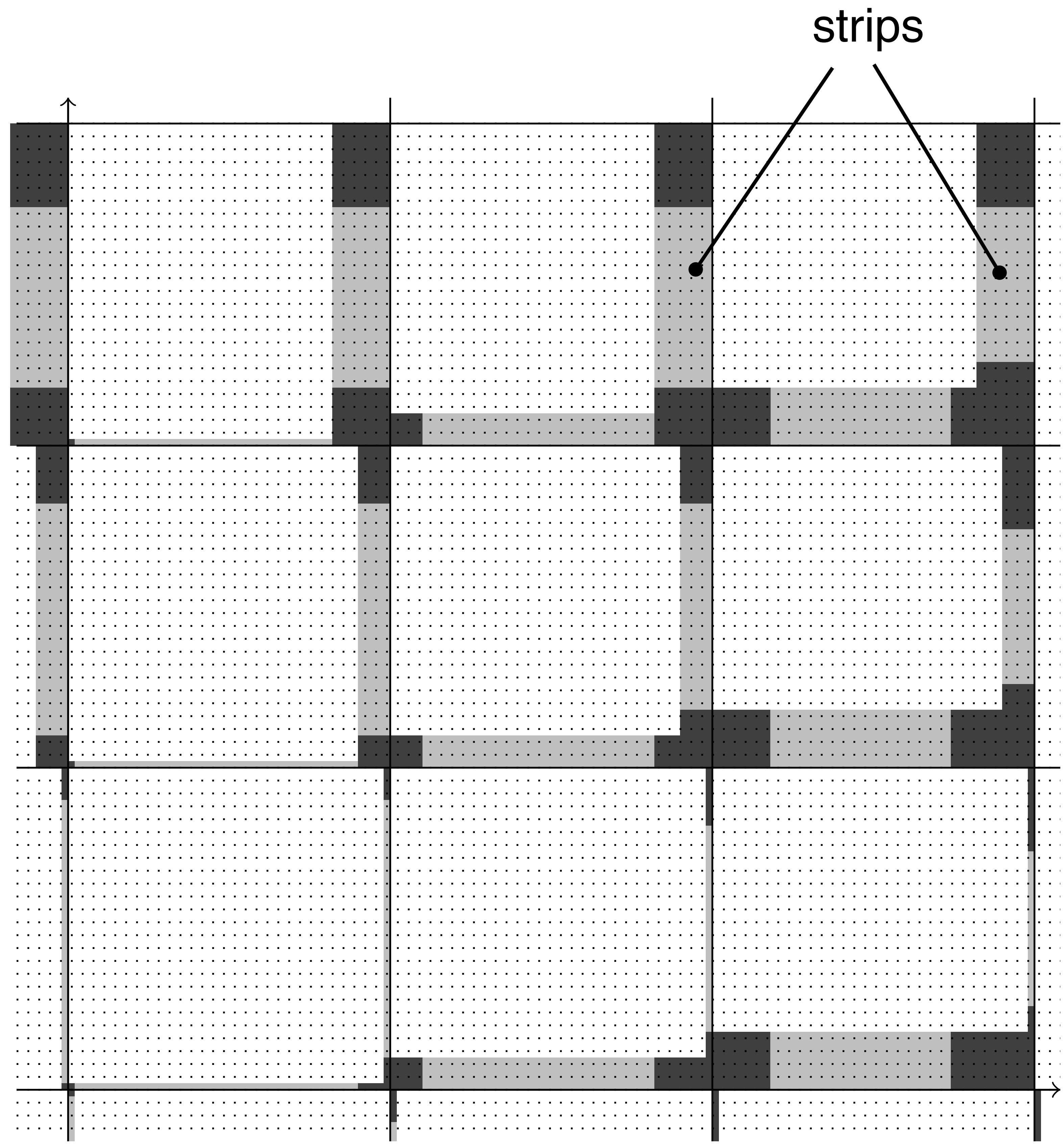

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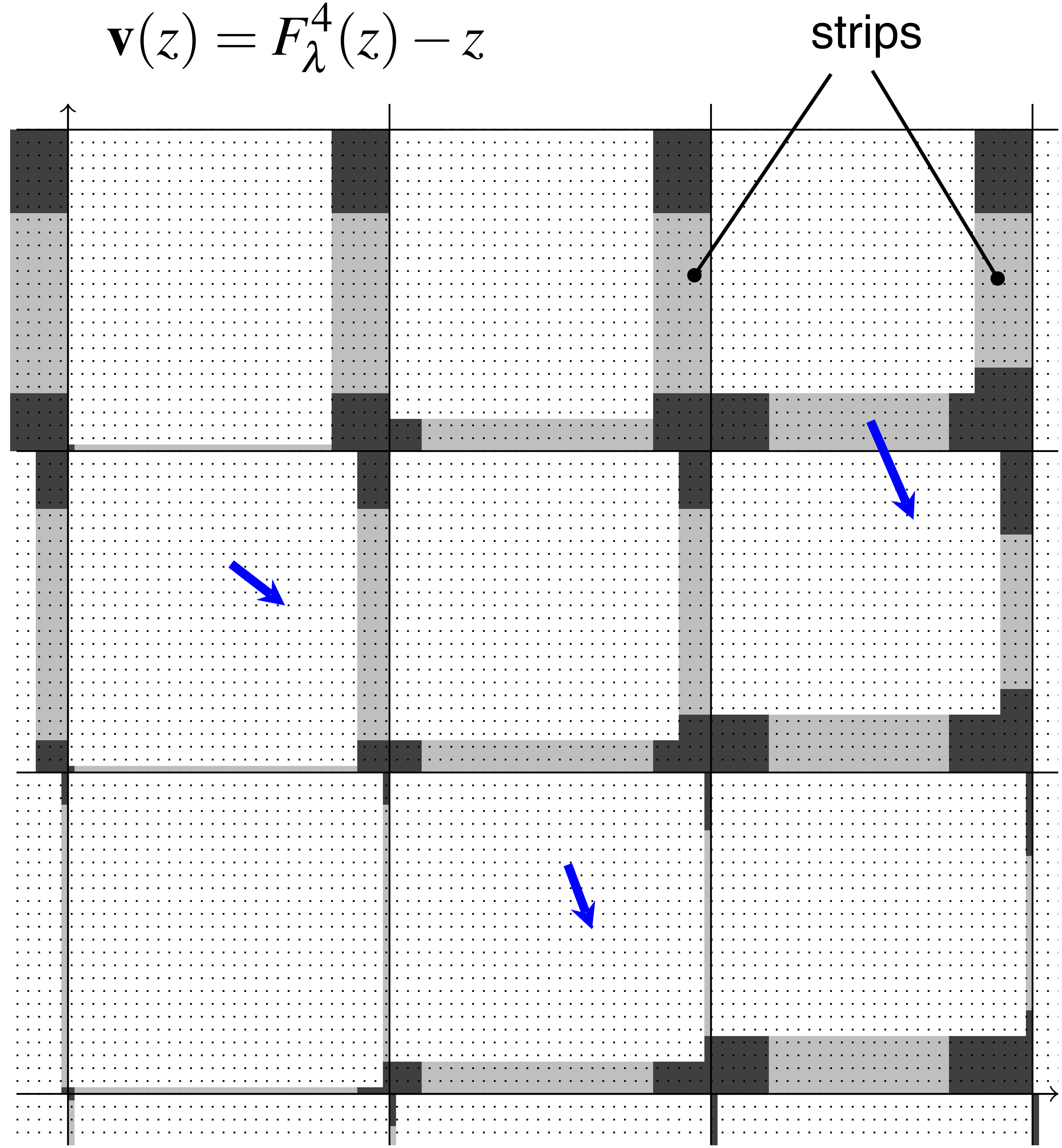
$$\mathbf{v}(z) = F_\lambda^4(z) - z$$



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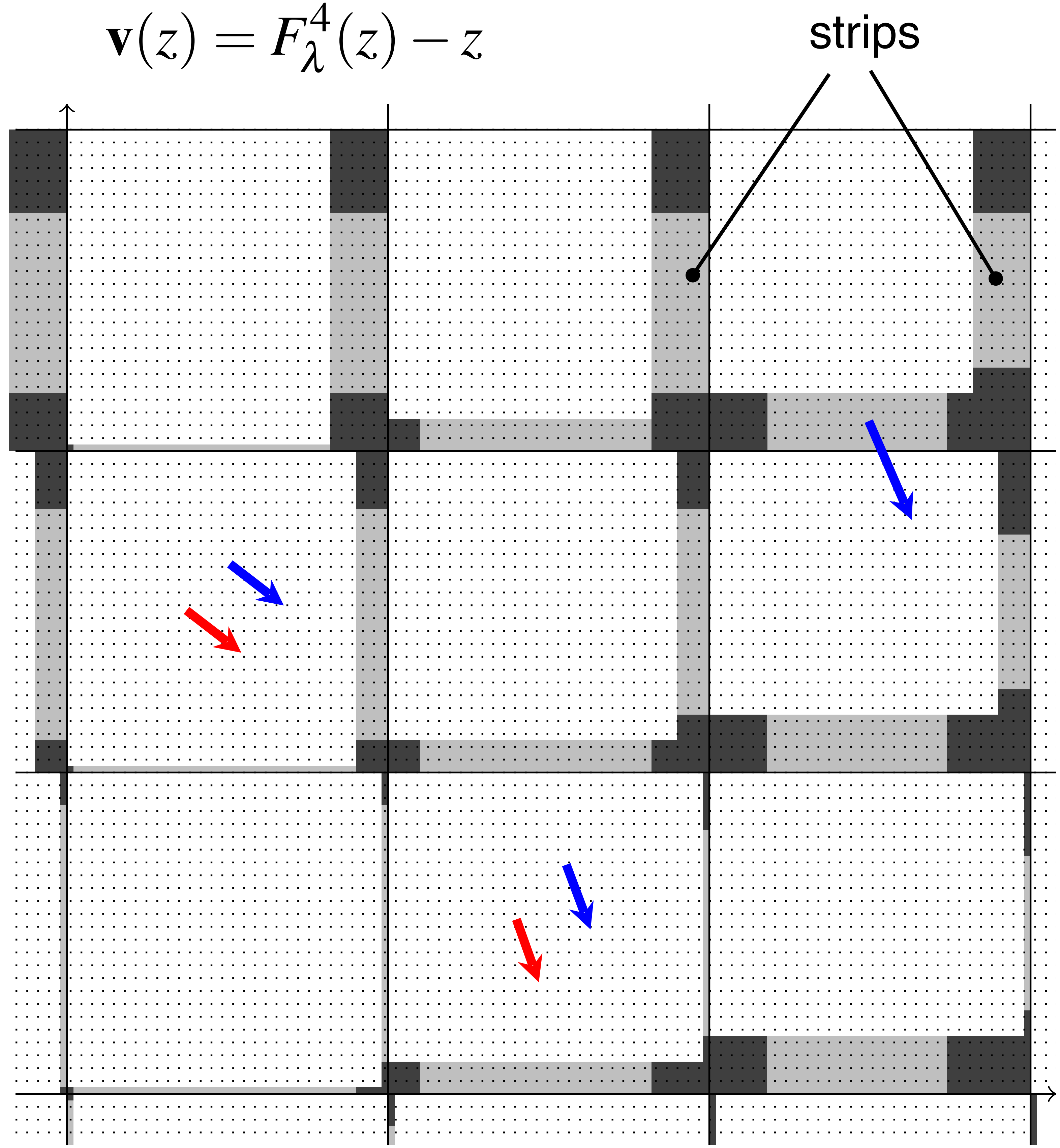

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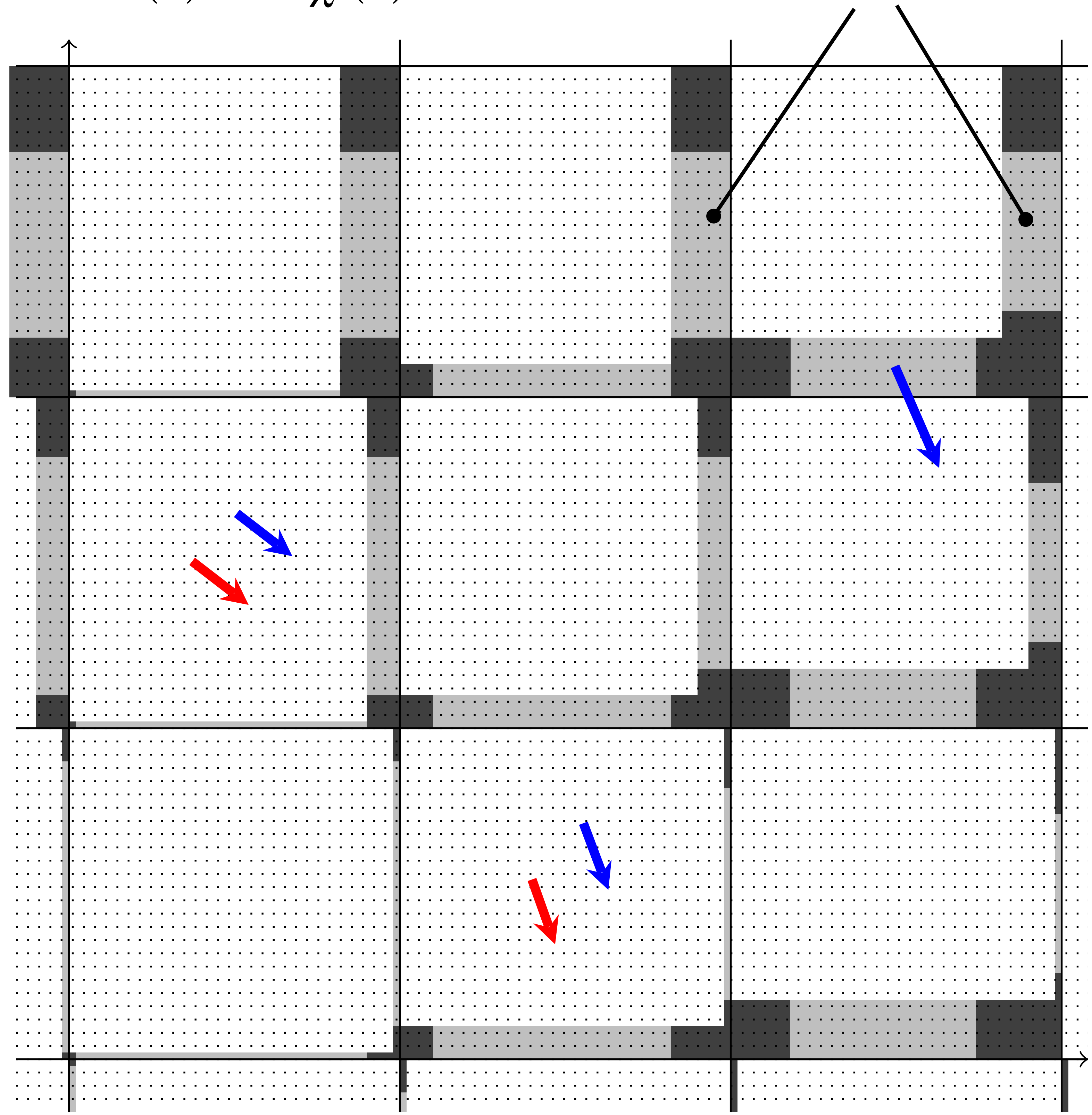

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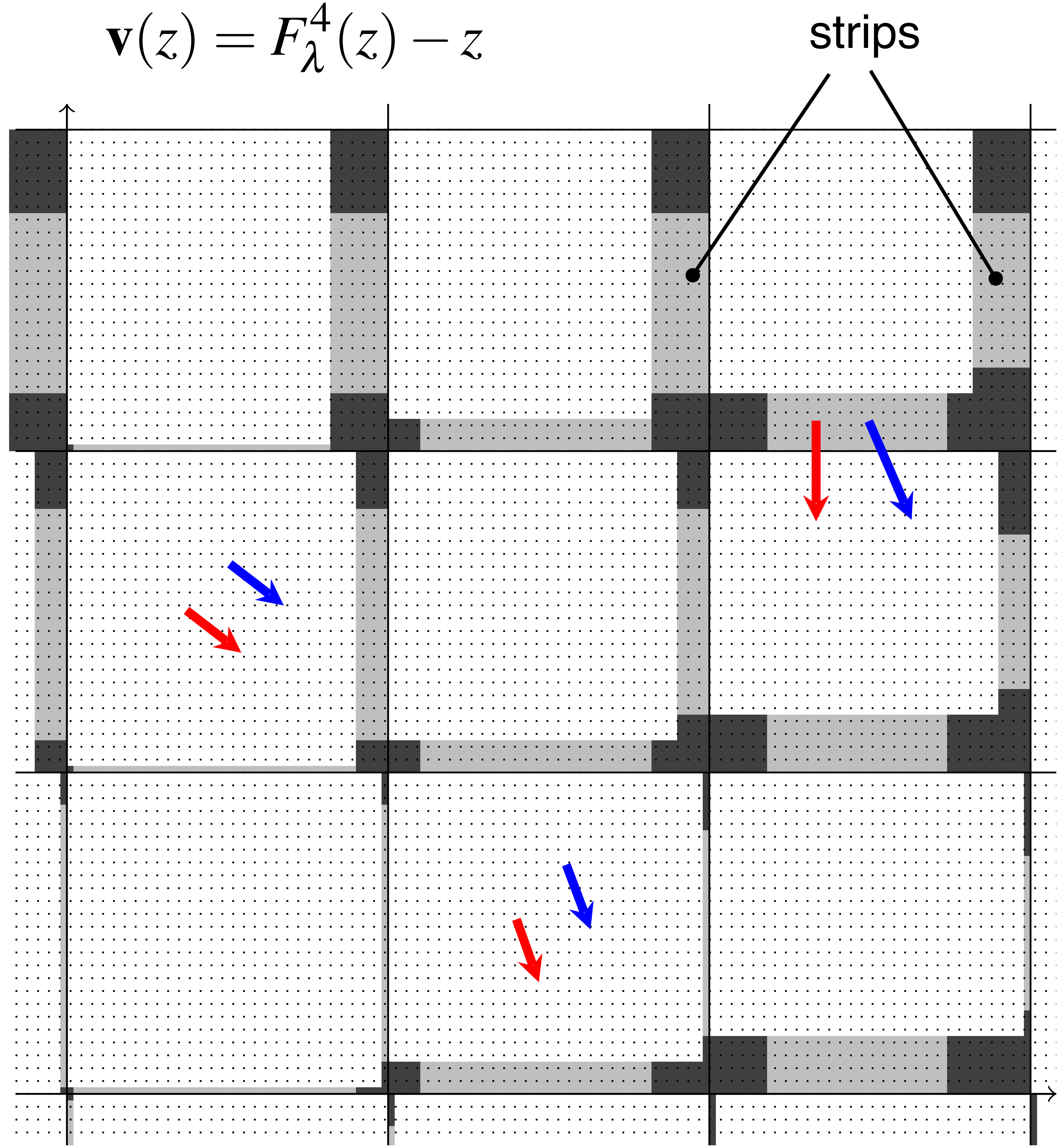

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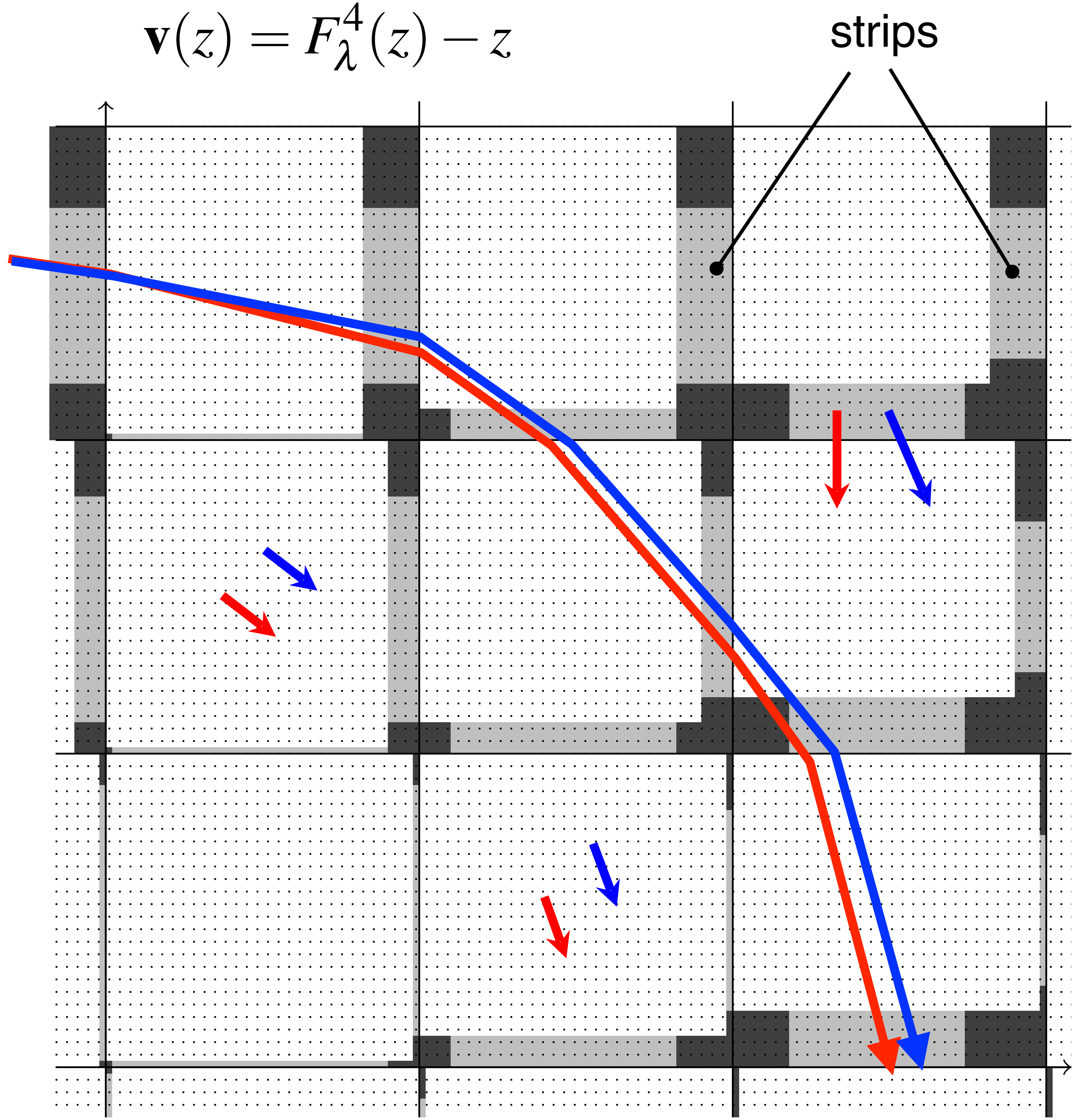
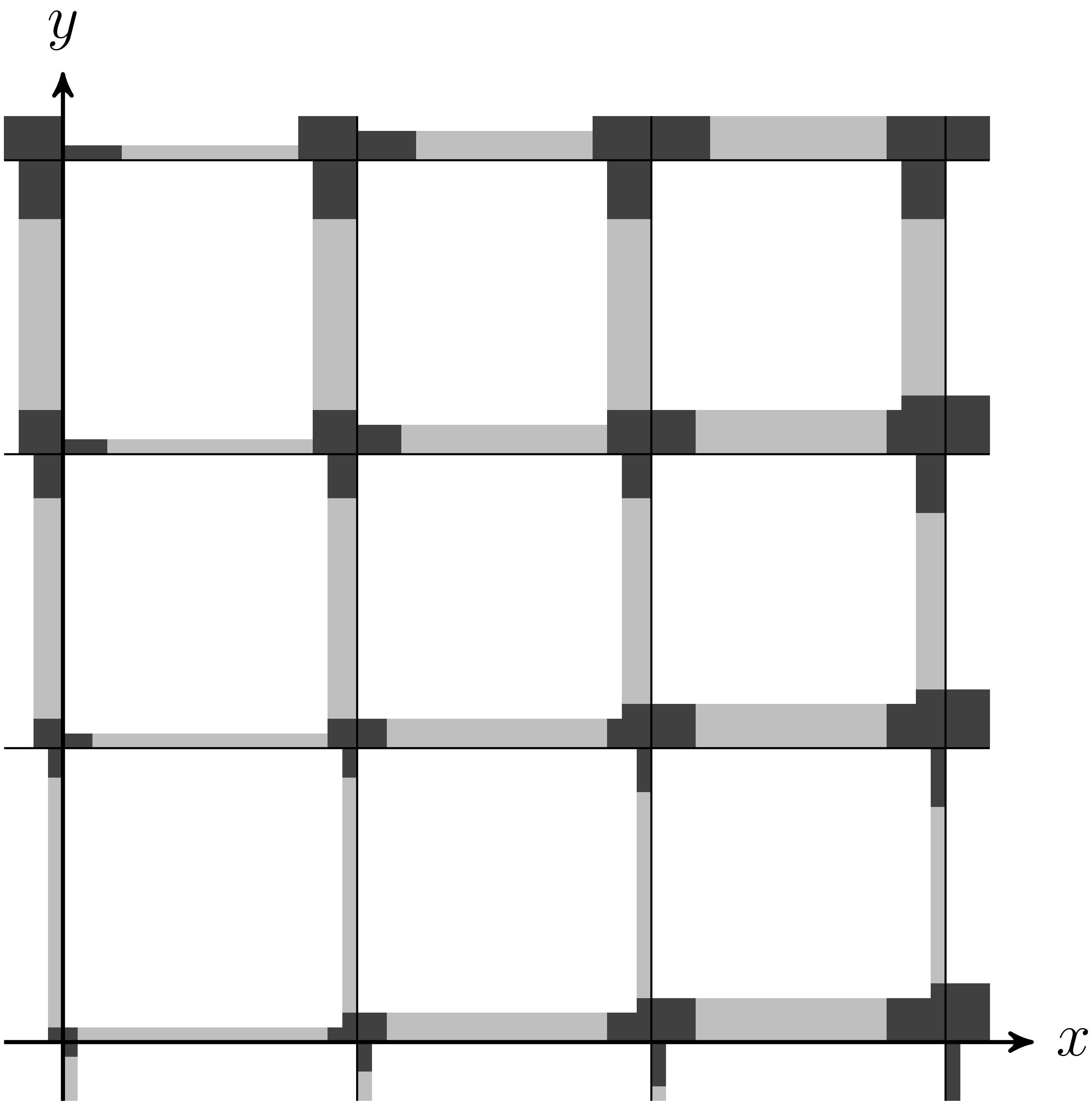
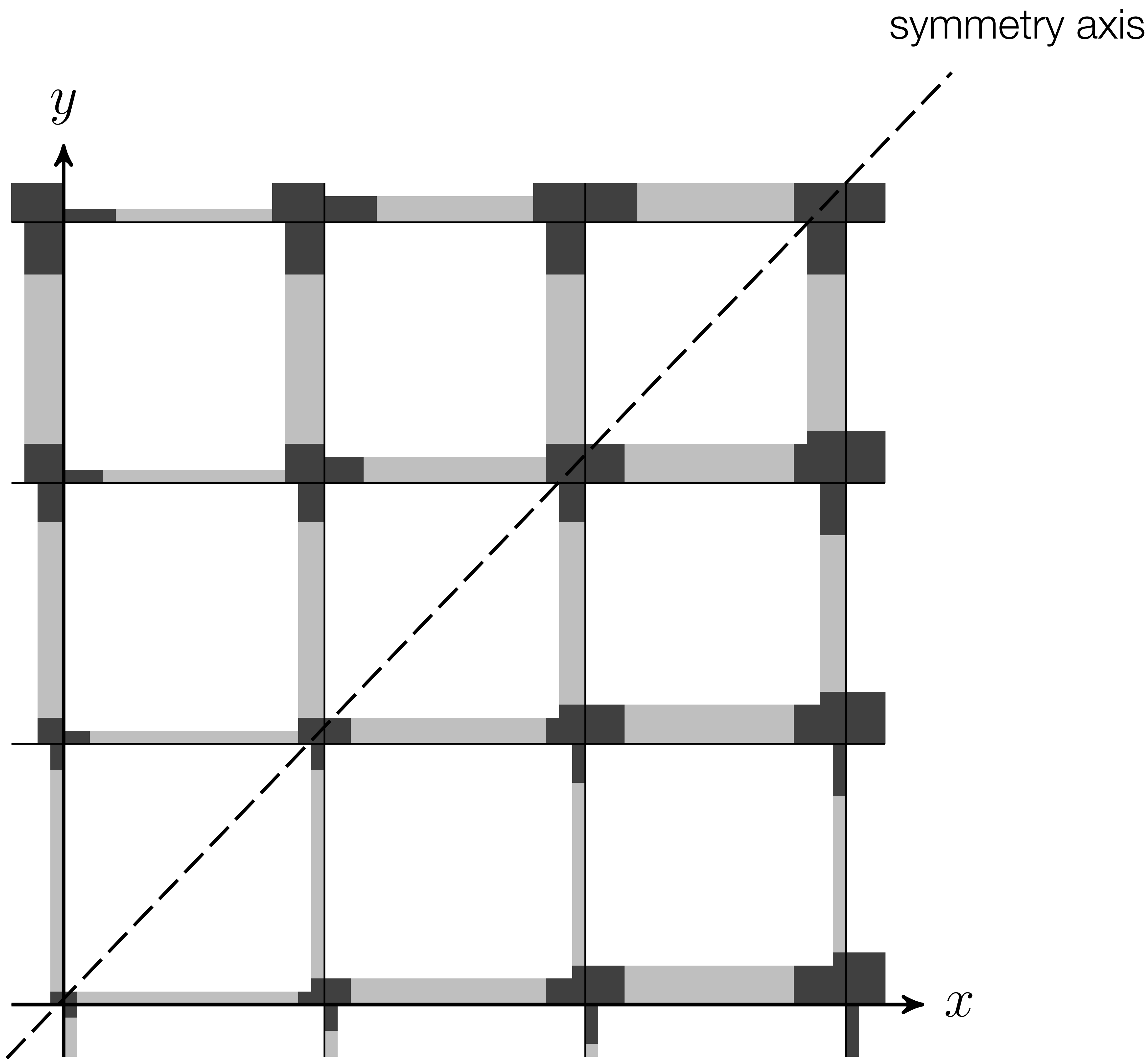
The **integrable** and **perturbed** orbits differ.



2D Poincaré section



2D Poincare´ section



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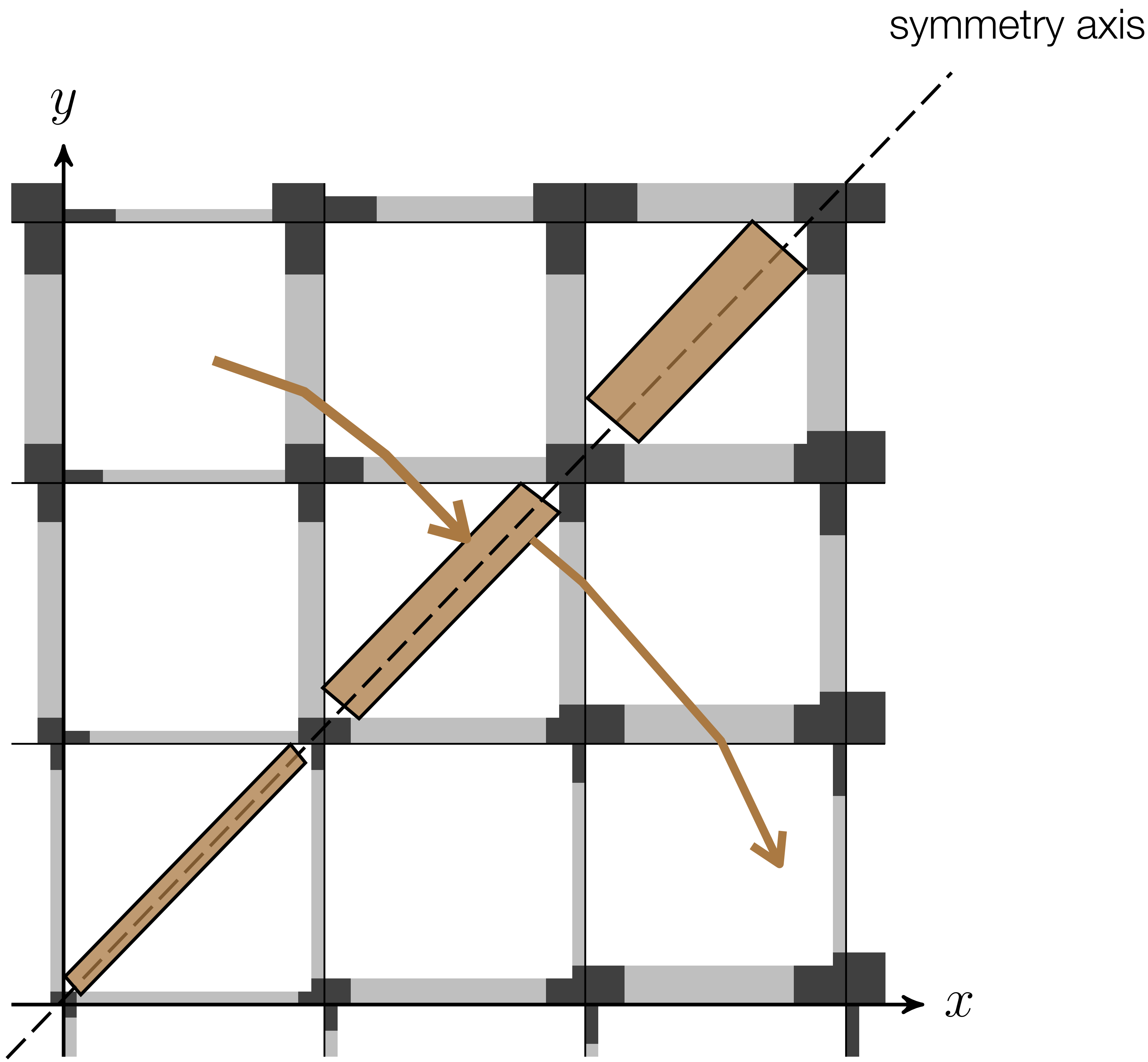
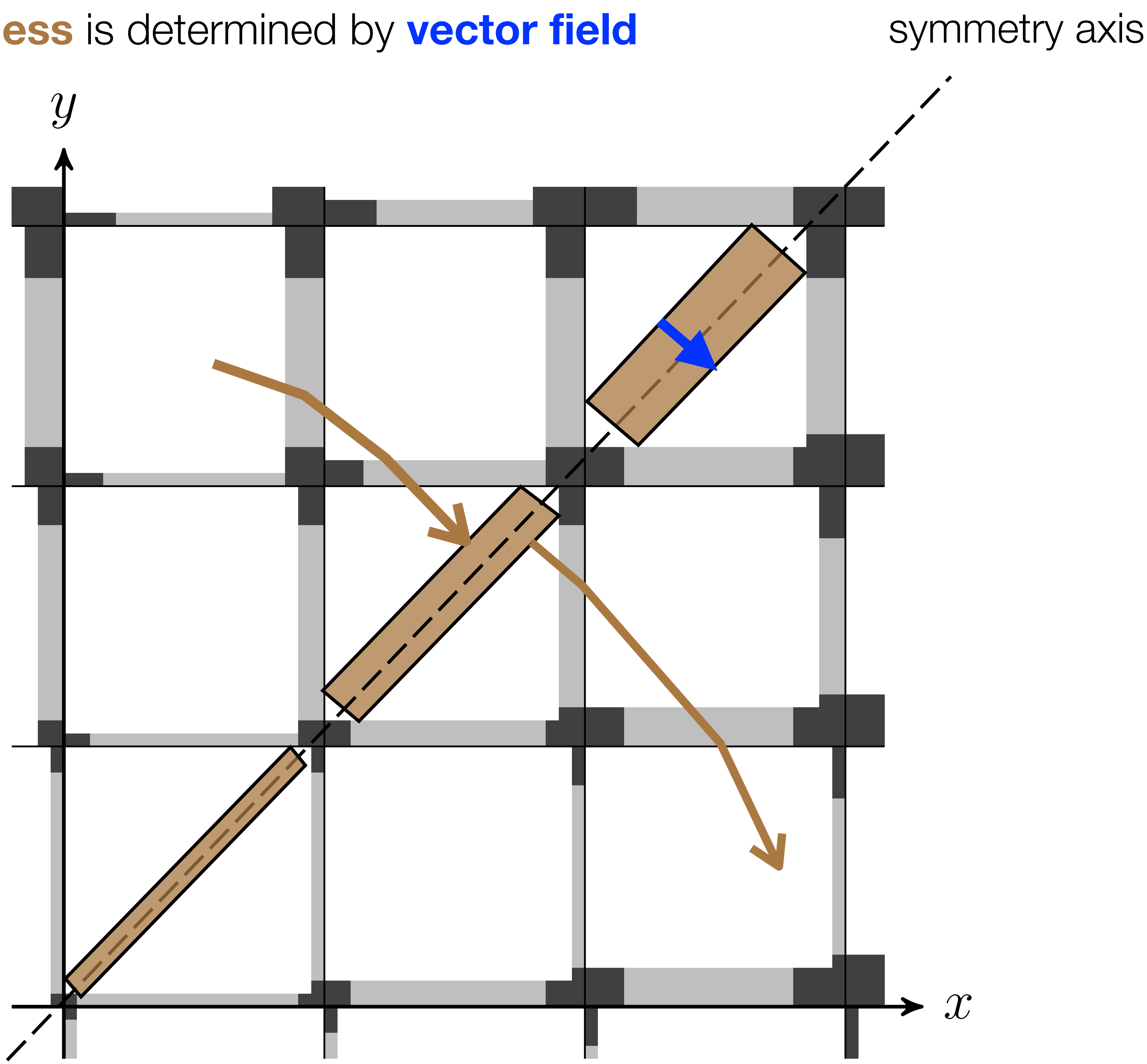


2D Poincaré section

Thickness is determined by **vector field**



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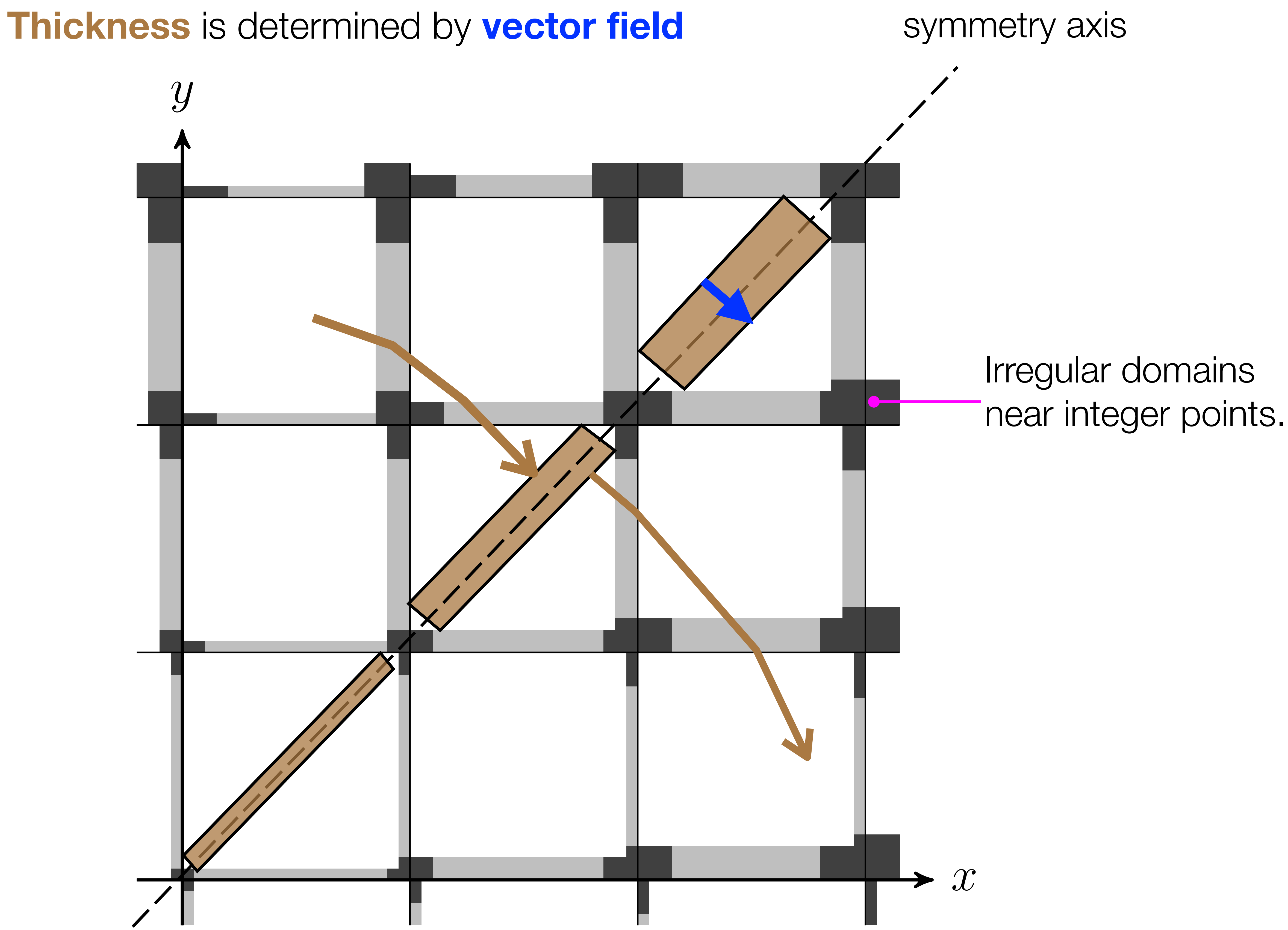

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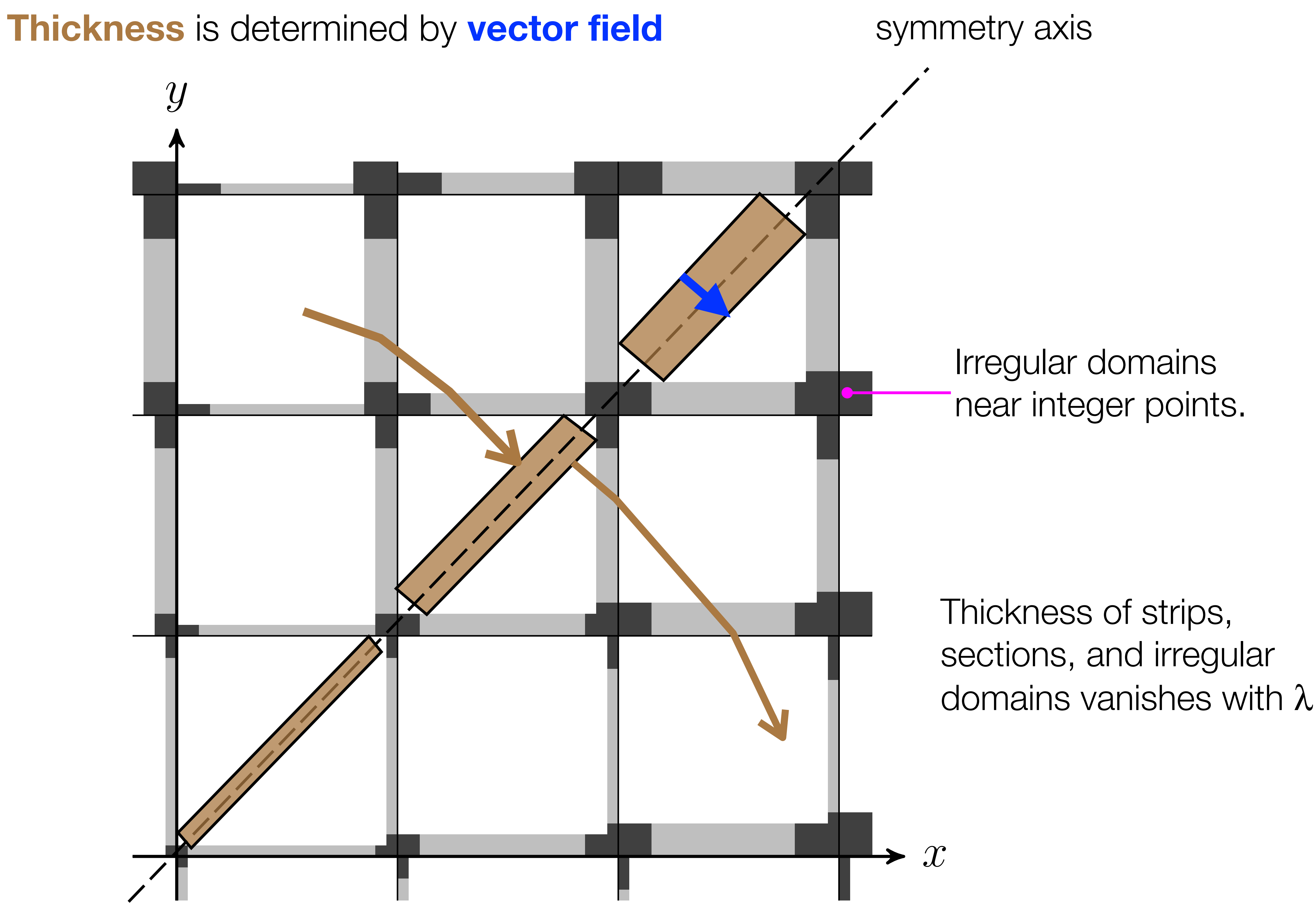

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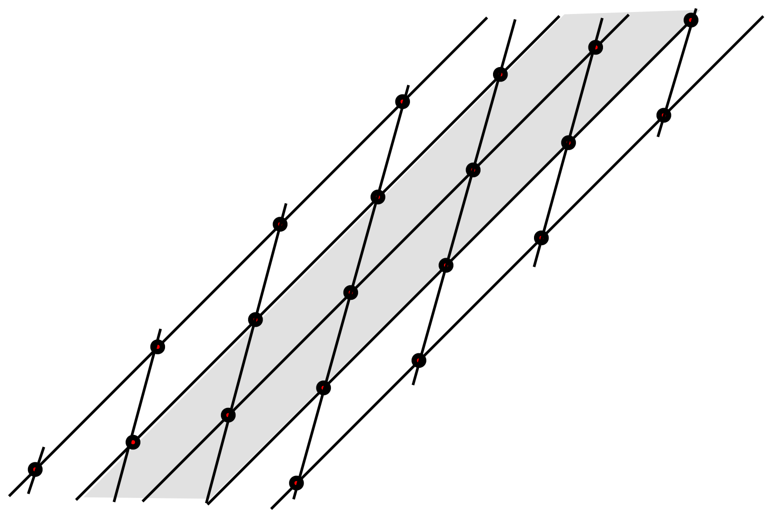

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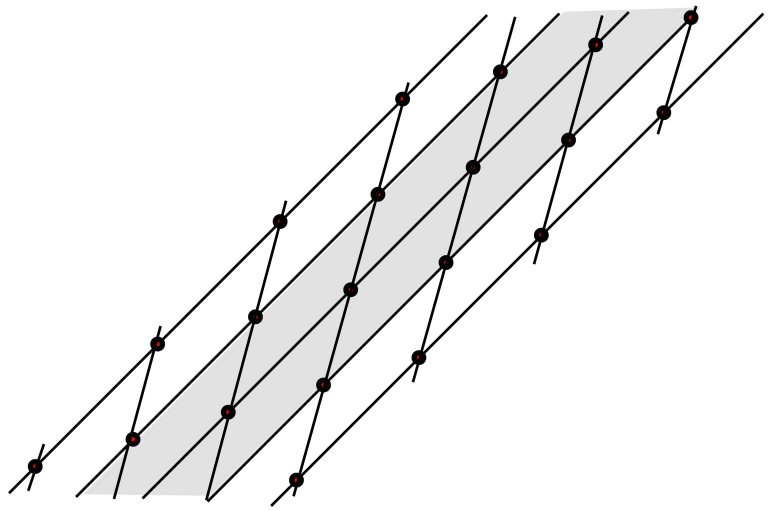

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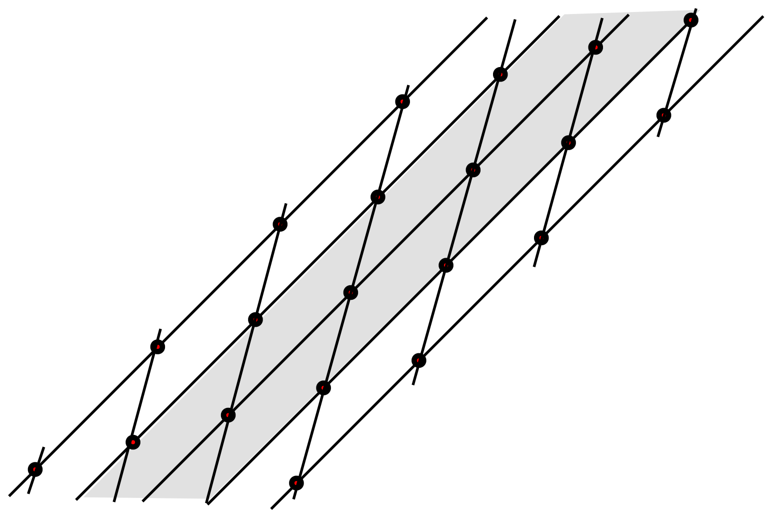

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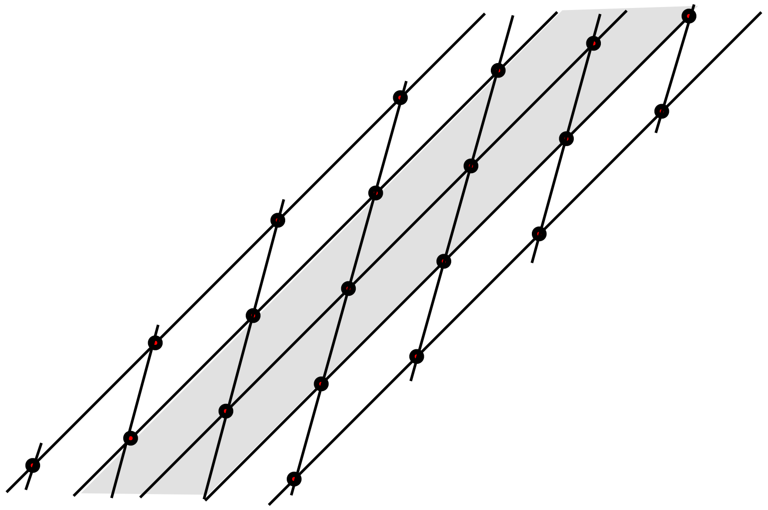

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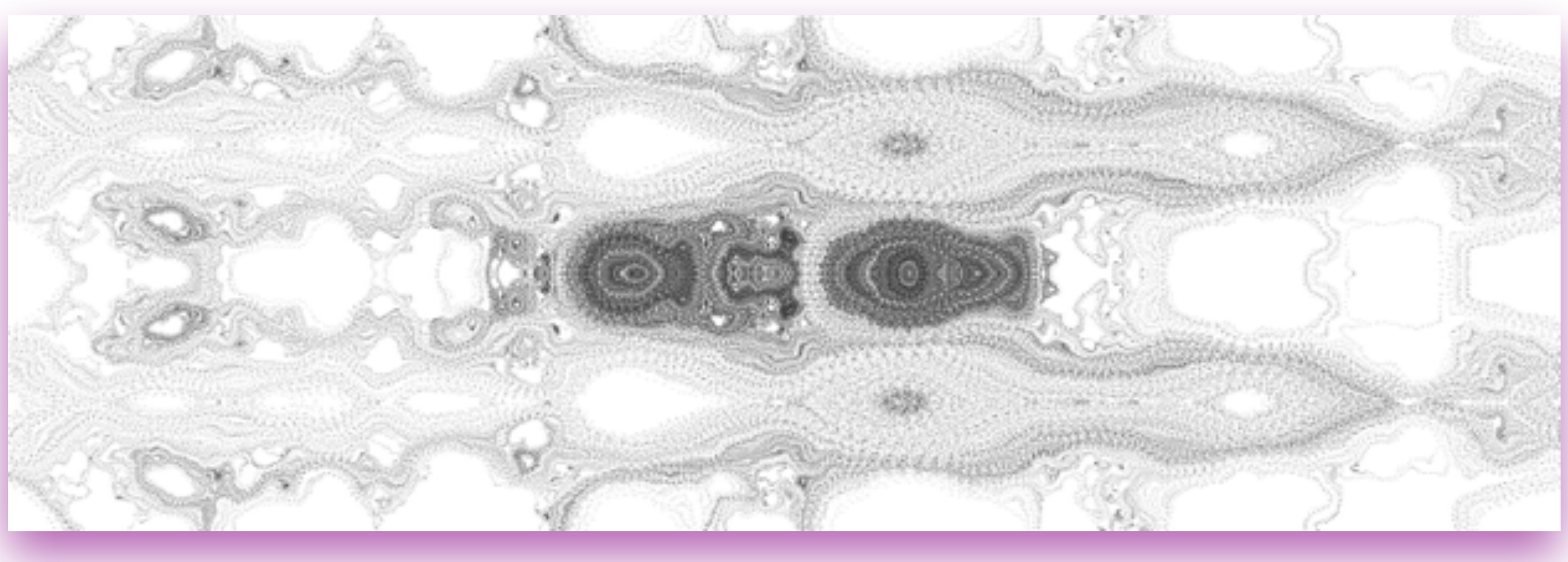

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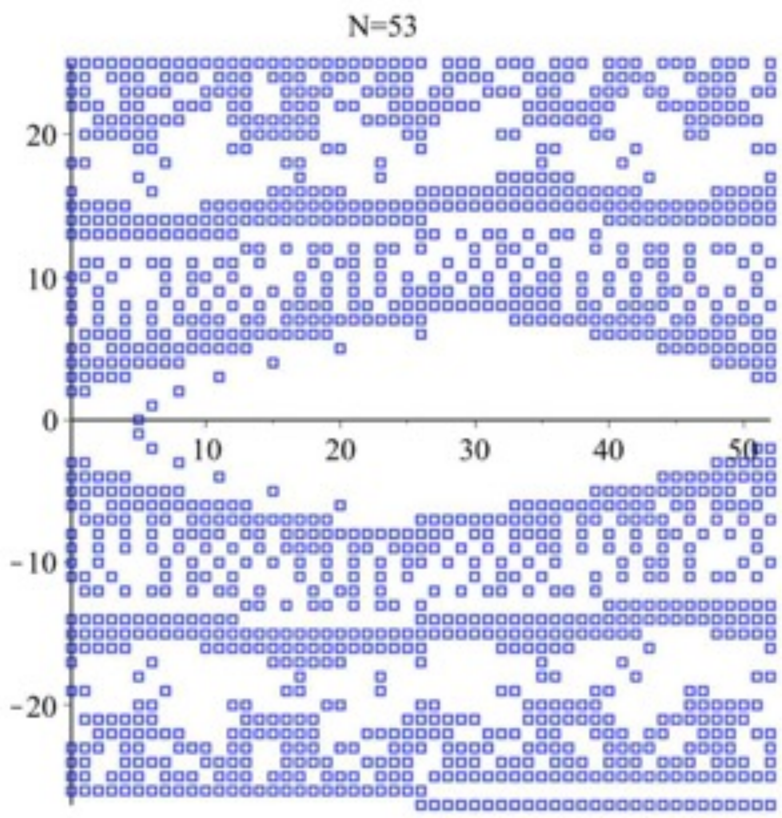

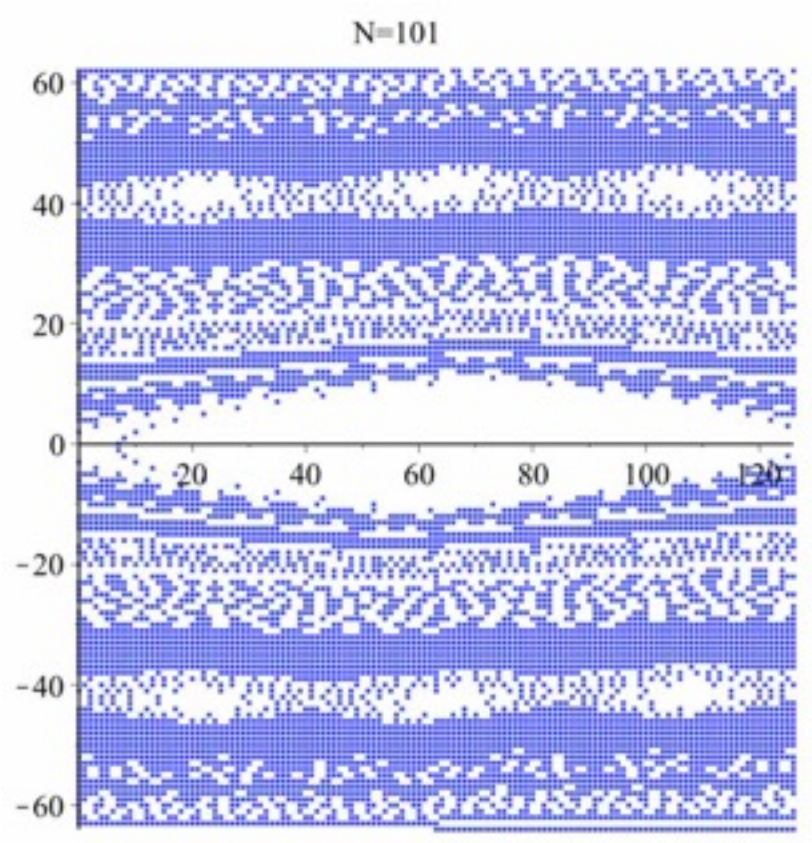

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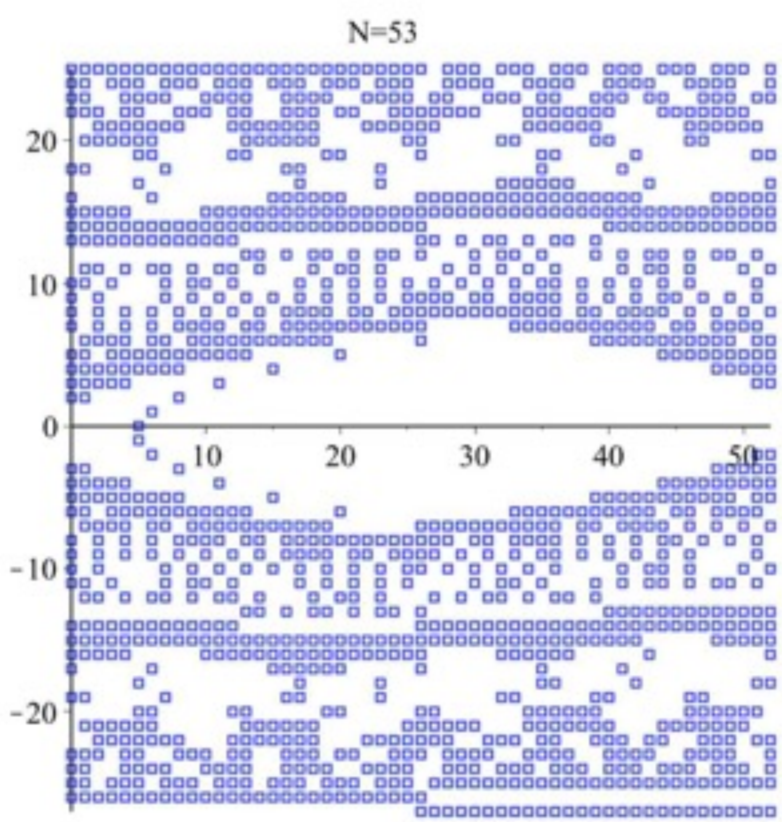

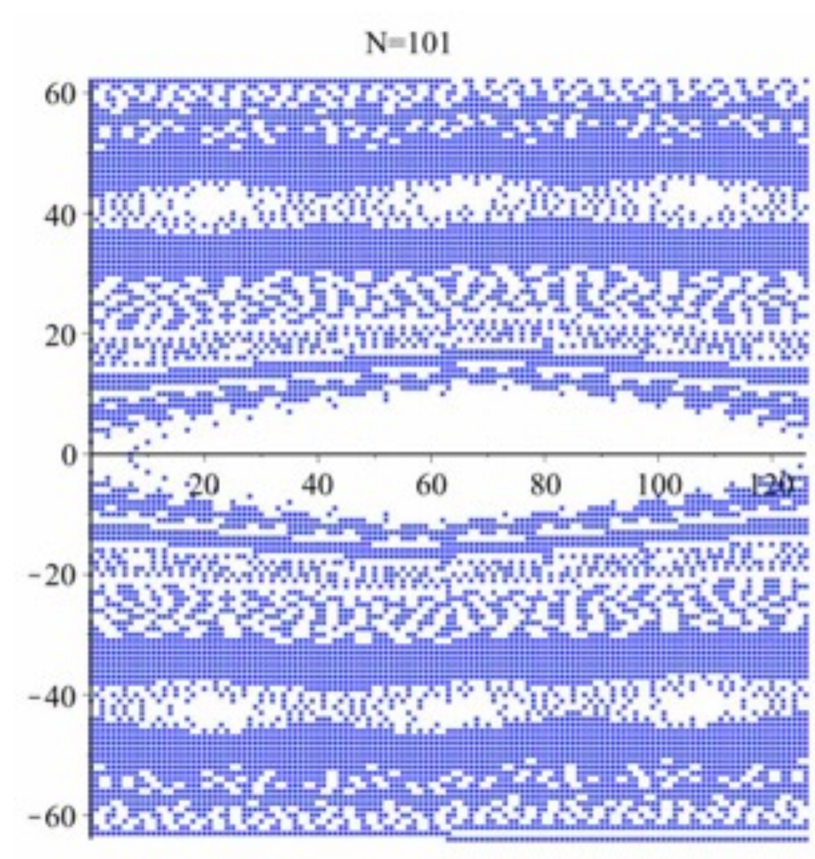

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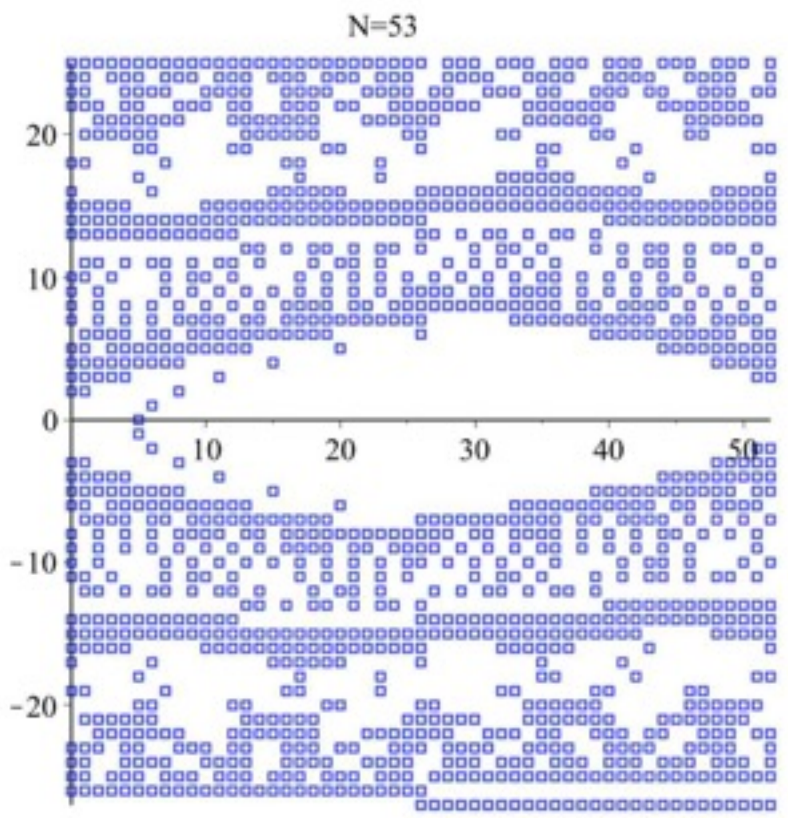

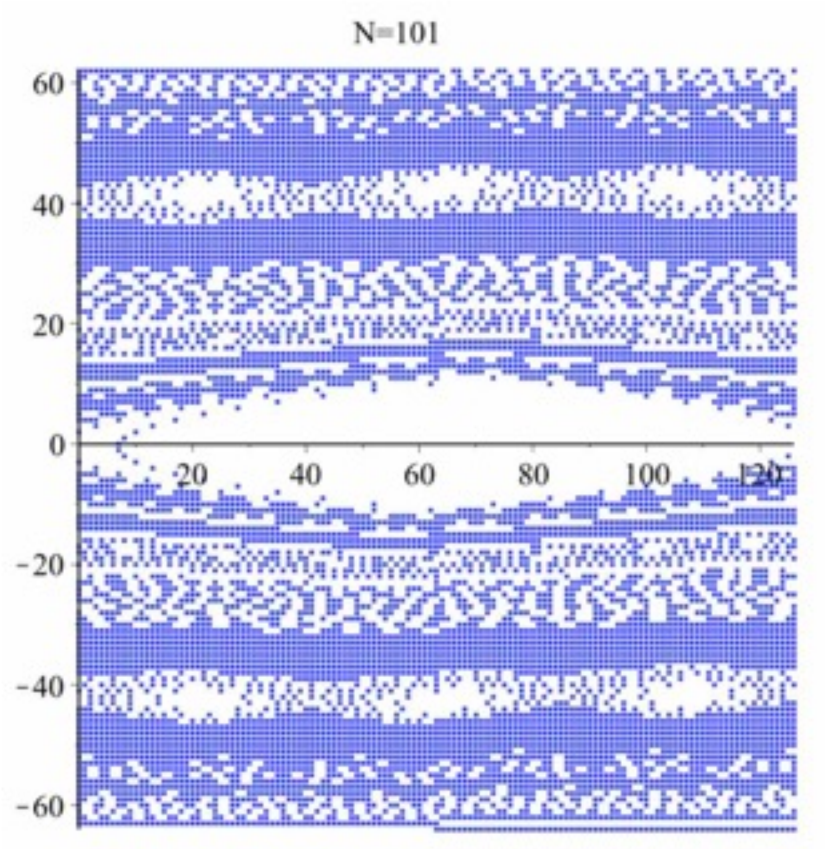

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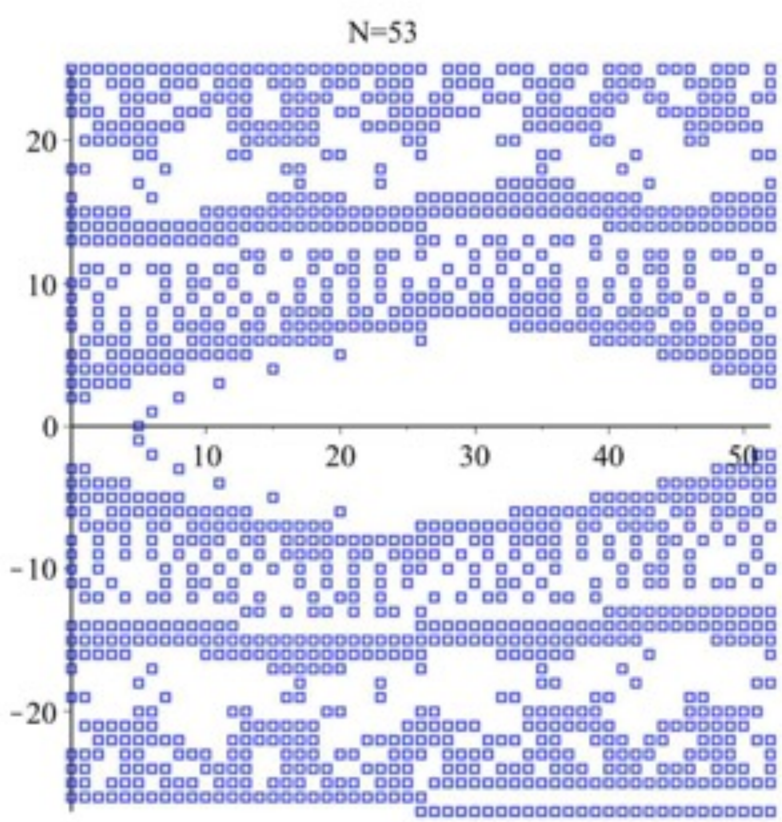

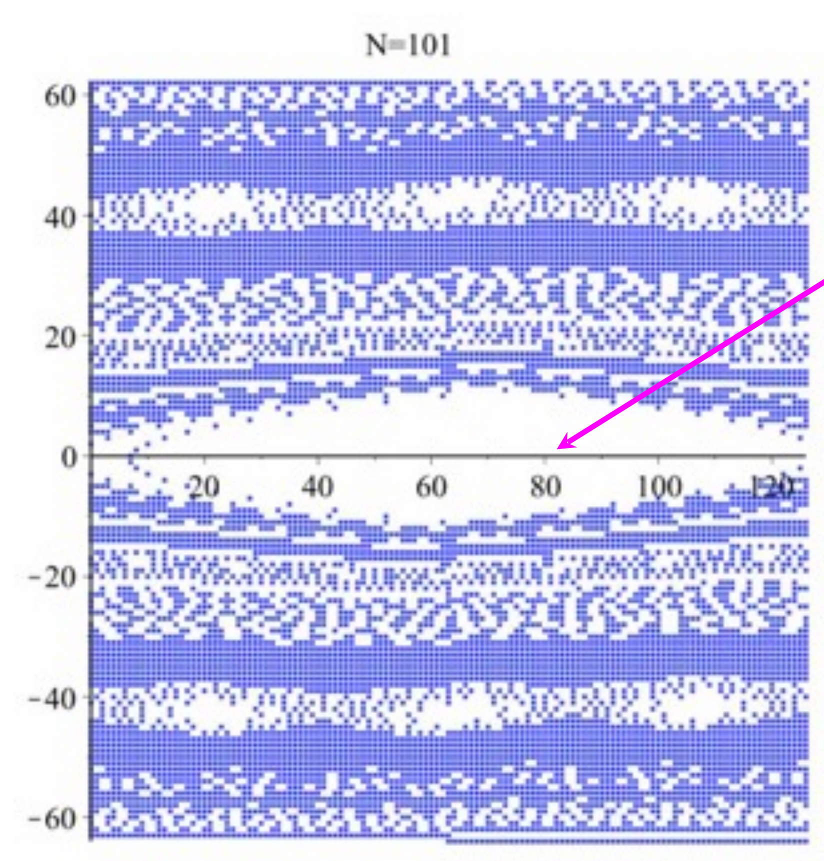

what is the dynamics inside an island?

- There is a small number of very long cycles.
- Islands of stability exist, for **some** rotation numbers.

The local map for an island

$$\mathscr{F} : \mathbb{Z}^2 \to \mathbb{Z}^2$$

$$\begin{aligned} y_{t+1} &= y_t - \operatorname{sign}(x_t) \\ x_{t+1} &= x_t + \alpha y_{t+1} + \beta \end{aligned}$$

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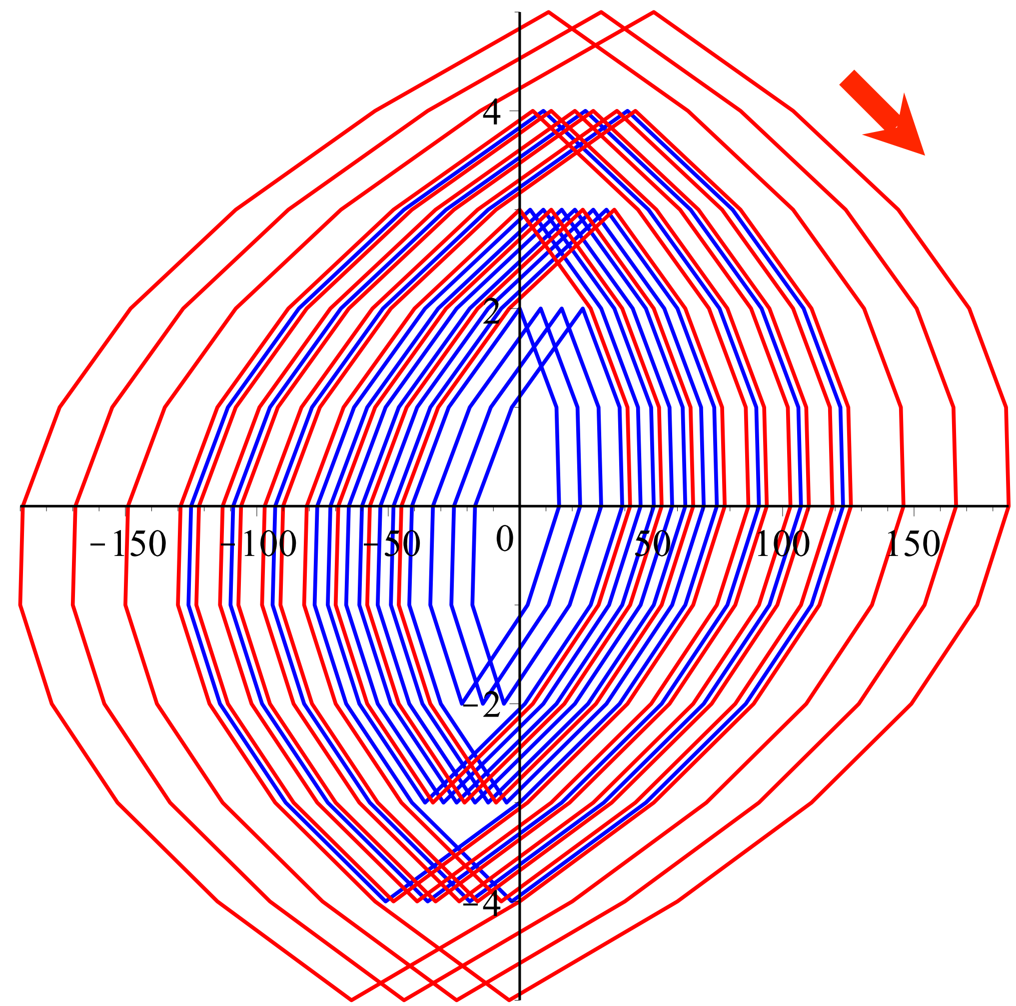
two overlapping orbits



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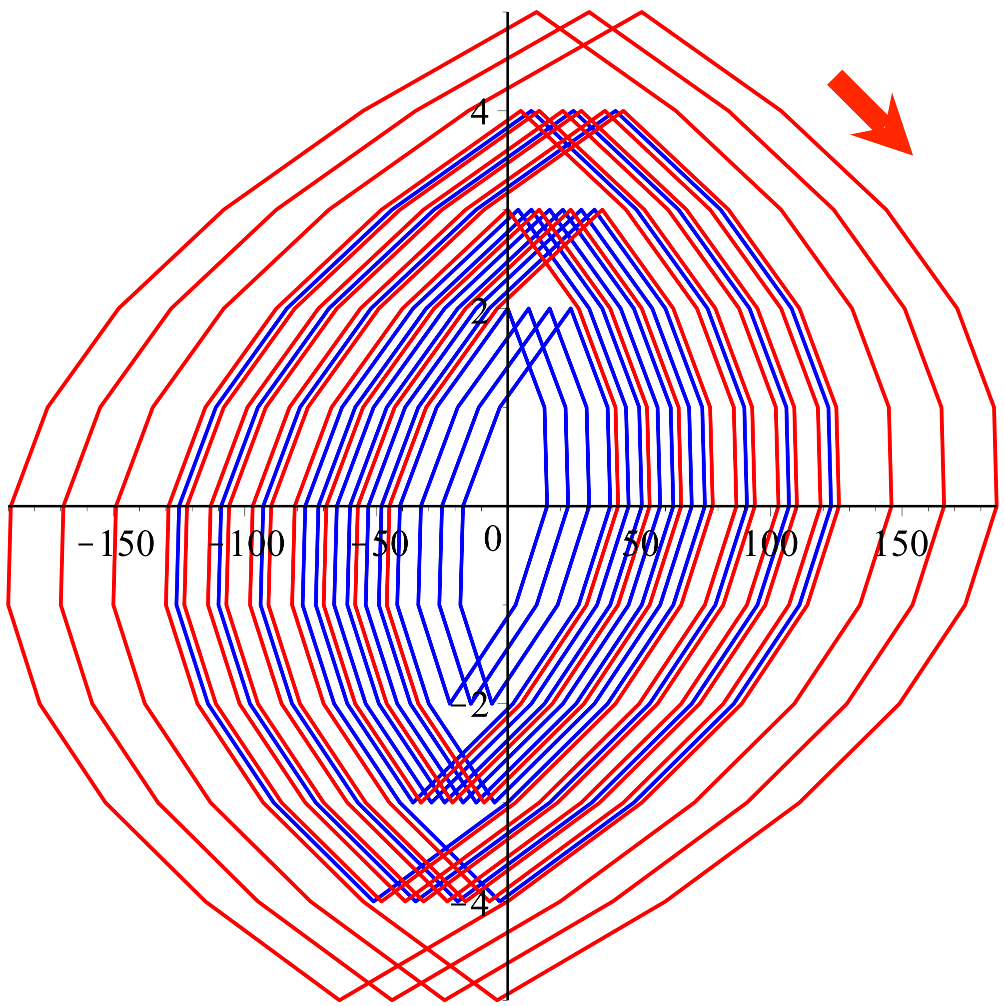

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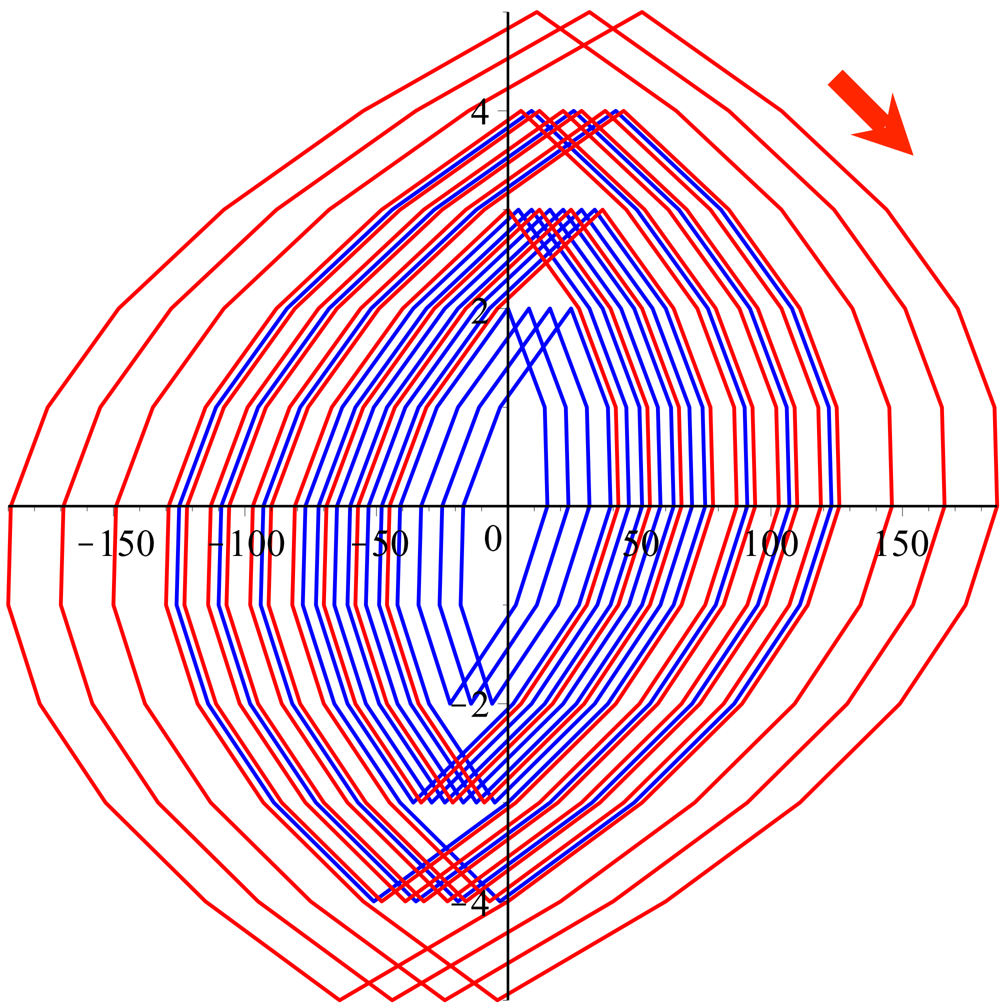
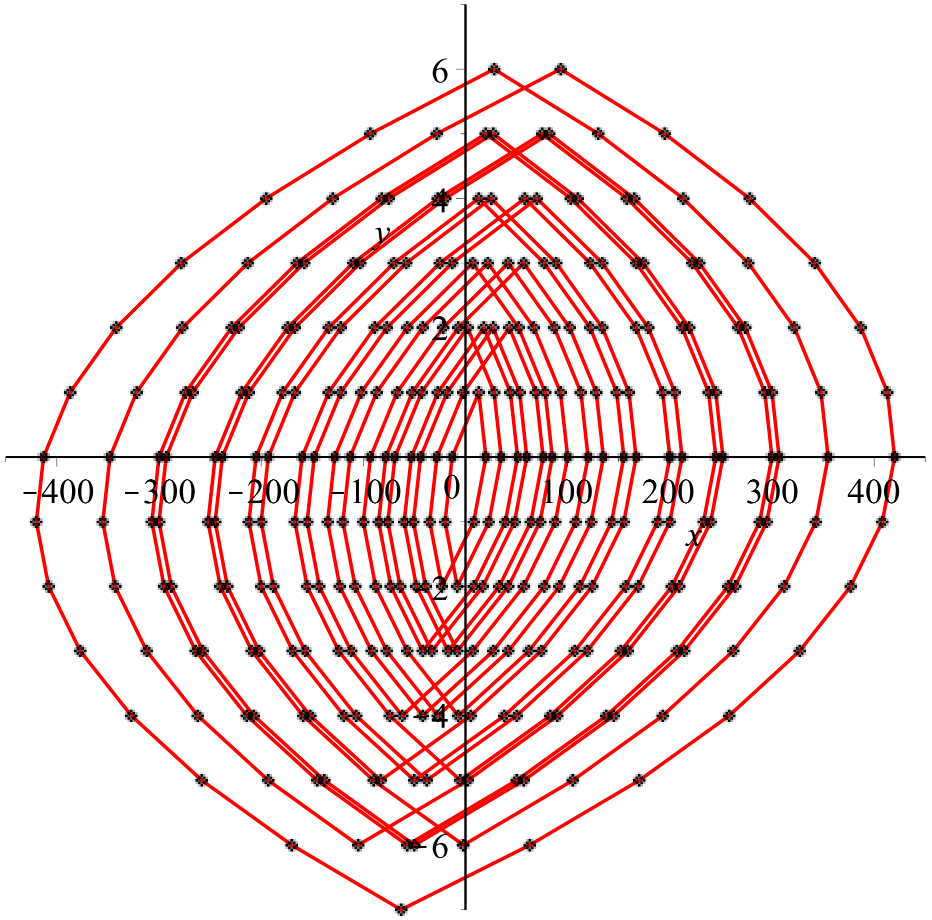
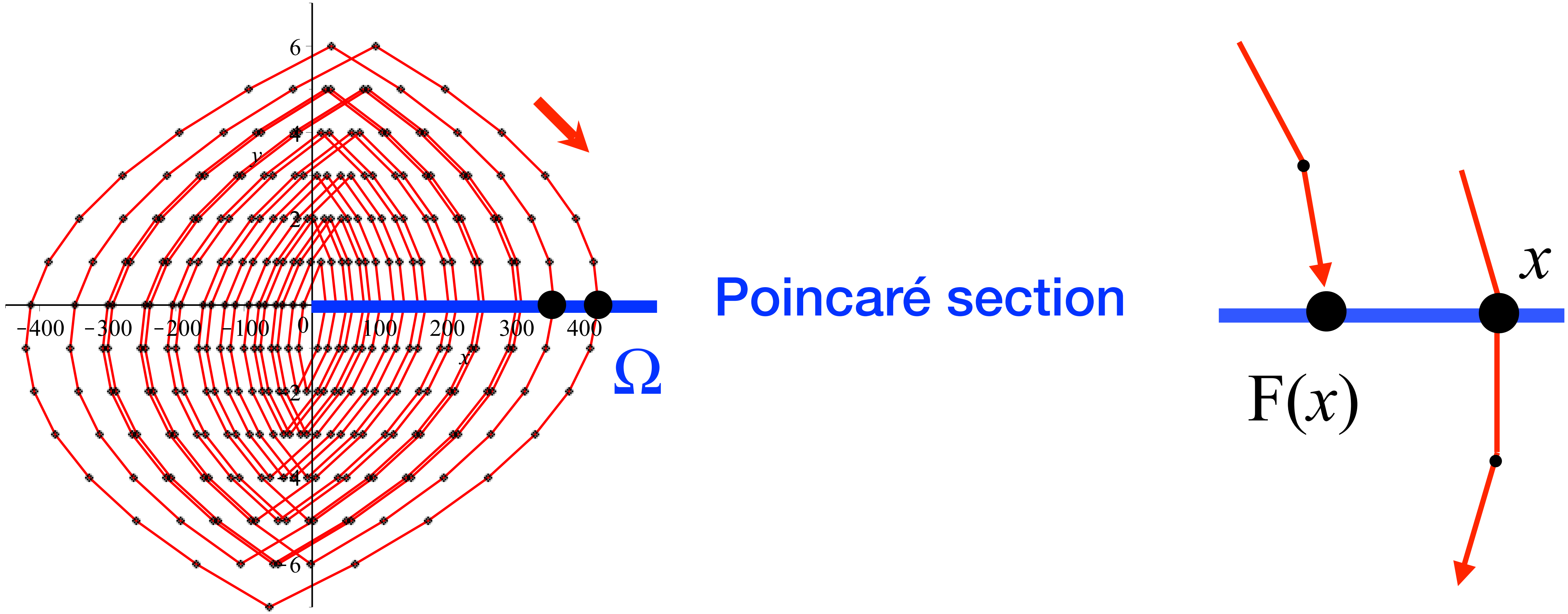


As the amplitude increases, the rotational period increases without bound.

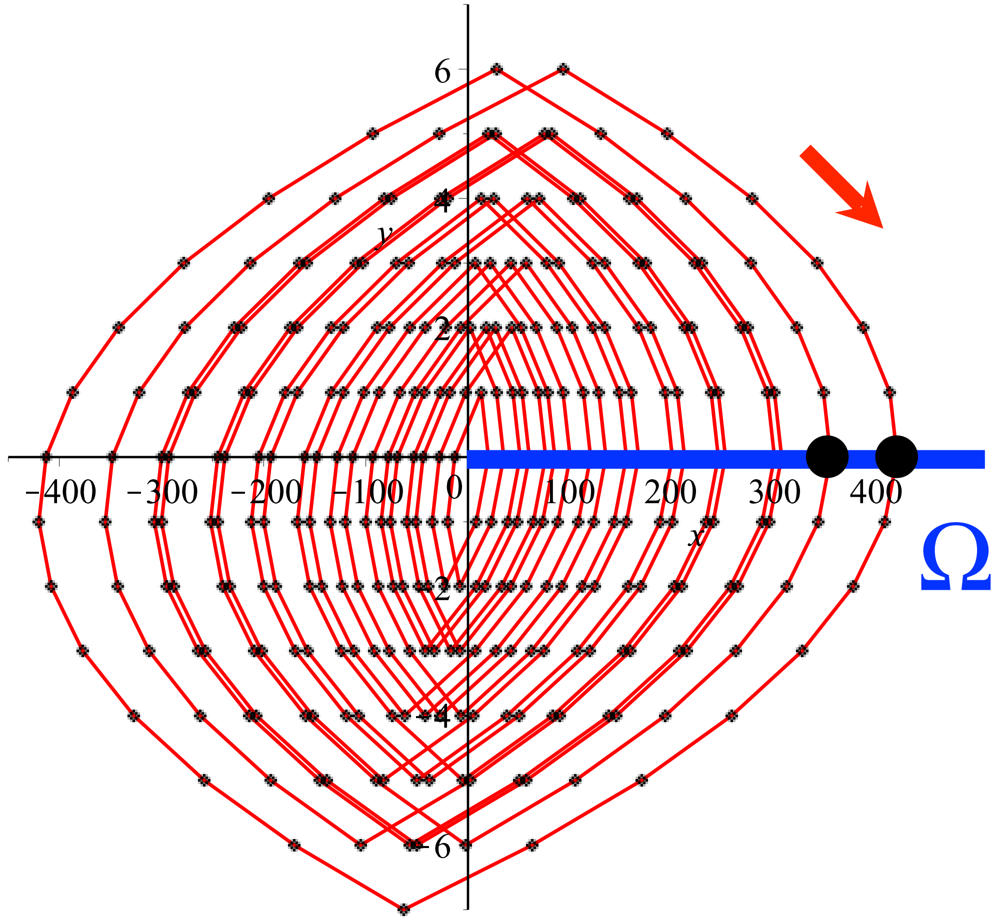
Observation. Depending on the parameter values, all orbits have the same character: they are either all periodic, or all unbounded.



Poincaré section



6
4
2
y
−2
−4
−6
−400
−300
−200
−100
0
100
200
300
400
x
Ω
Poincaré section
x
F(x)



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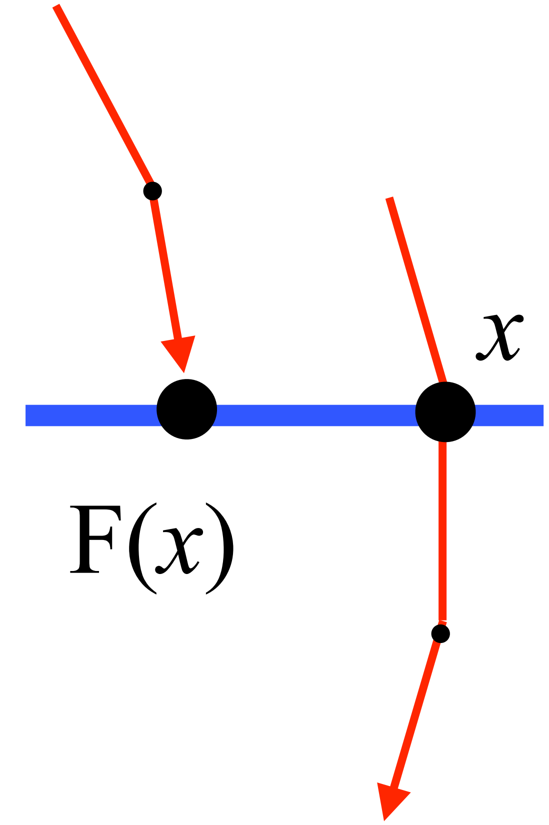
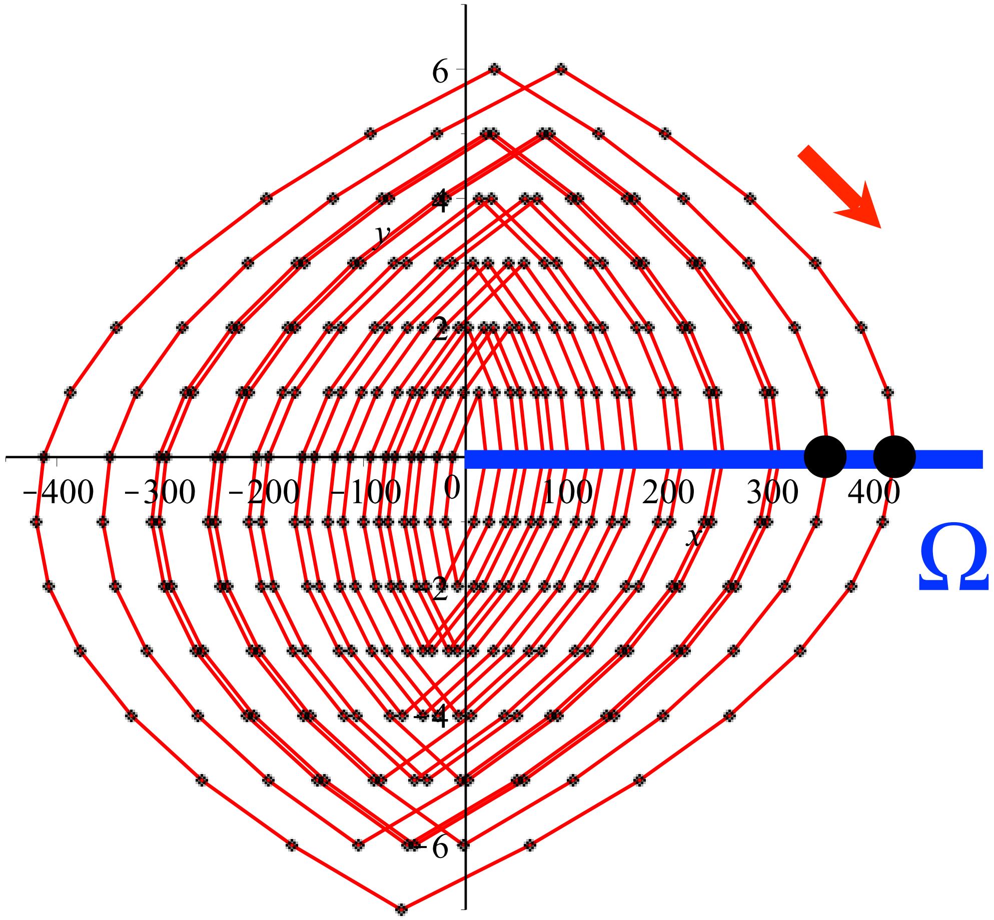


F is the first-return map to the non-negative integer abscissa axis Ω.



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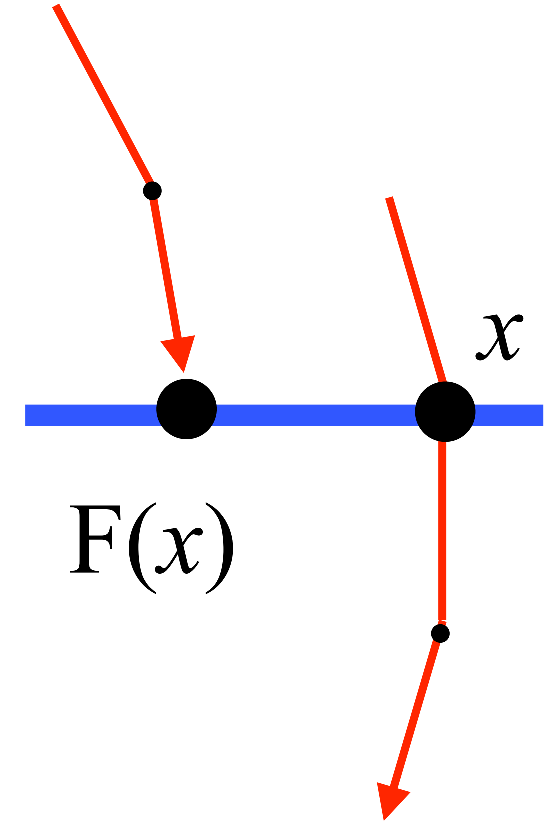

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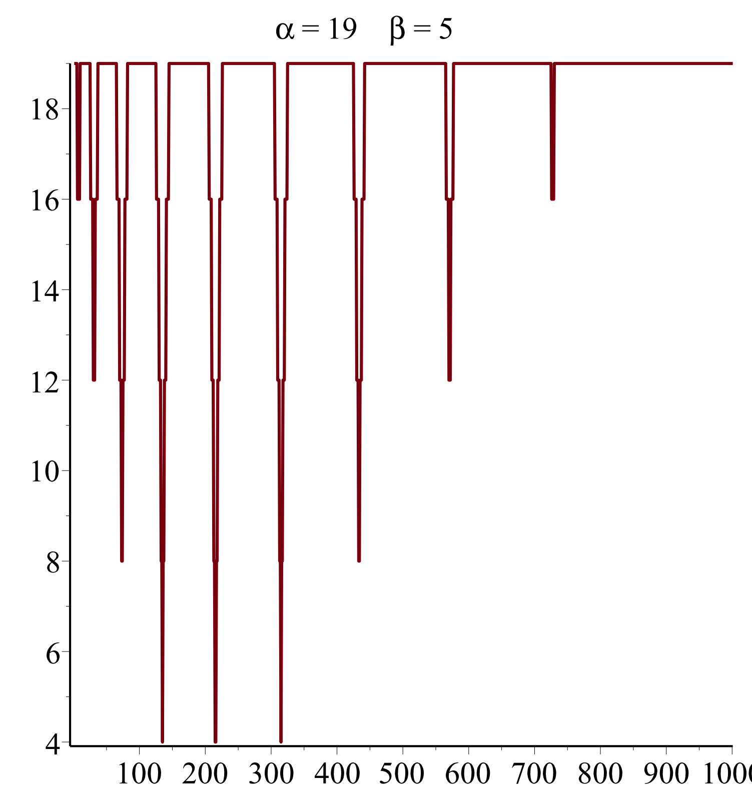
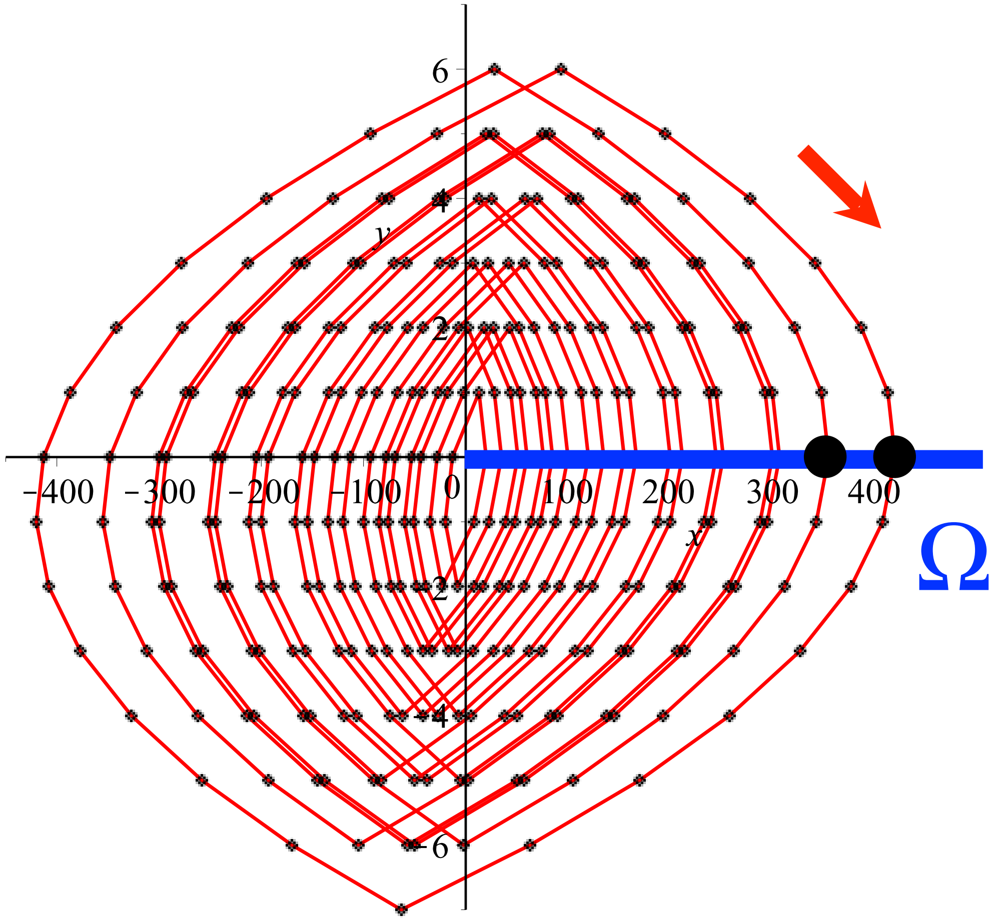
Observation. If the orbits are periodic, then, close to the origin, their period depends nontrivially on the initial condition.





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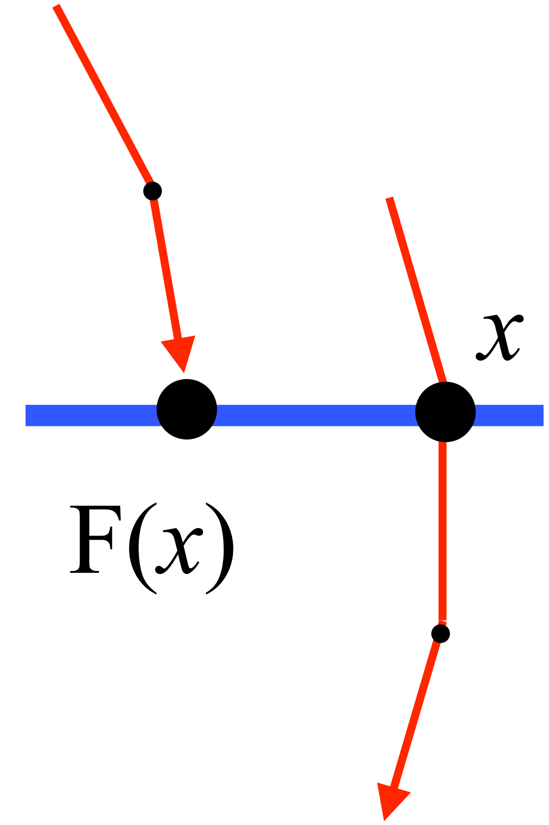

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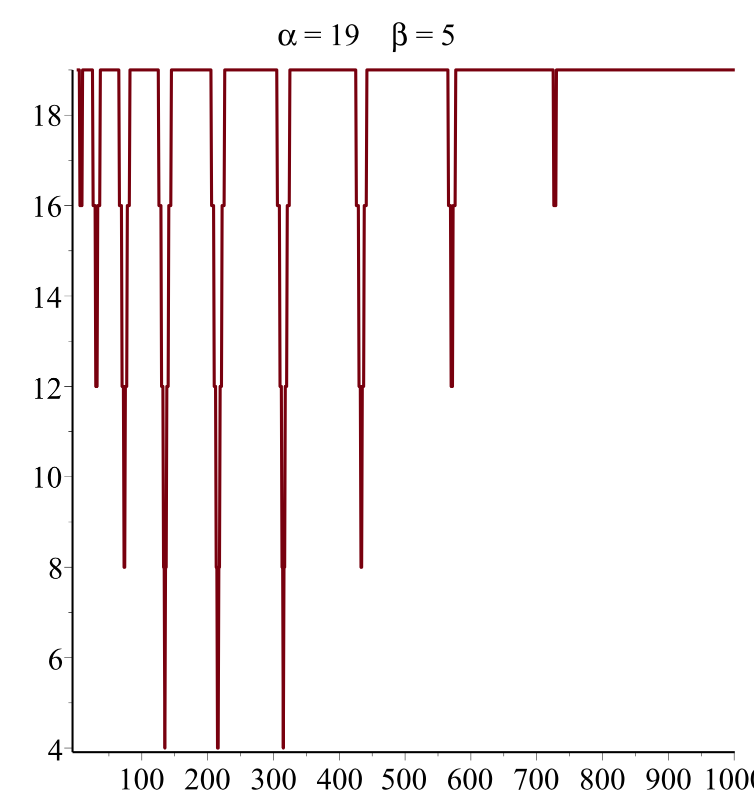
Far from the origin, the periods appear to stabilise.

period



initial condition

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Conjecture: **P**$=\Omega$ and **P**$=\varnothing$, respectively.

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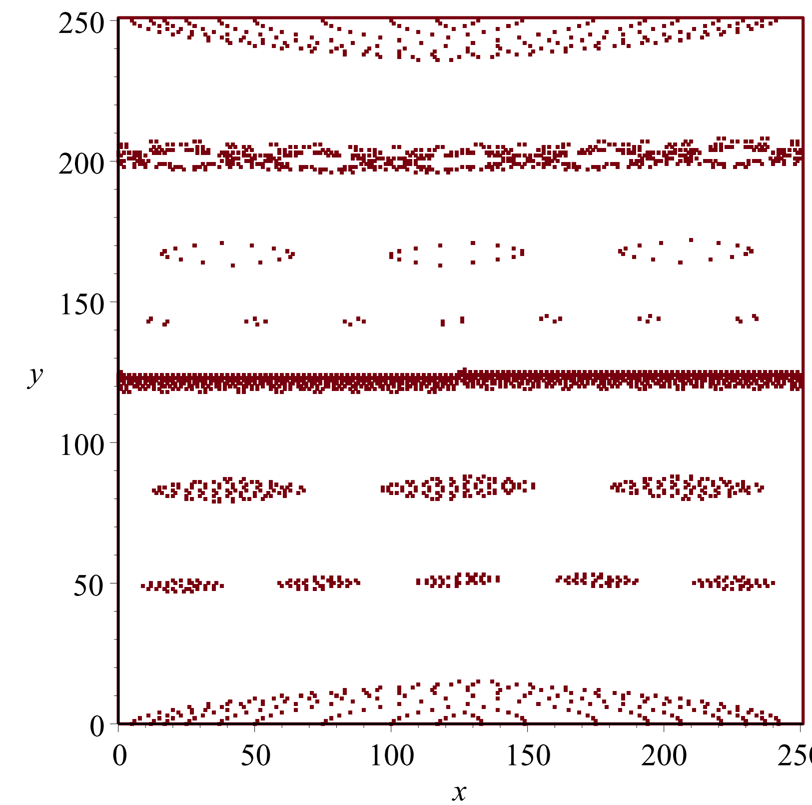

Only islands of odd order can exist, if N is large enough.

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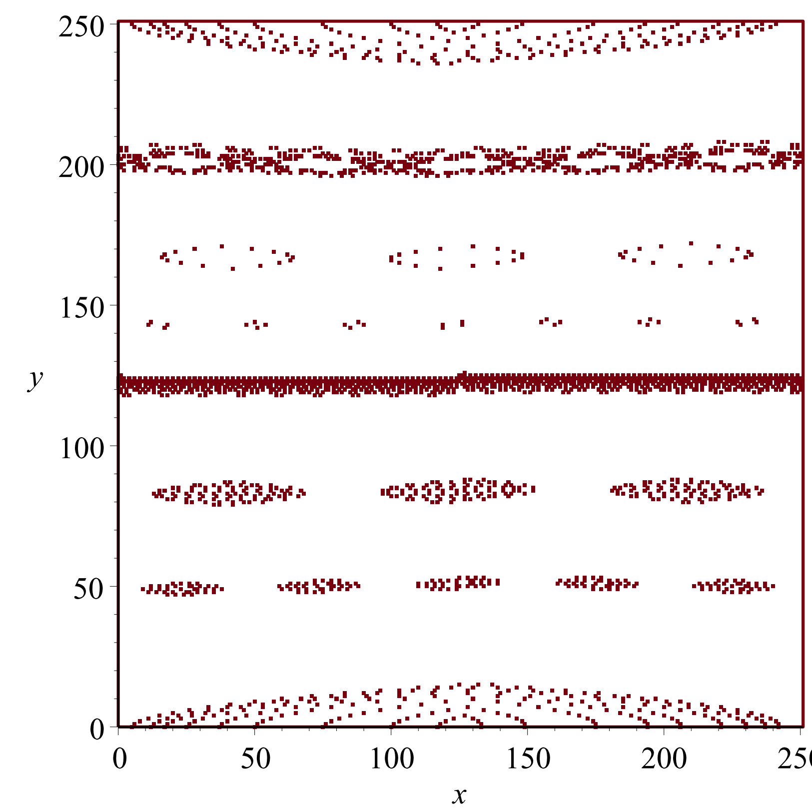

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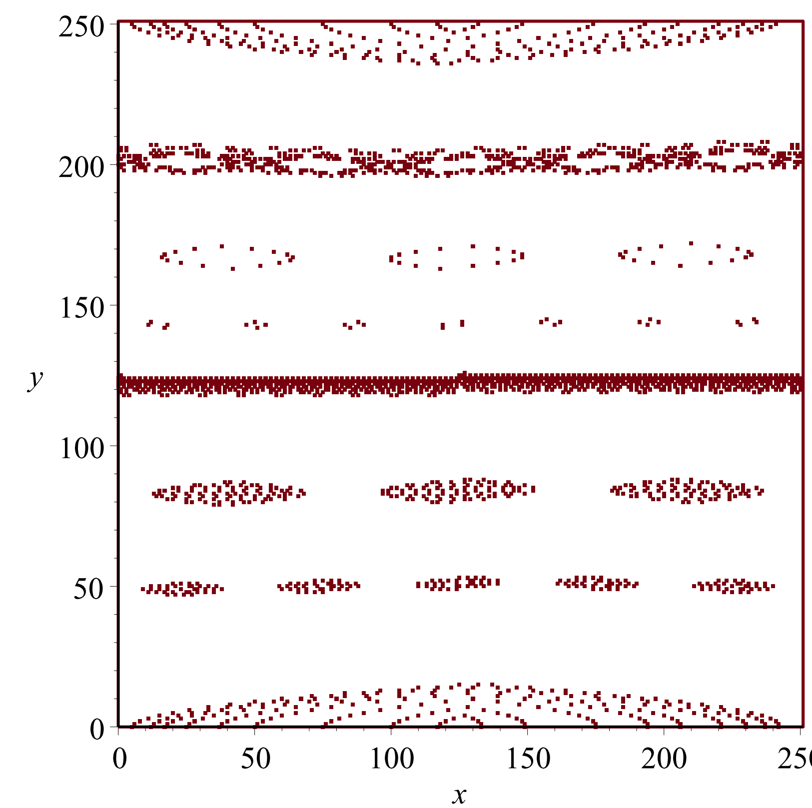

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The dynamics for non-coprime pairs can be reduced to that of coprime pairs.

Trivial pairs: $(\alpha,0)$, $(2\beta,\beta)$

Conjugate pairs: (α,β), $(\alpha,\alpha-2\beta)$

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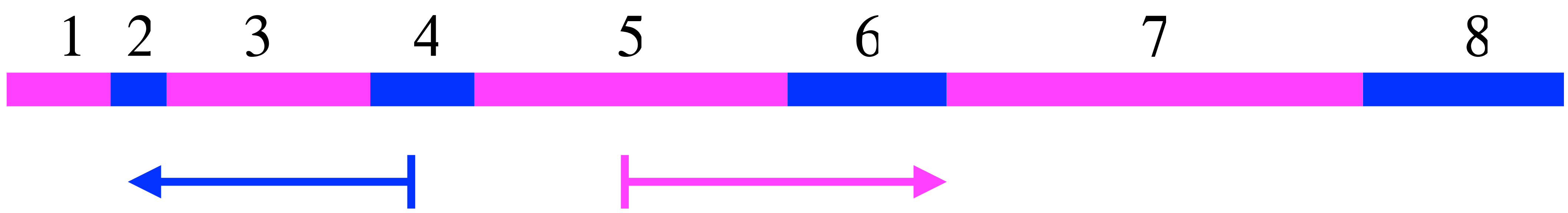

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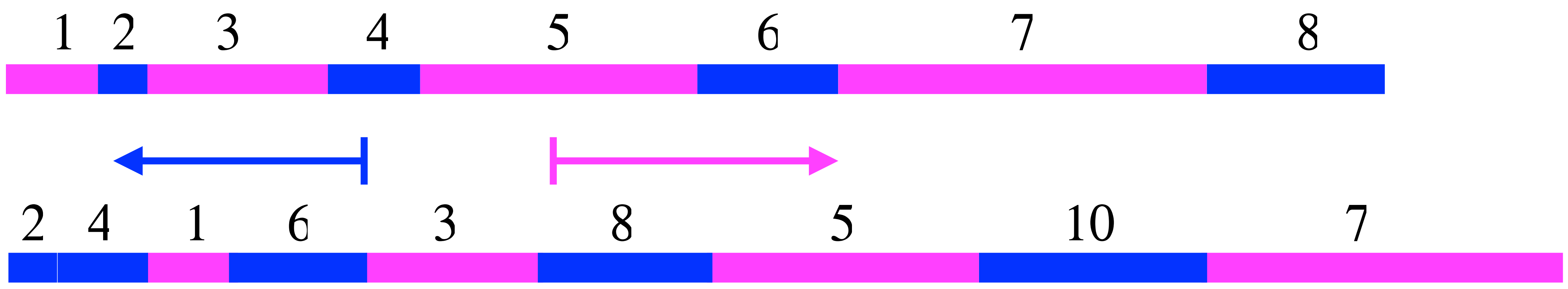

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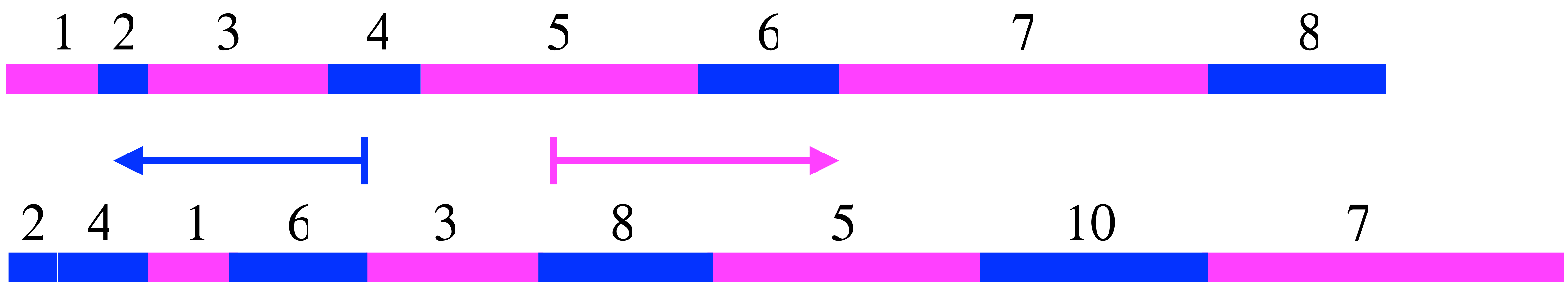

Combinatorial data: $\sigma(1,2,3,4,5,6,7,8,\ldots)=(3,1,5,2,7,4,9,6,\ldots)$

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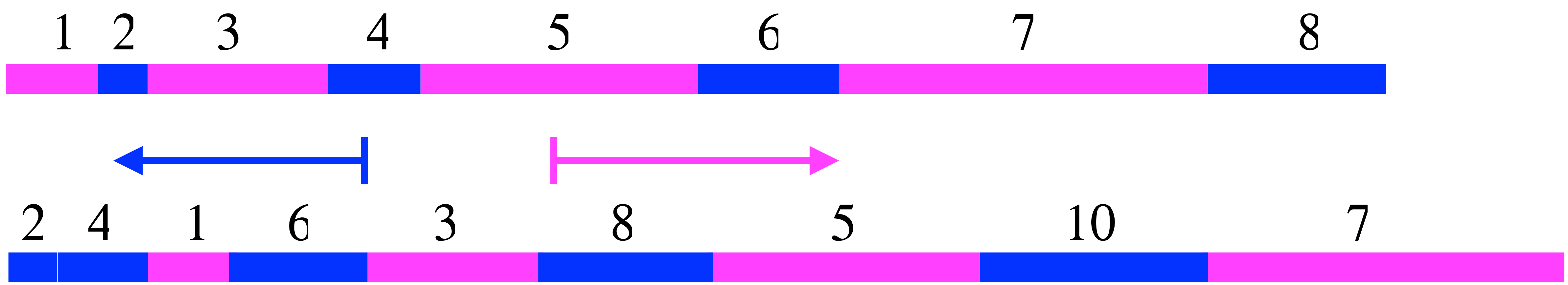

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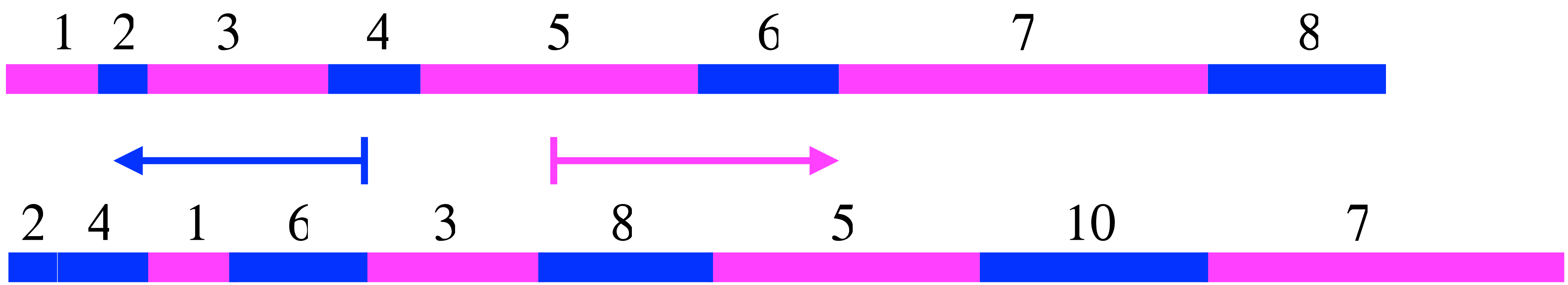

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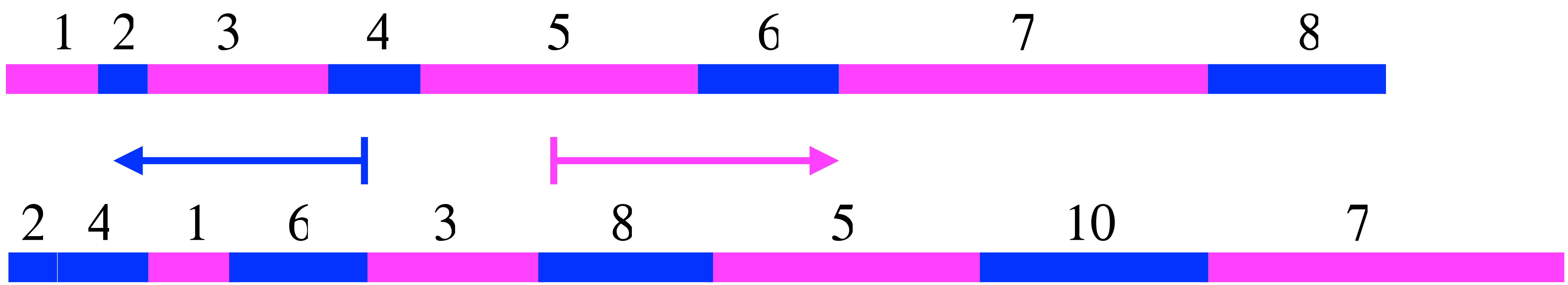

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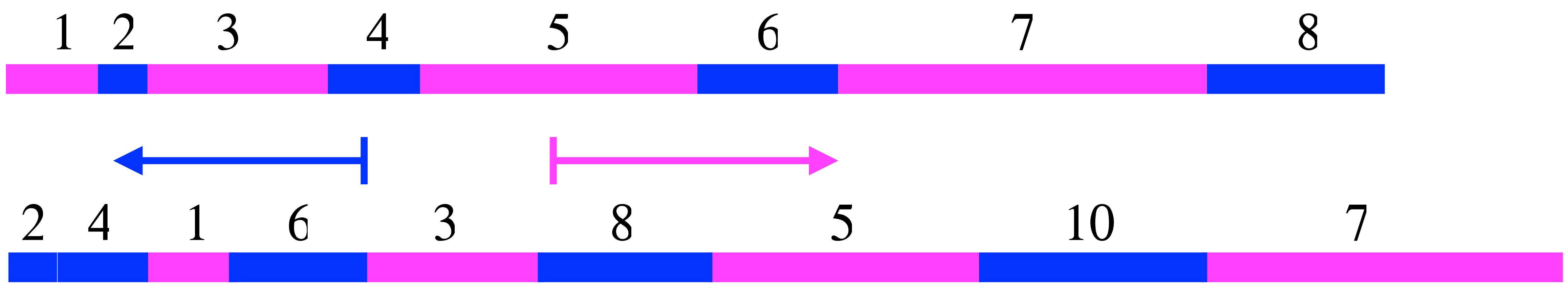

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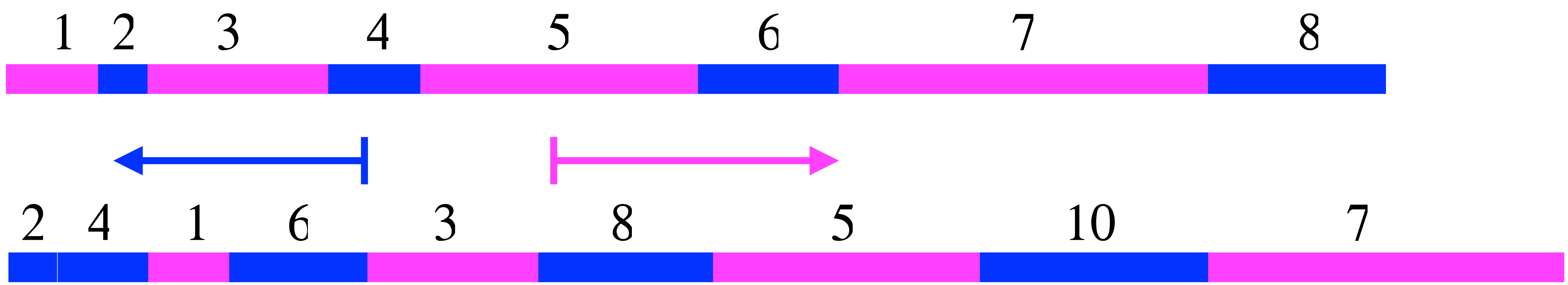
Far from the origin:

- The combinatorics is regular: $\sigma(2m)=2(m-1)$, $\quad \sigma(2m-1)=2m+1$

The Poincaré return map is an **interval-exchange transformation** over the integers, with infinitely many intervals Δ_n, $n=1,2,3,\ldots$



Combinatorial data: $\sigma(1,2,3,4,5,6,7,8,\ldots)=(3,1,5,2,7,4,9,6,\ldots)$

Metric data: $|\Delta_{2m}| \sim 2\beta m$ $\quad |\Delta_{2m-1}| \sim (\alpha-2\beta)m$

Symbolic dynamics: the sequence of labels of the intervals Δ_n.

Far from the origin:

- The combinatorics is regular: $\sigma(2m)=2(m-1)$, $\quad \sigma(2m-1)=2m+1$
- The size of many cylinder sets appears to grow linearly with the order of the intervals.

Scaling the interval lengths, we to obtain the **reduced system** F´,

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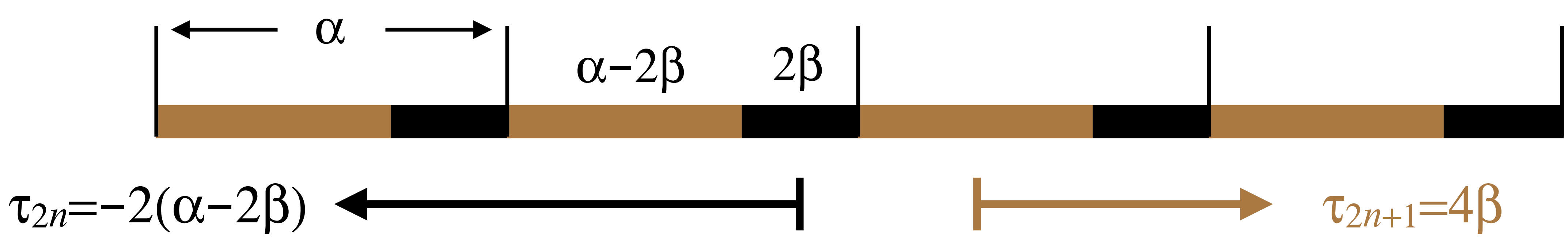

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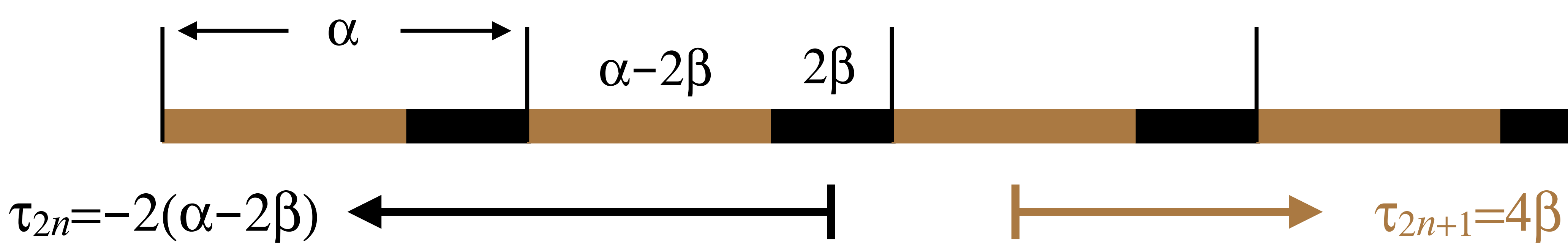
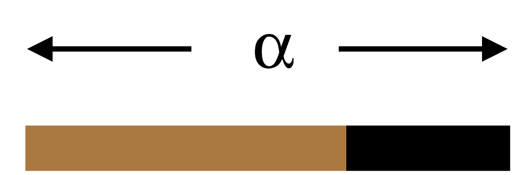


- Spatially periodic: the basic periodic structure is a **block**.



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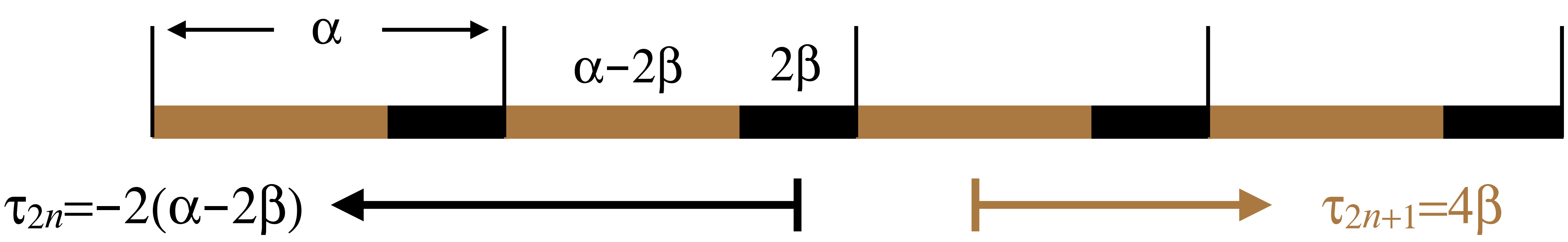

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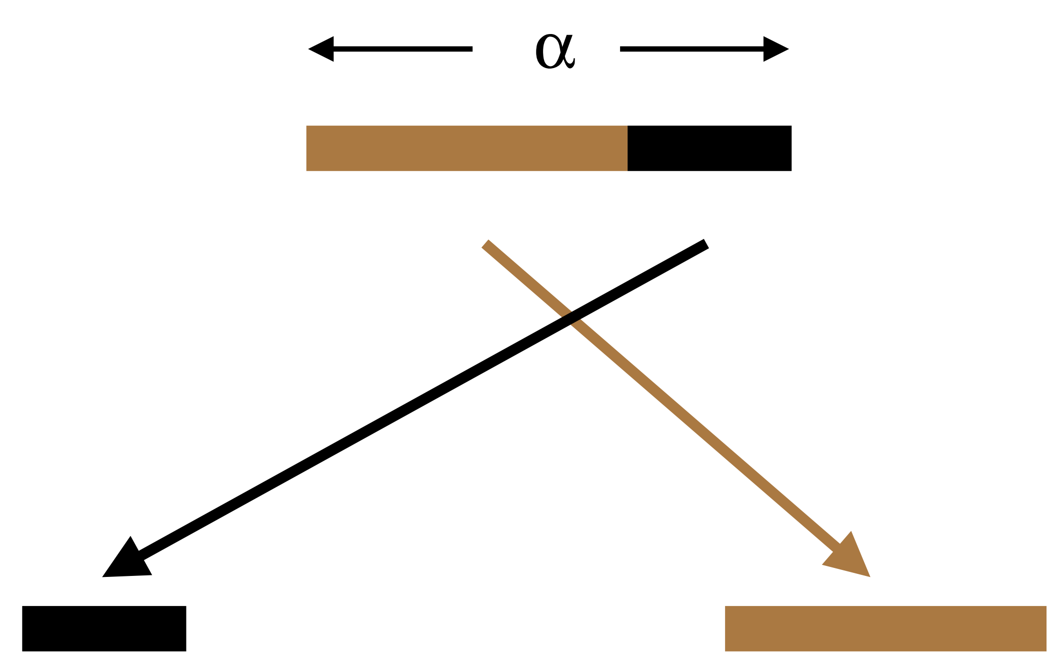

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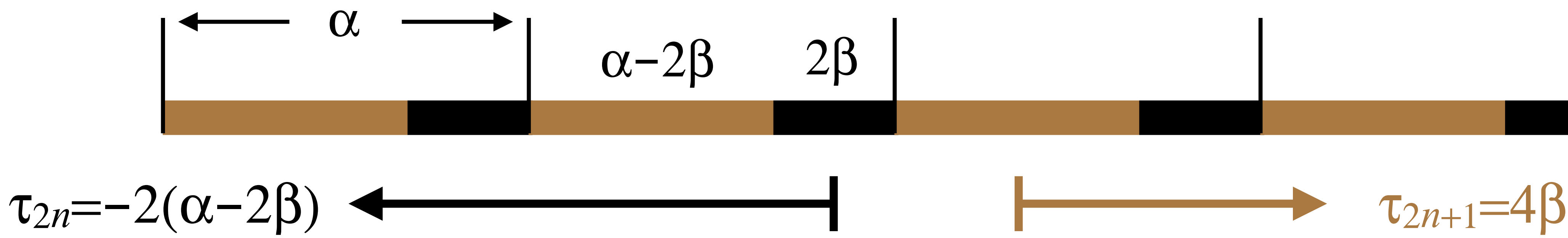

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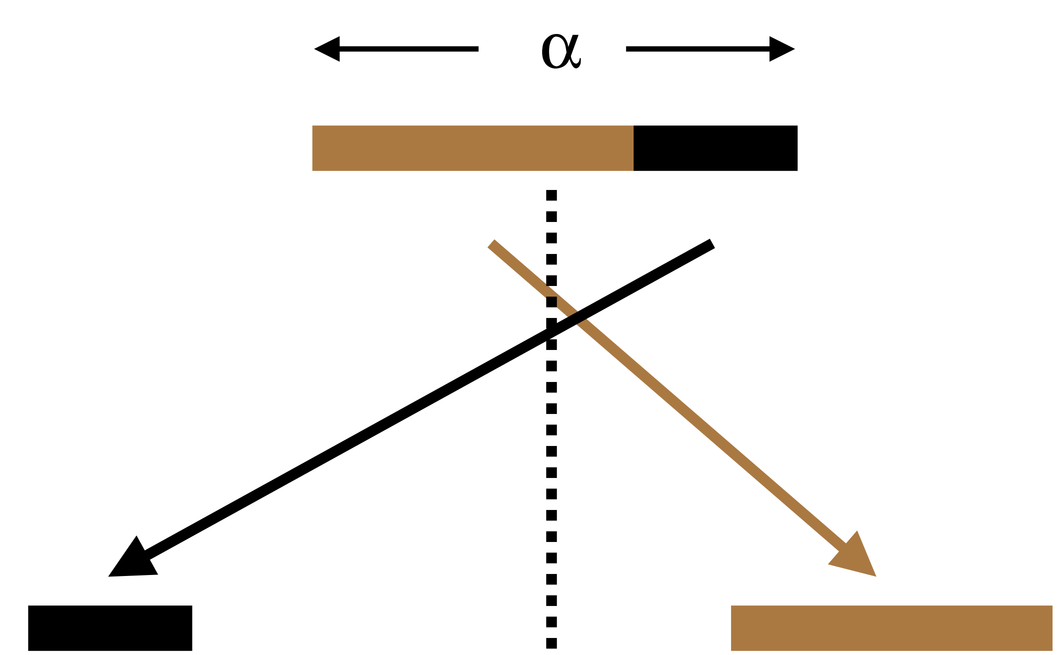
- The centre of mass of a block is preserved under iteration.



Transitions between blocks

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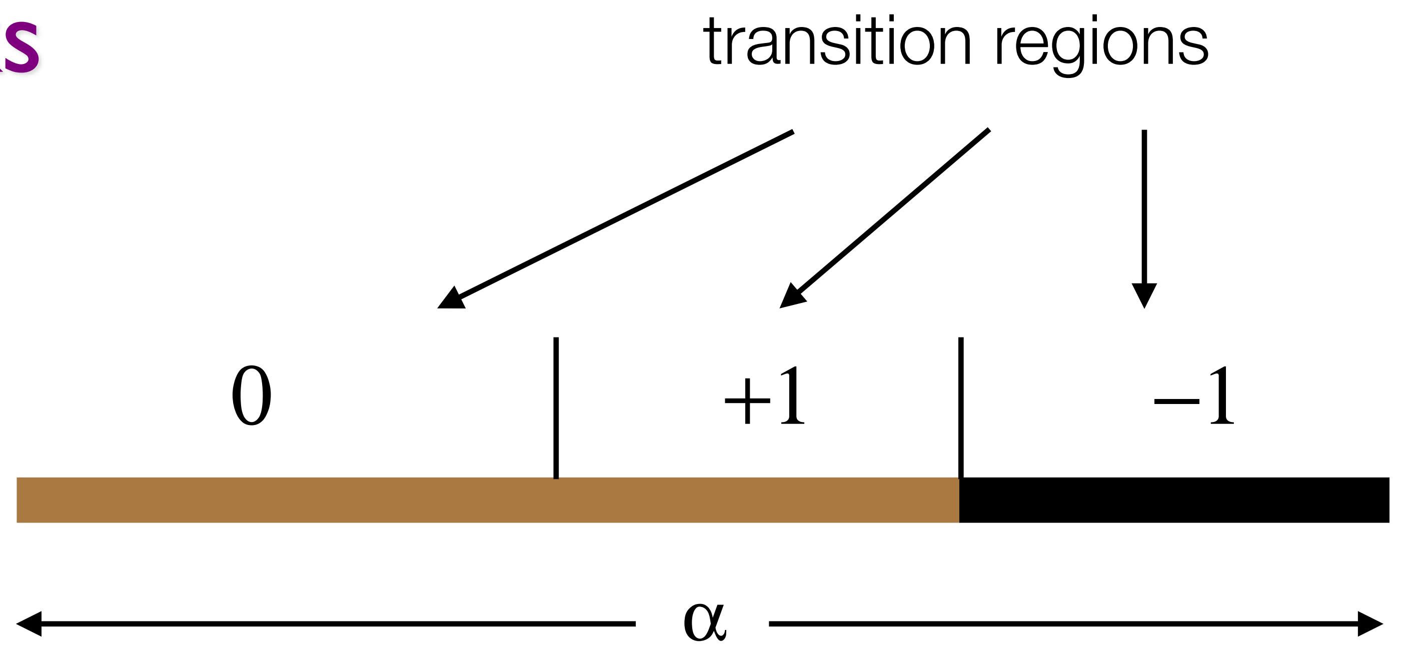

Transitions between blocks

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+1: move to next block on the right

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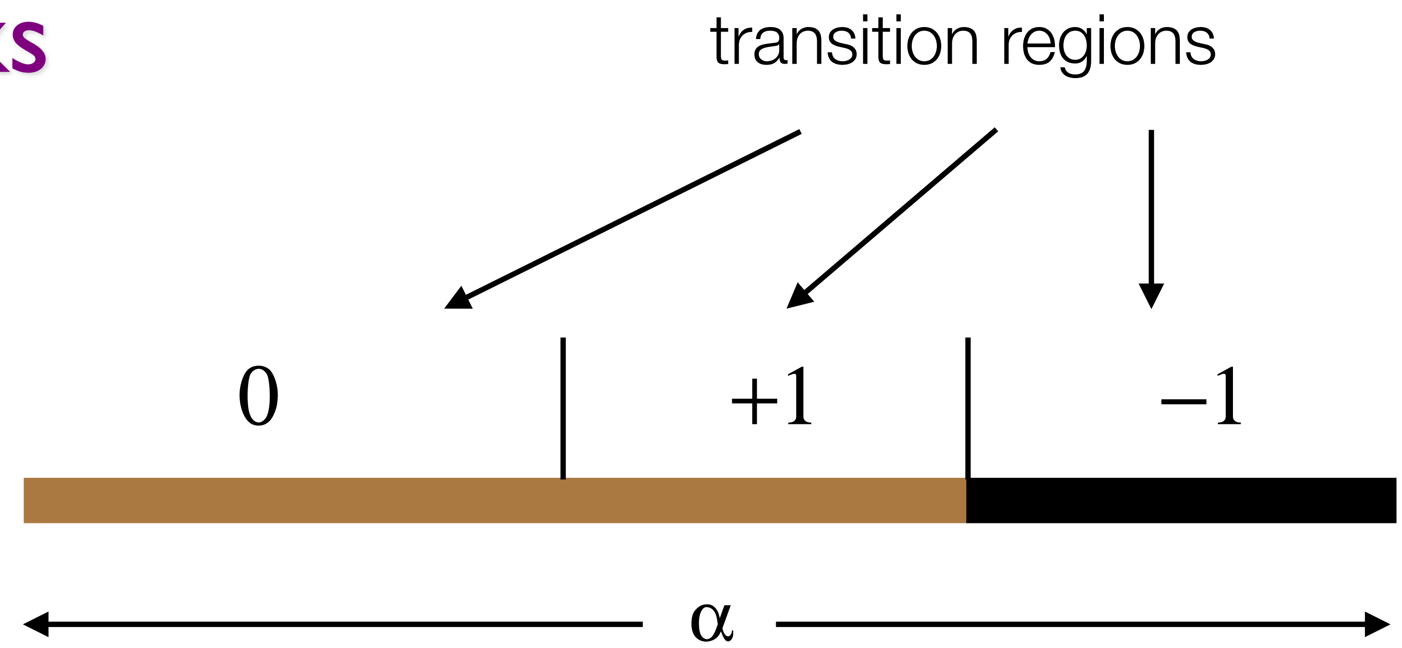

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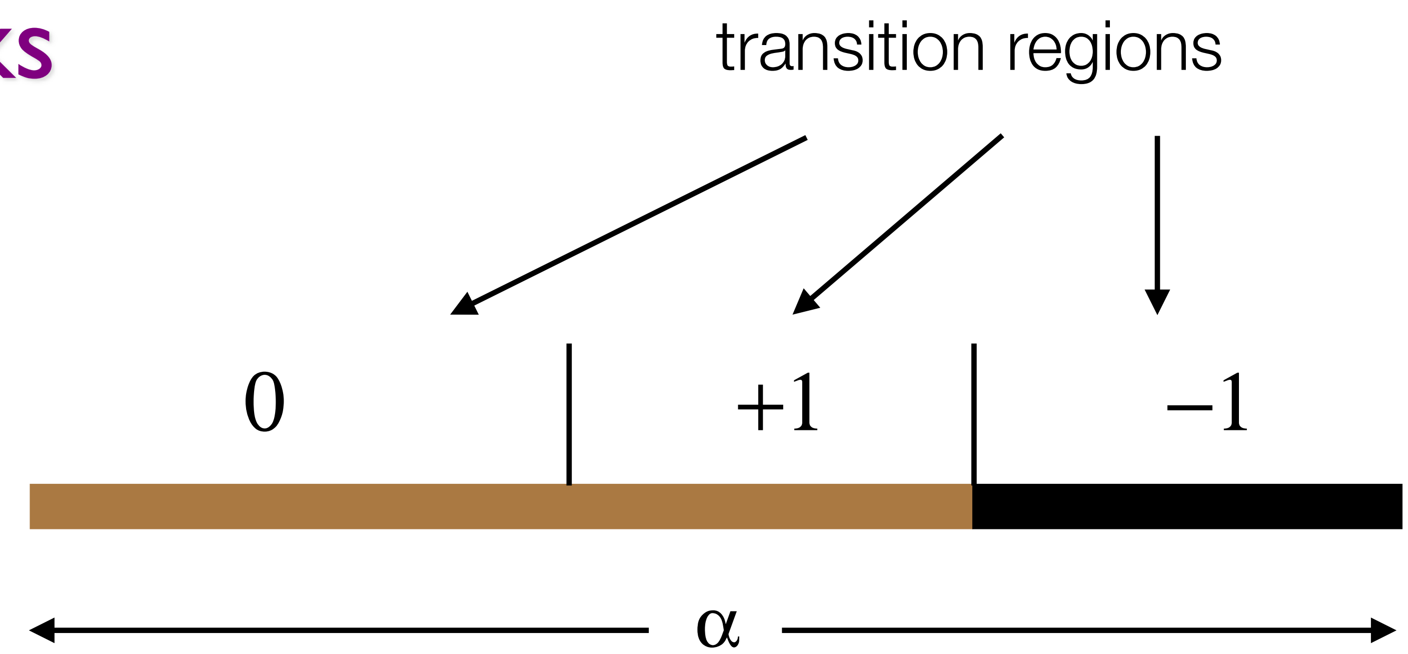
±1 regions have the same length



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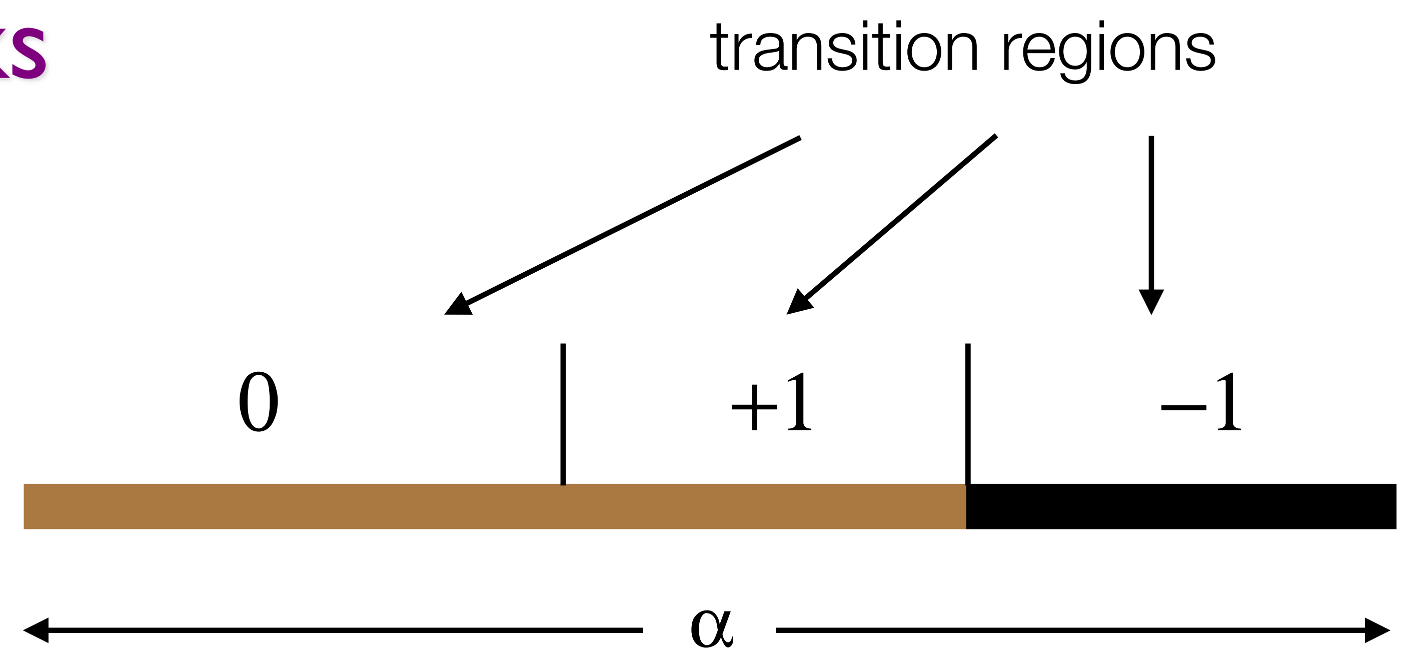

Dynamics modulo α: $z_{t+1} \equiv z_t + 4\beta \pmod{\alpha}$

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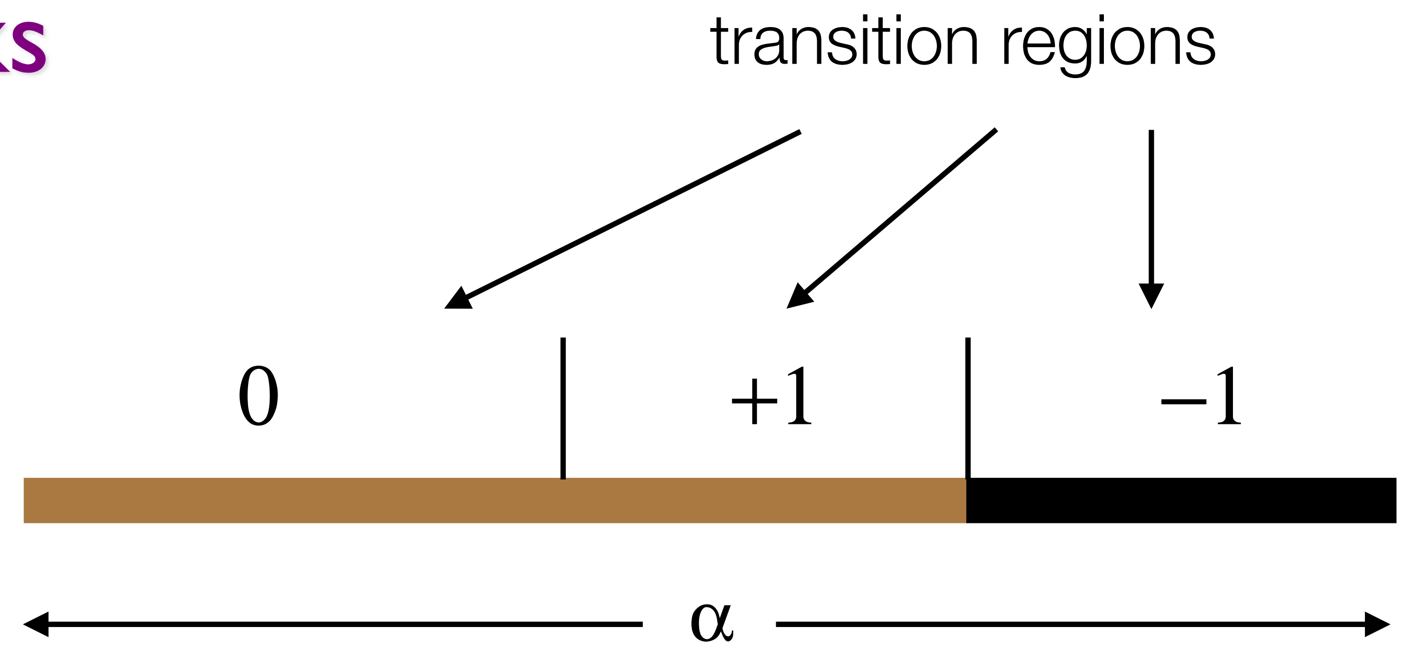

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The reduced Poincaré map is a **skew system**, a walk on the integers driven by a rotation.

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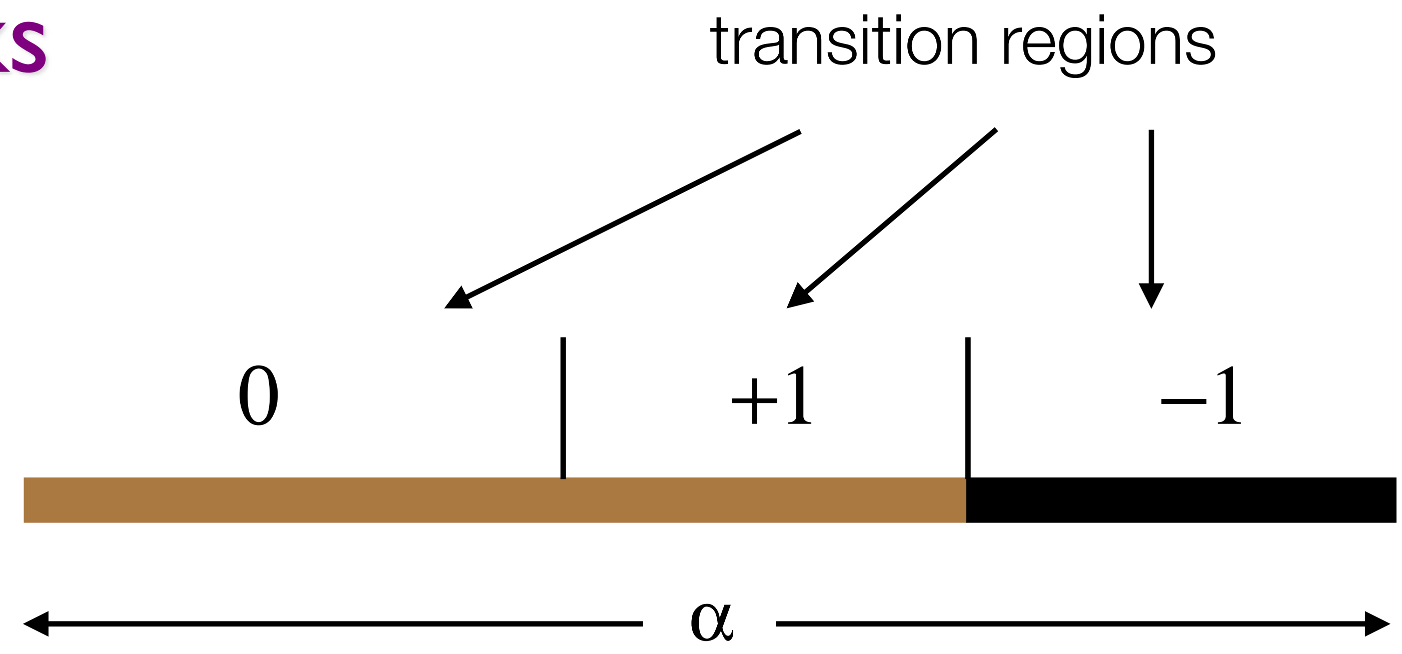

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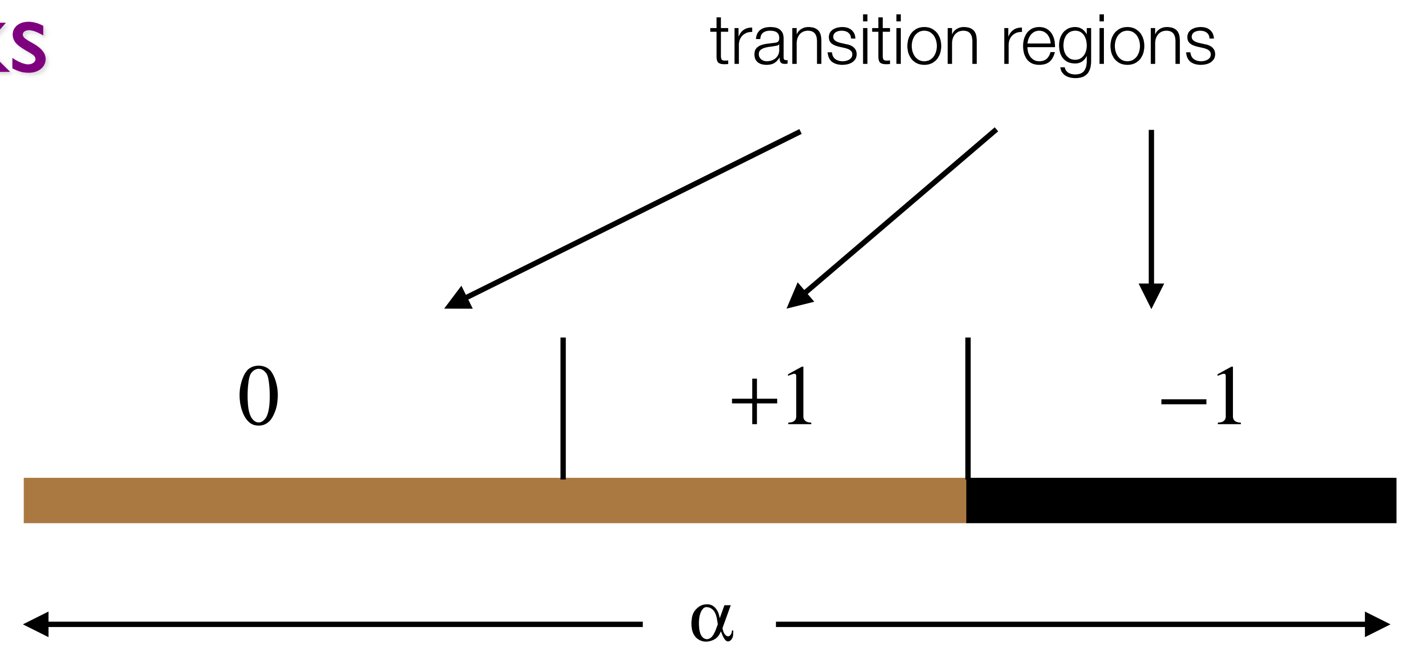

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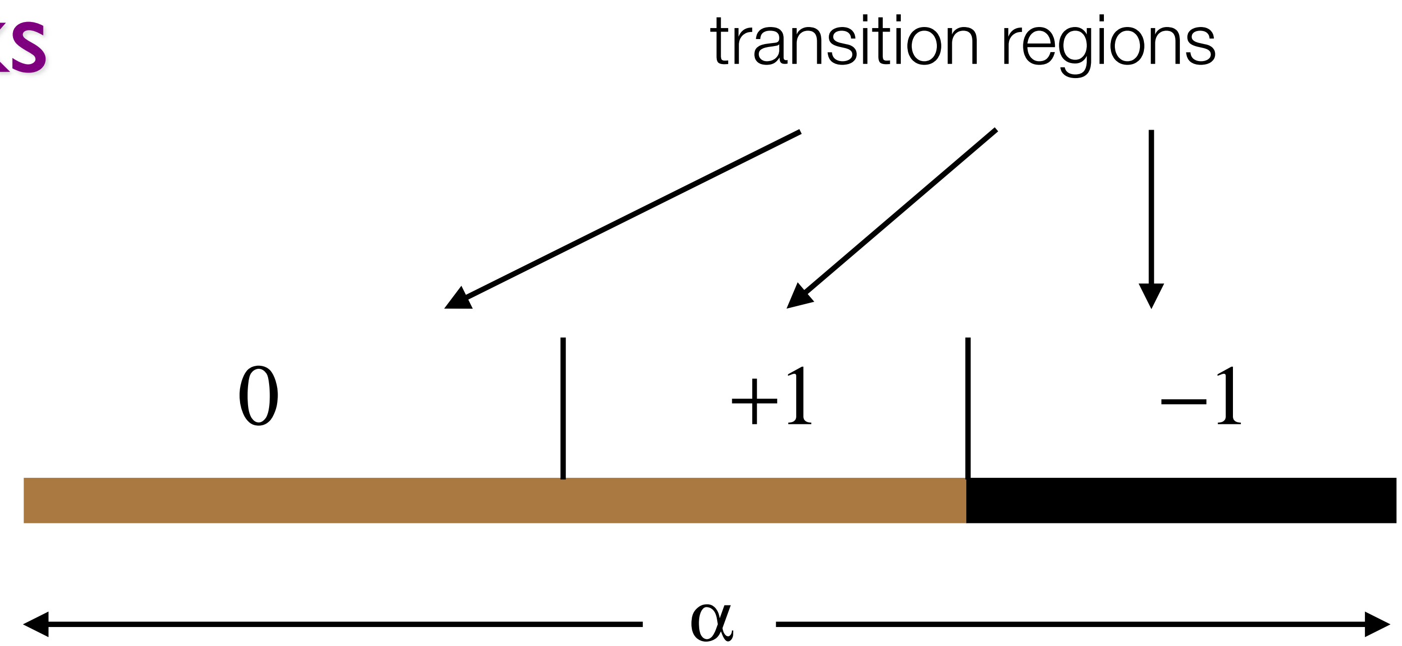

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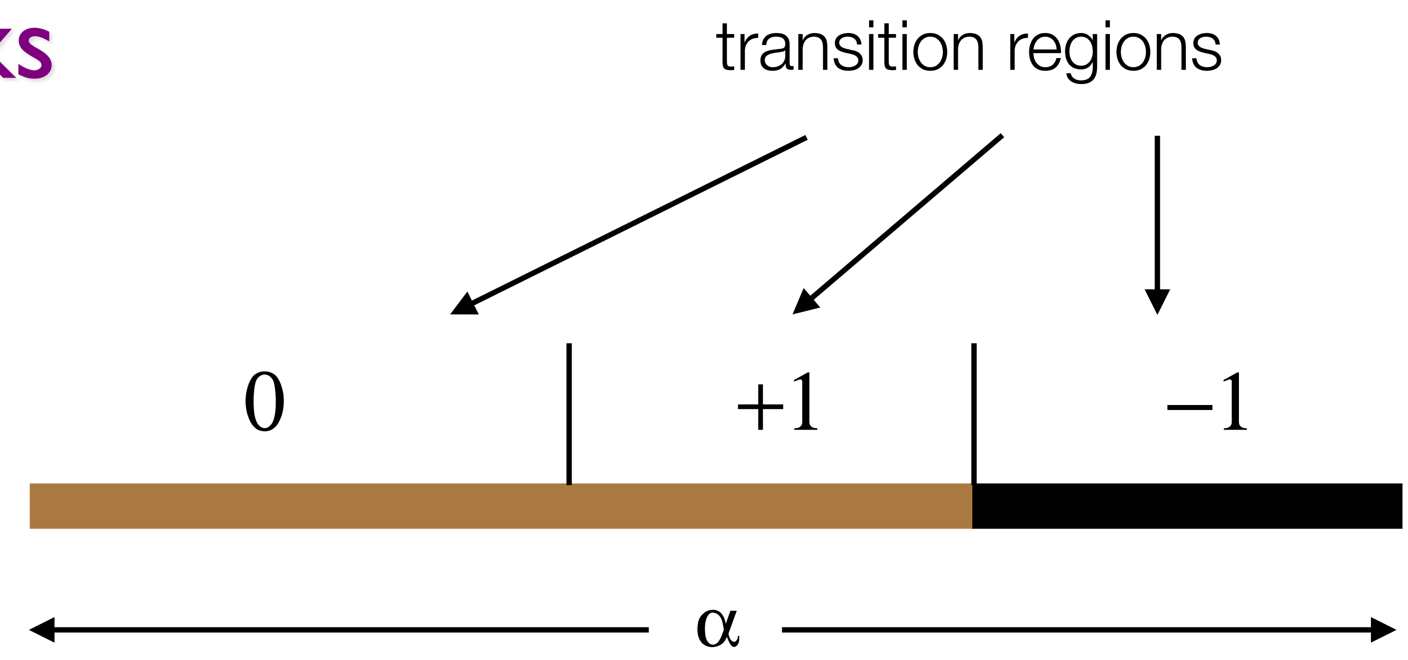

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The conjecture holds for the reduced system.

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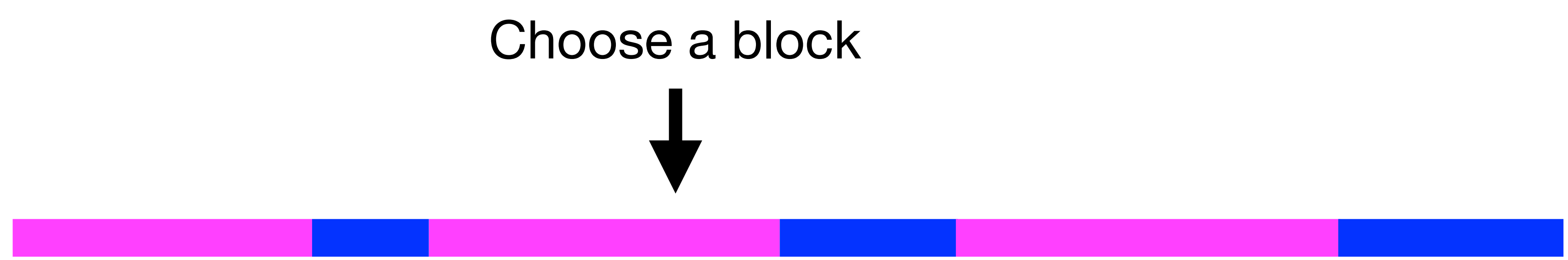

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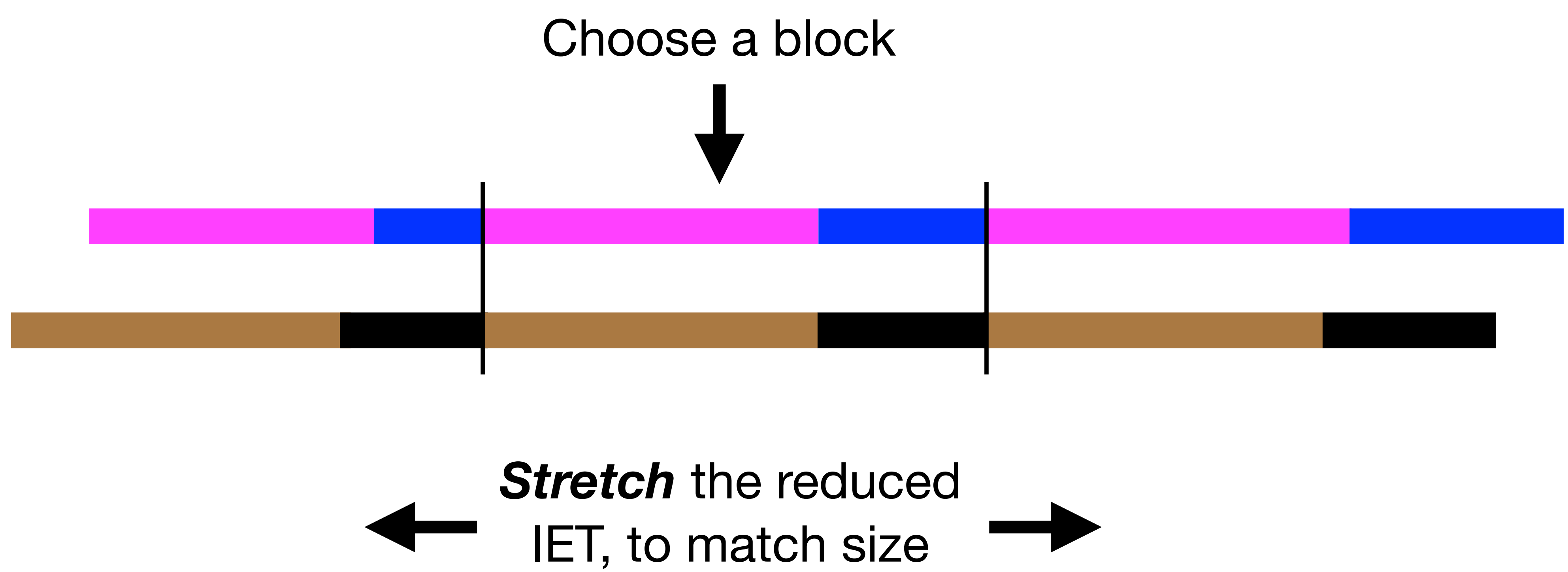

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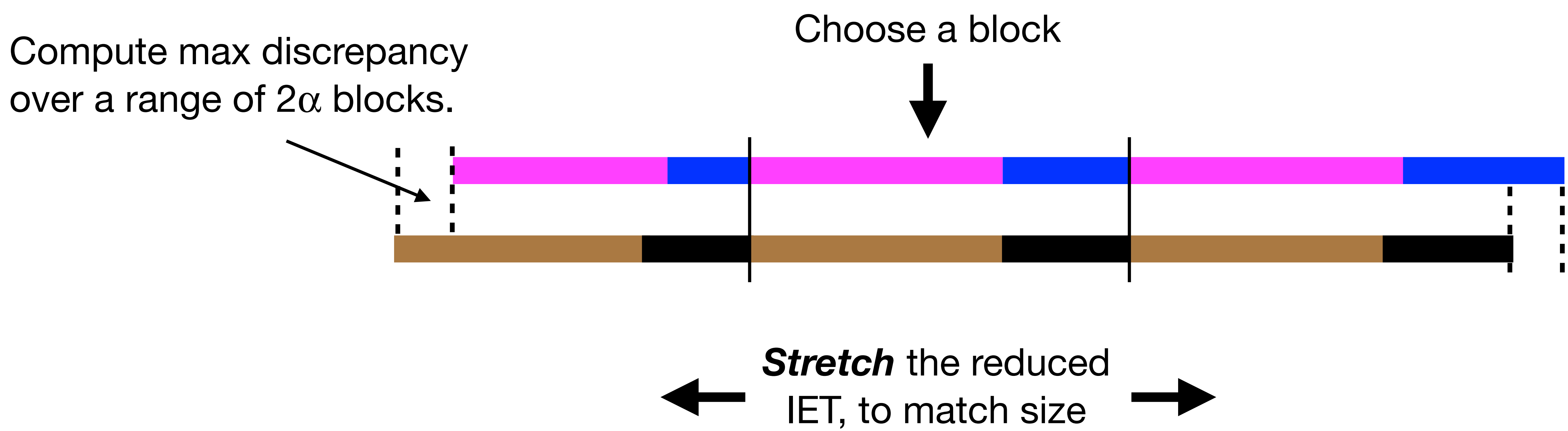

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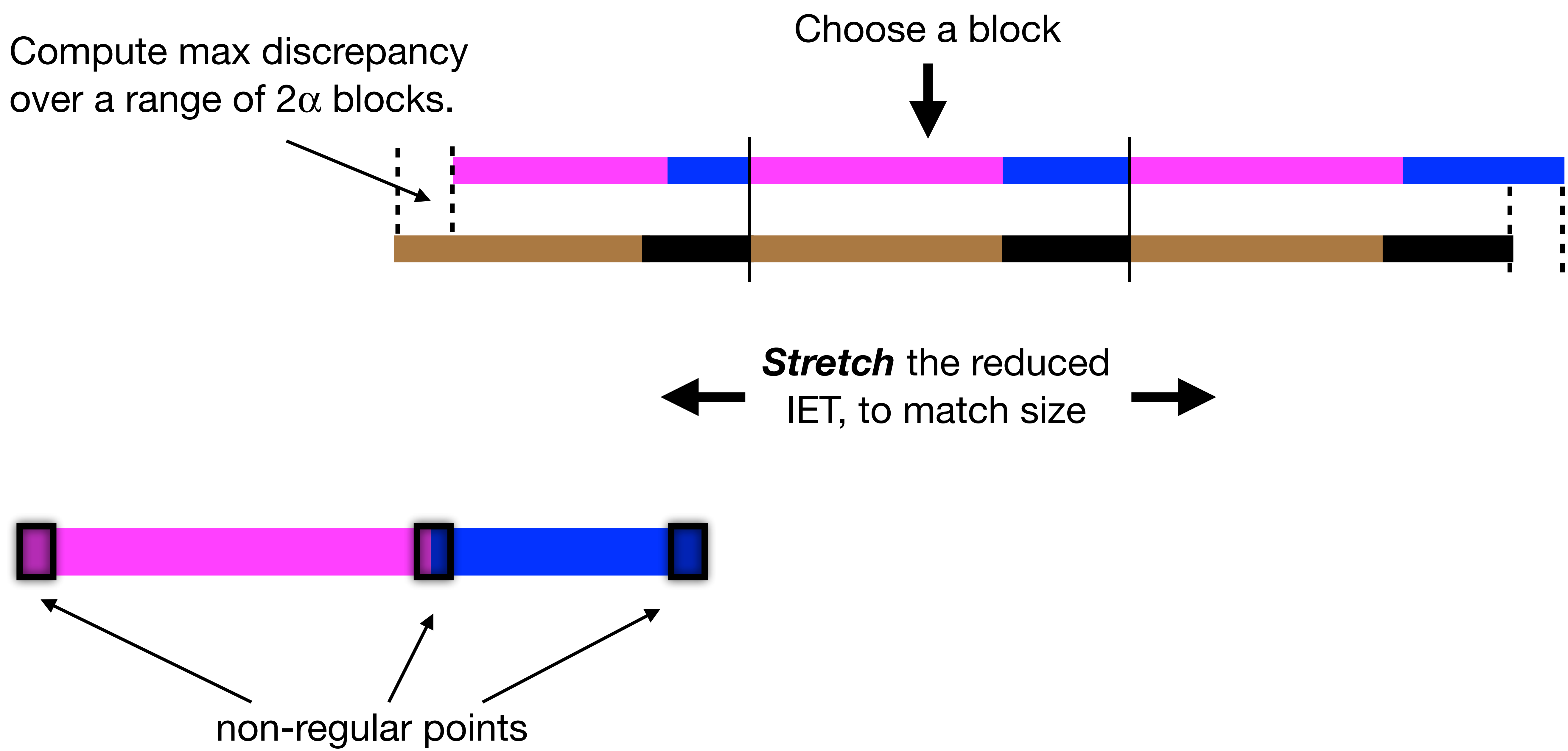

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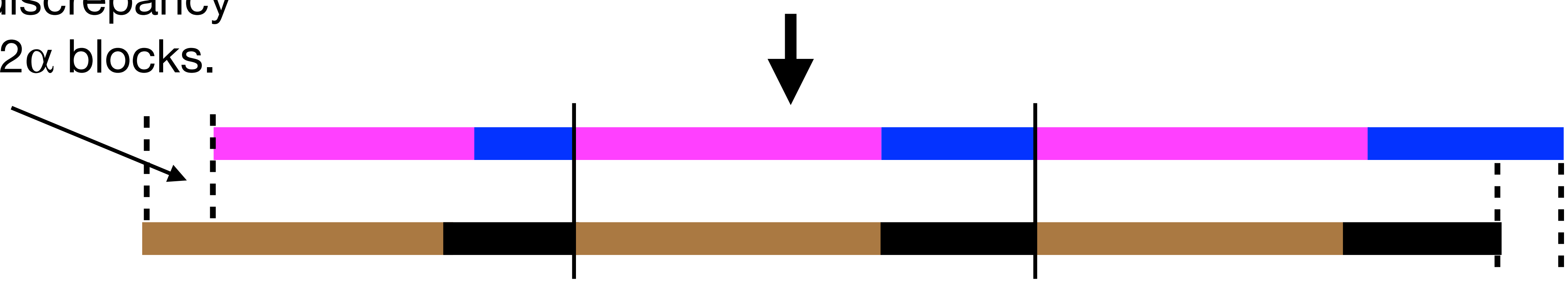
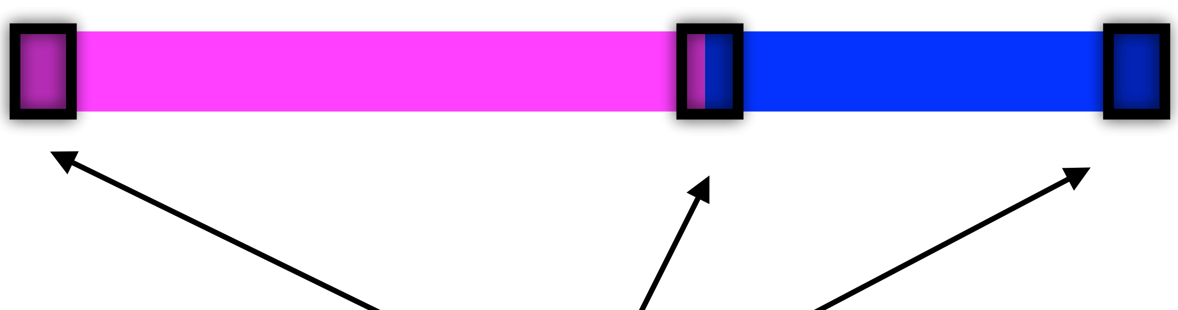
Compute max discrepancy over a range of 2α blocks.

Choose a block



Stretch the reduced IET, to match size



non-regular points

Lemma. The set of regular points of the Poincare´map F has full natural density in Ω.

From the reduced IET to the original IET

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Total translation for a regular point x of F, after α iterates: $$\sum_{c \in C(x)} \tau(c)$$

translation

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To compute it, we must evaluate the following sum:

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- We find that the total translation is what it should be (zero if α' is odd, and equal to the block length if α' is even).

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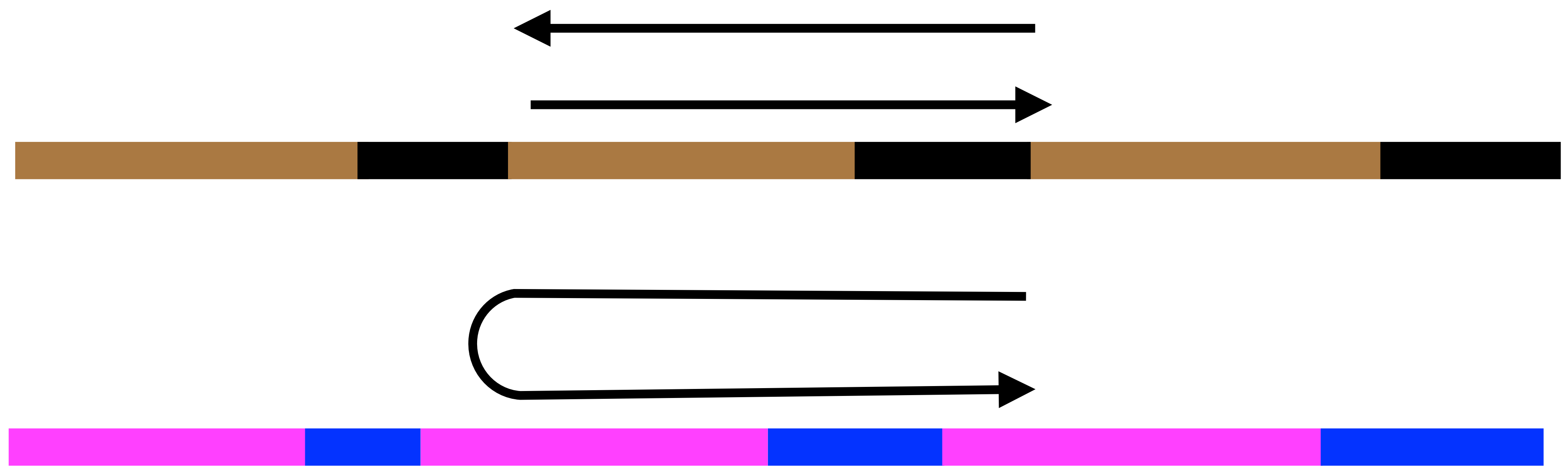

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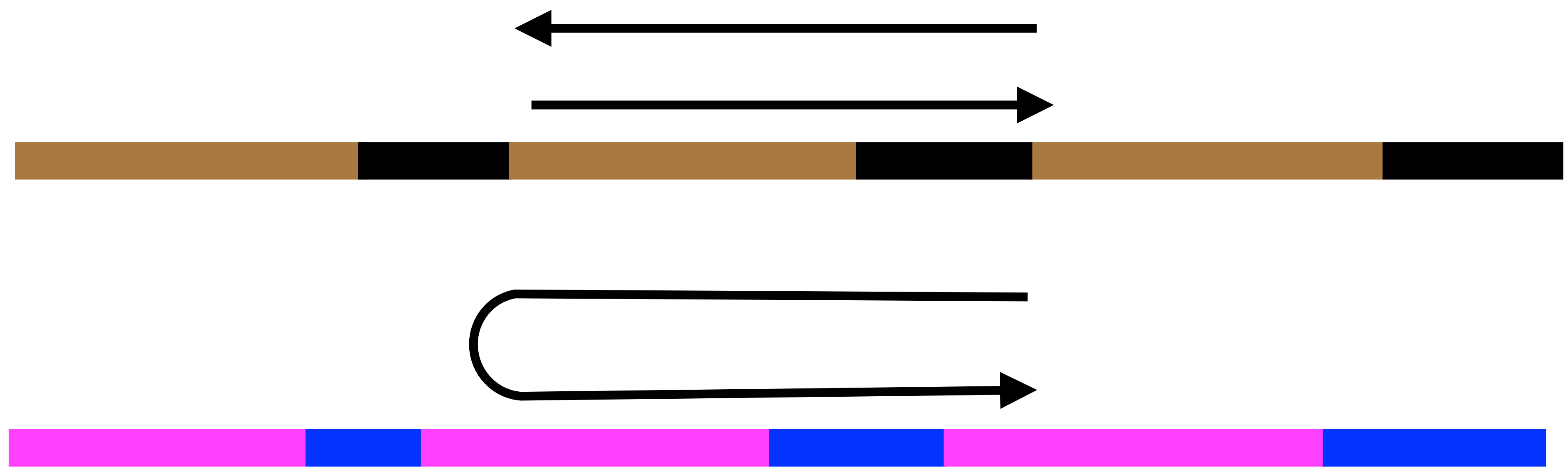

- In the periodic regime, we believe that all points are regular.

Thank you for your attention