Regular motions
and
anomalous transport
in a piecewise isometric system

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Queen Mary, University of London

with
J H Lowenstein (NYU, New York)

Piecewise isometries

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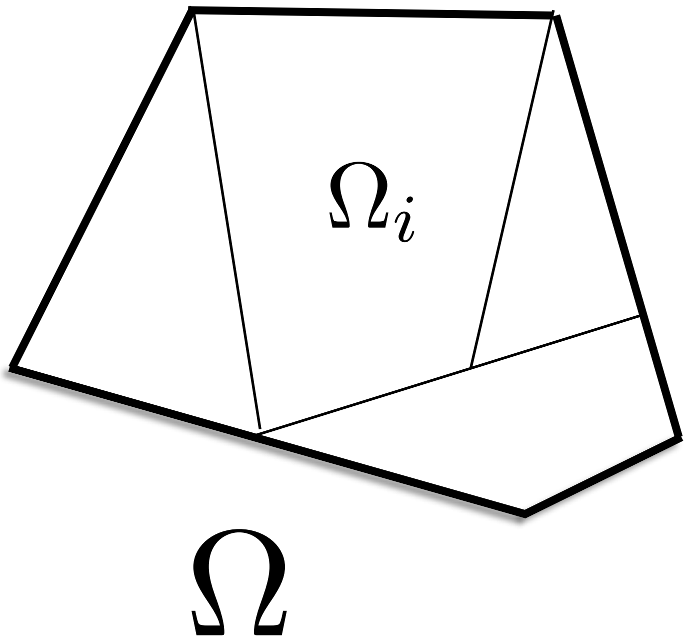
$$\Omega = \overline{\bigcup \Omega_i}$$



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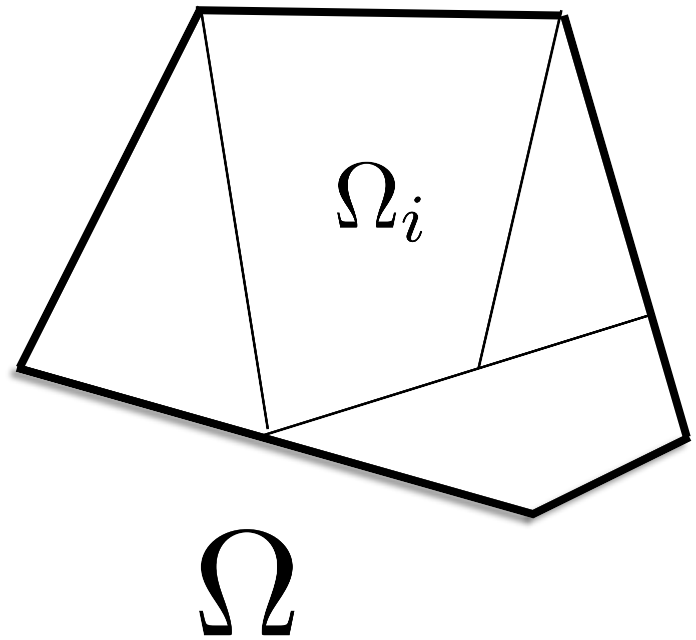

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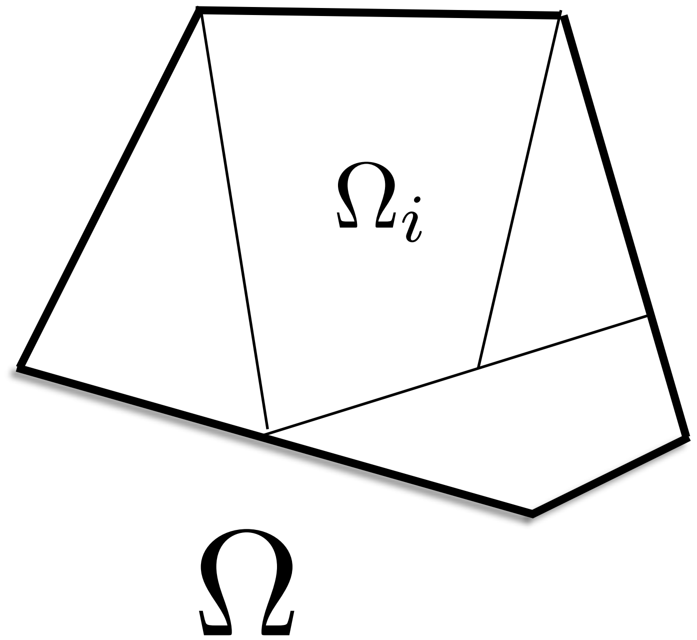

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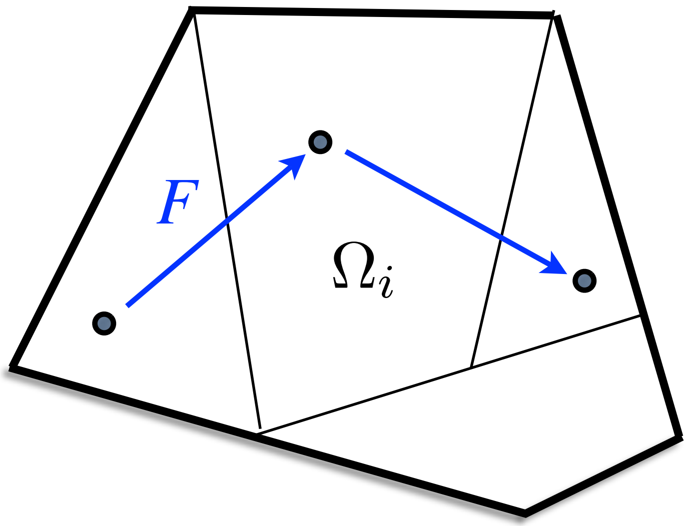
Theorem (Gutkin & Haydin 1997, Buzzi 2001)
The topological entropy of a piecewise isometry is zero.

Cells

The points of an orbit visit atoms in succession, defining a symbolic dynamics.



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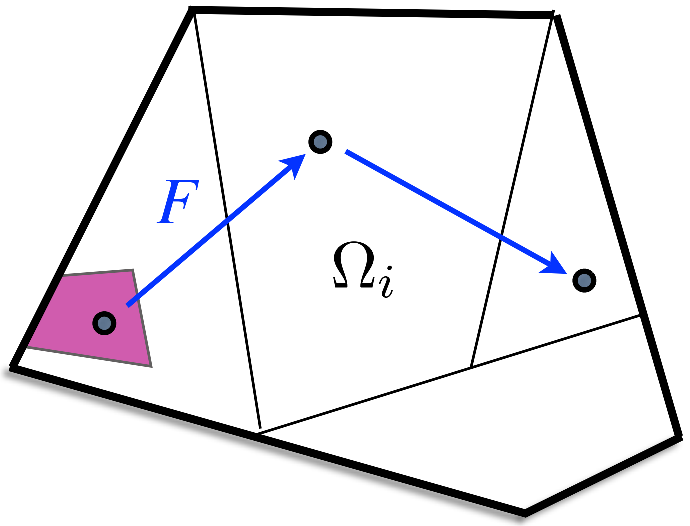

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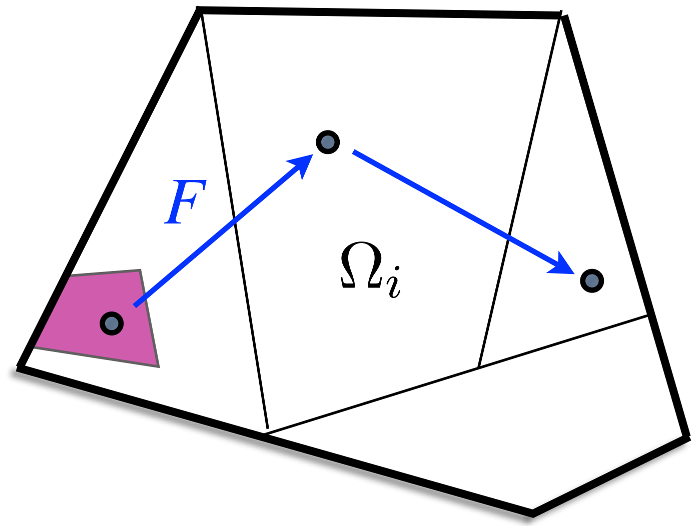
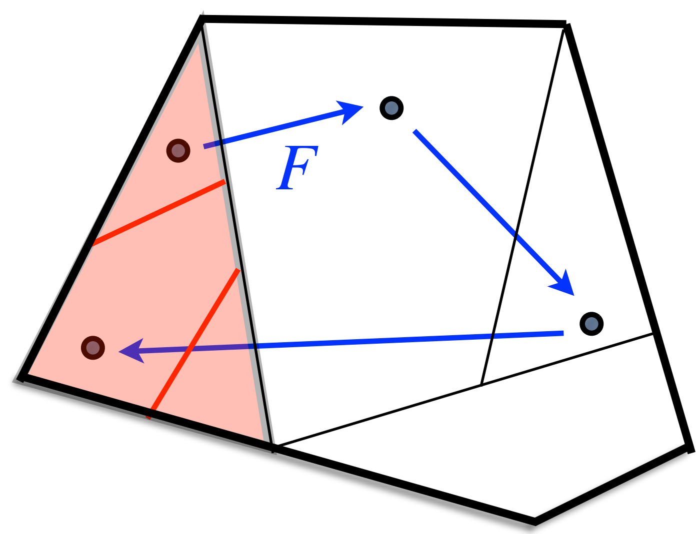


Induced maps

The first return map to an atom defines a new PWI on a smaller domain. This process may be continued recursively.



Topology

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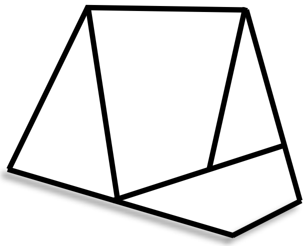

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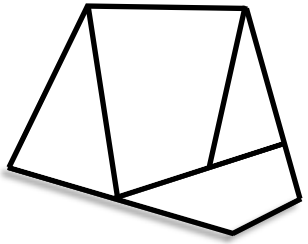

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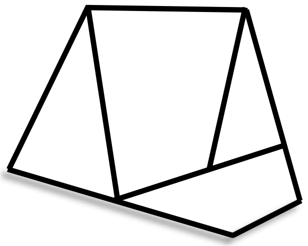

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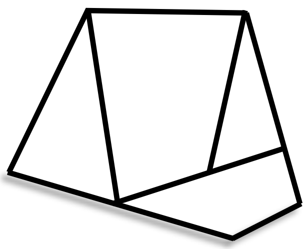

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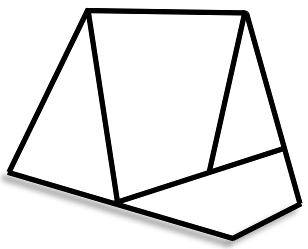

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Conjecture **(Goetz 1998).** *If* $\overline{\mathcal{D}} \neq \Omega$, *then* $\overline{\mathcal{D}}$ *has empty interior.*

Rotations on tori: the standard model

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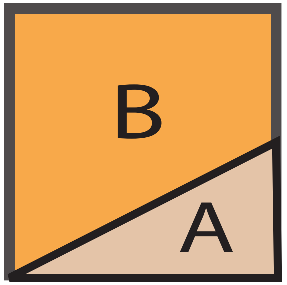

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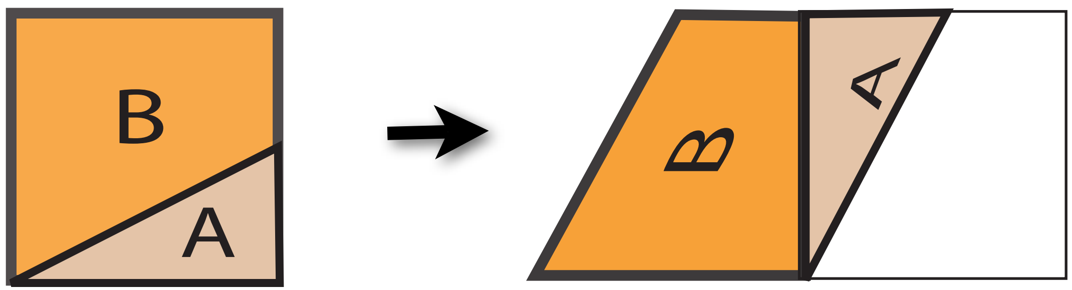

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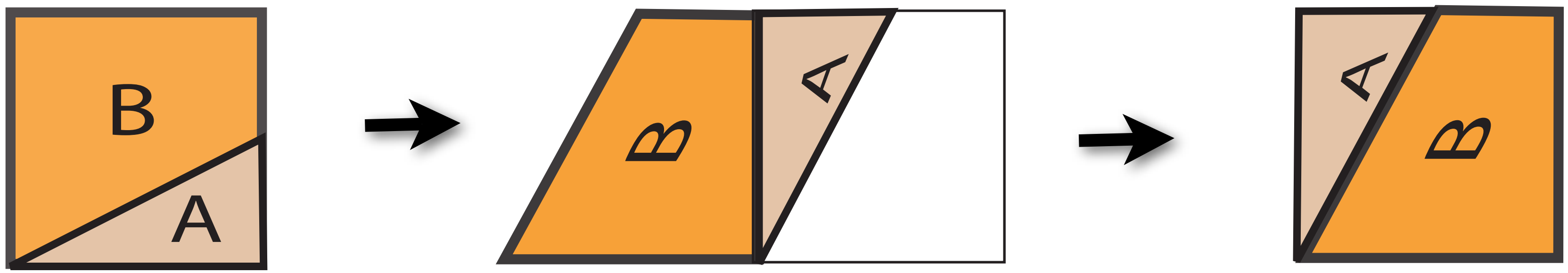

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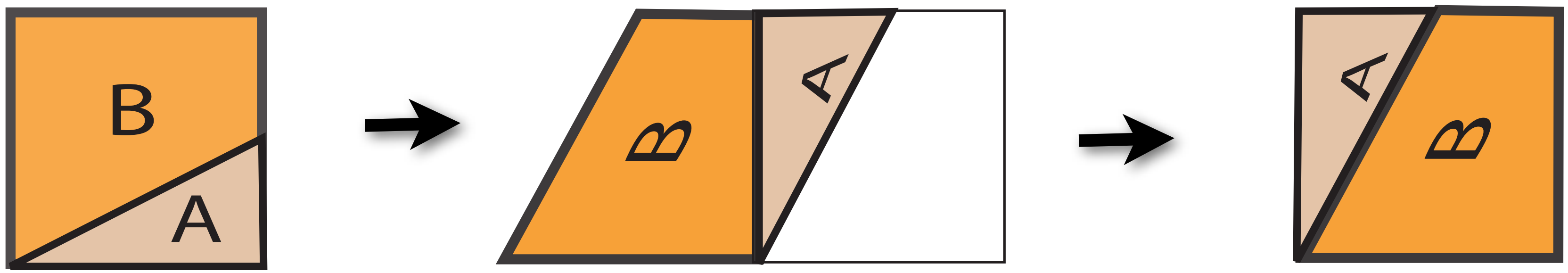

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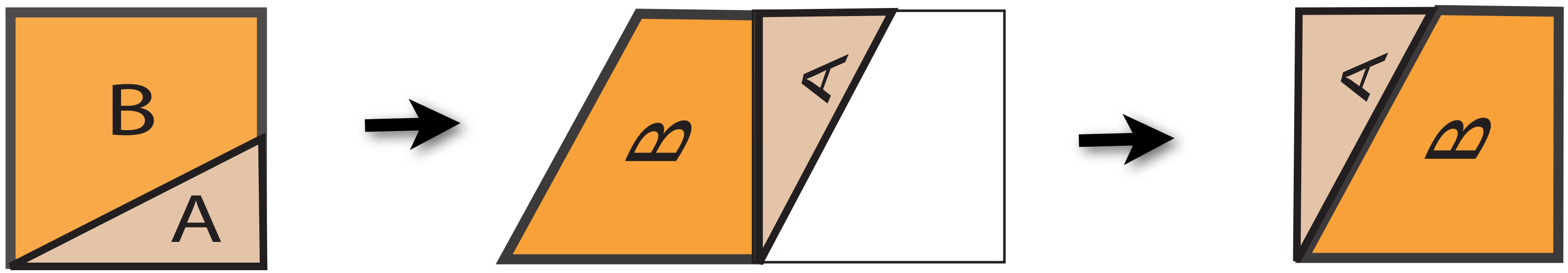

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"What surprised us most about these maps, is how quickly we ran out of cases which are amenable of any detailed analysis"

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In this talk, we consider near-rational behaviour.

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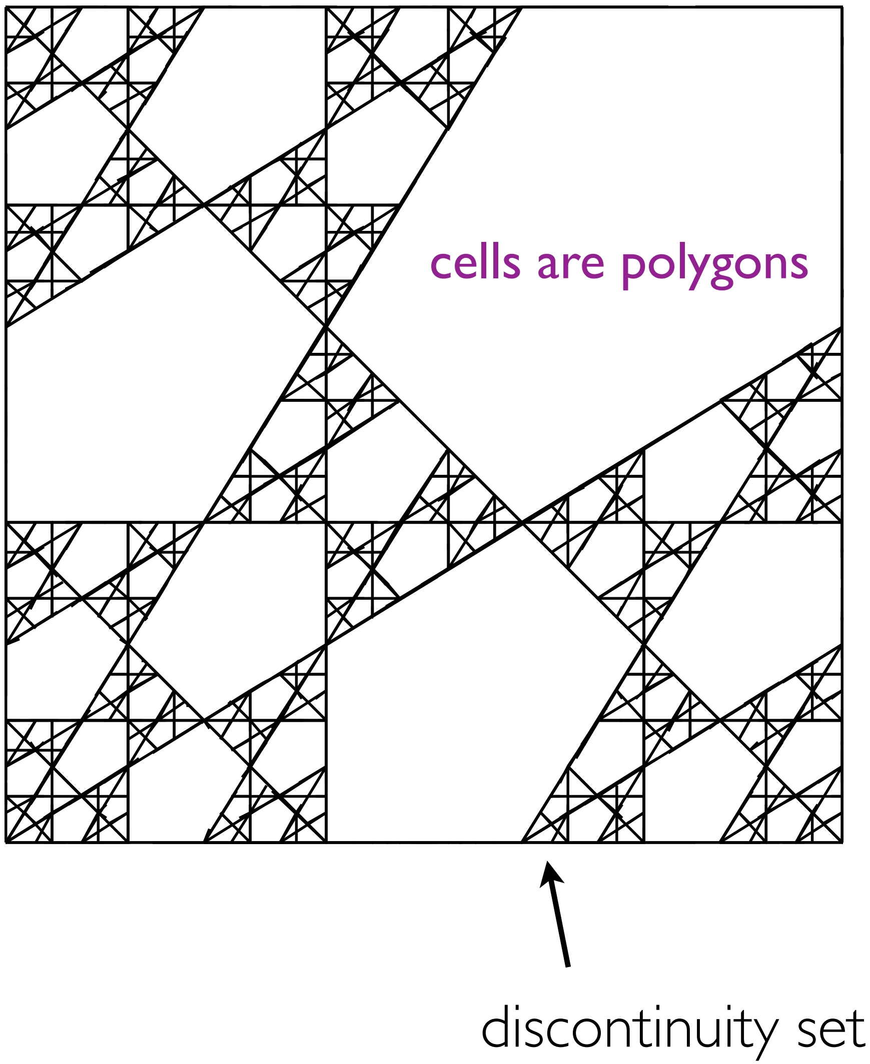

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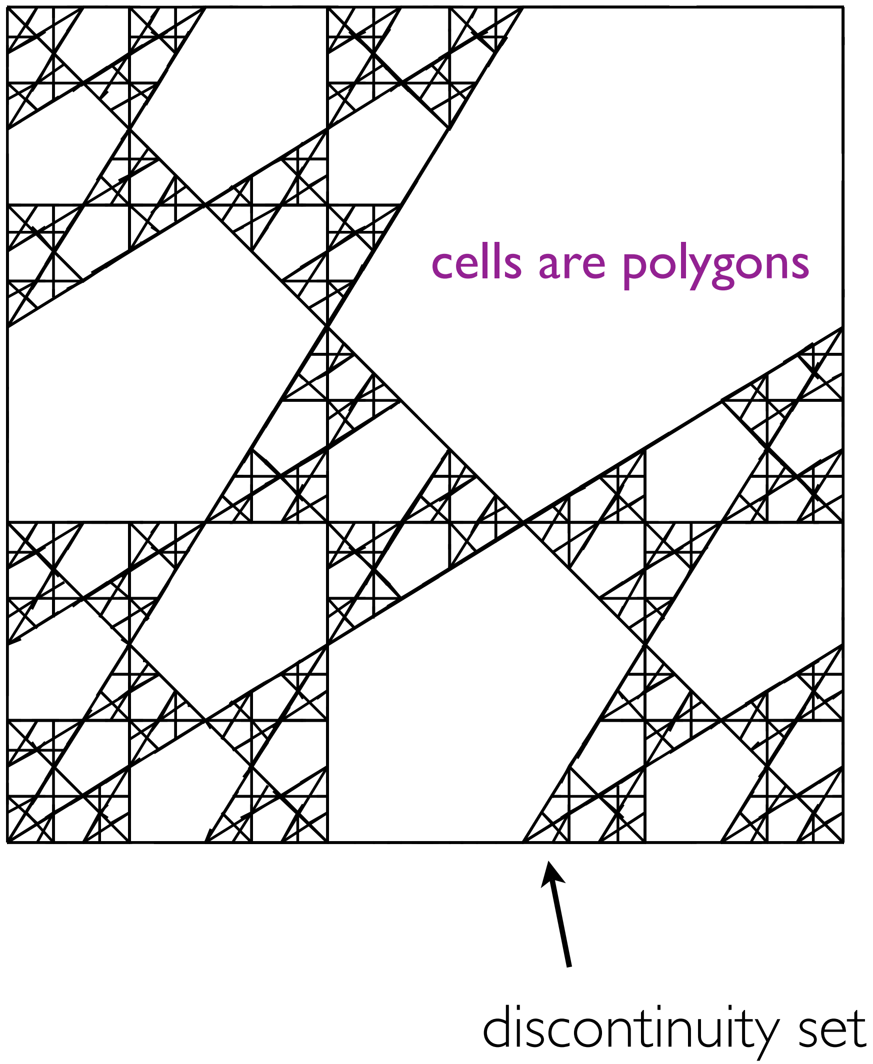

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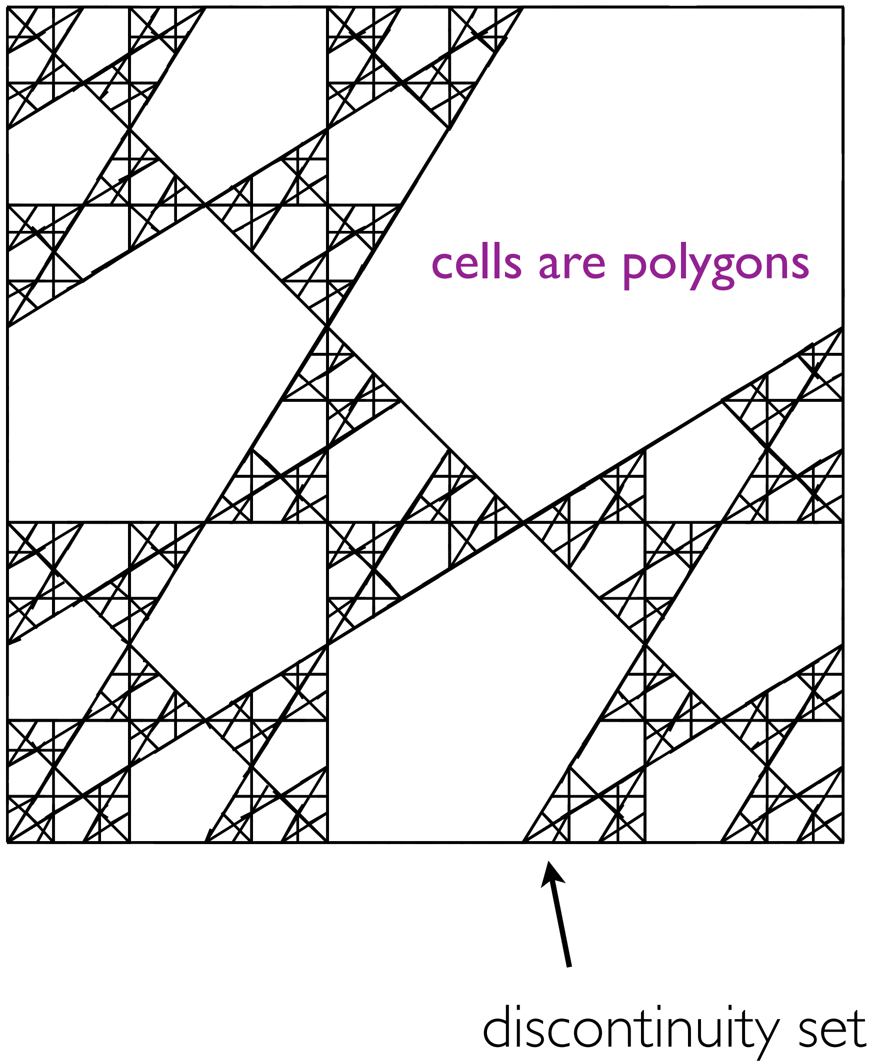

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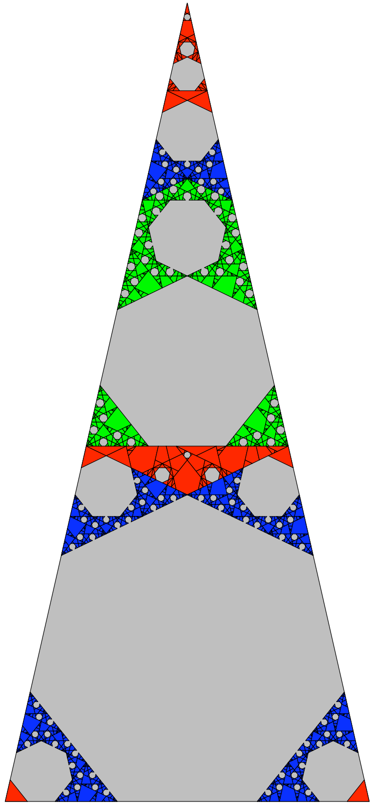
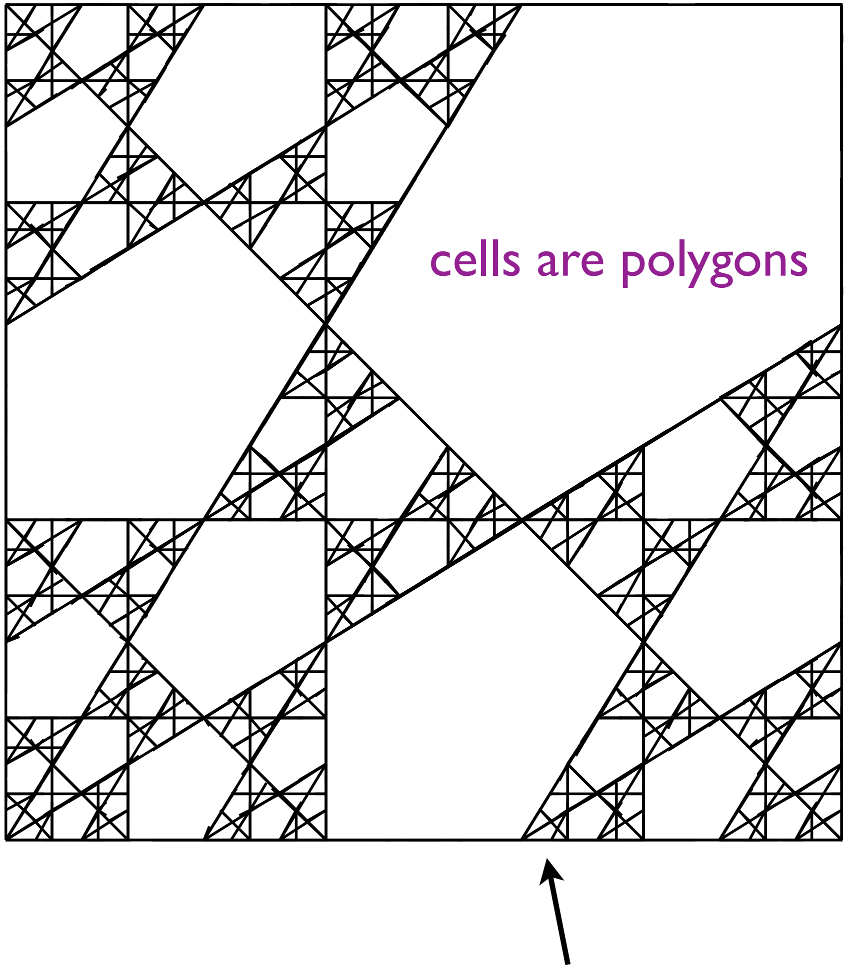


A few cubic cases: finitely-generated renormalization structure



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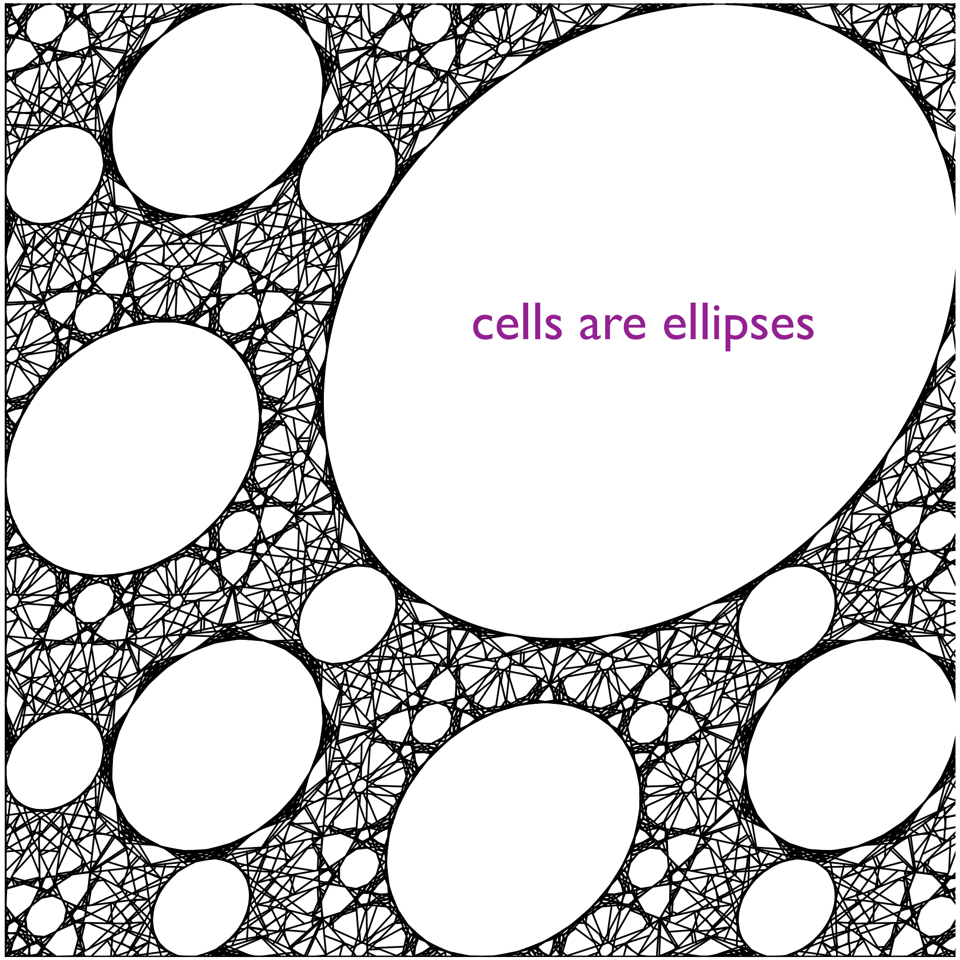

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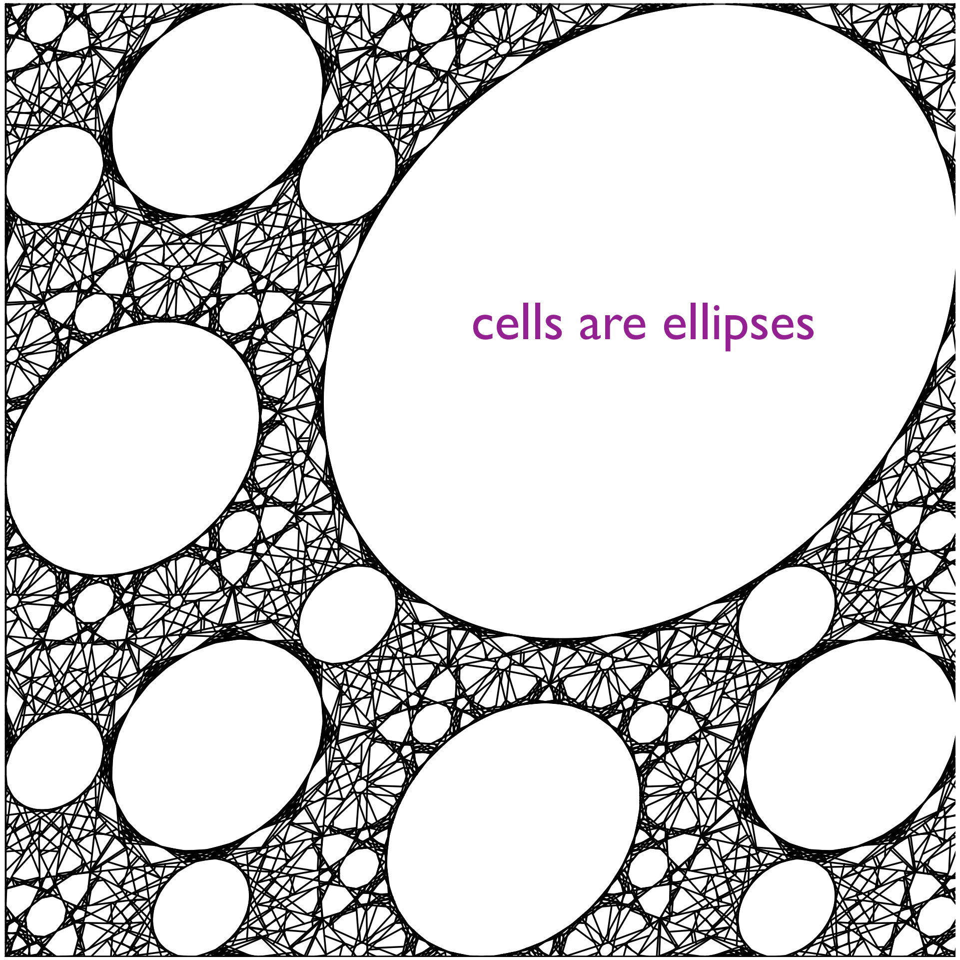

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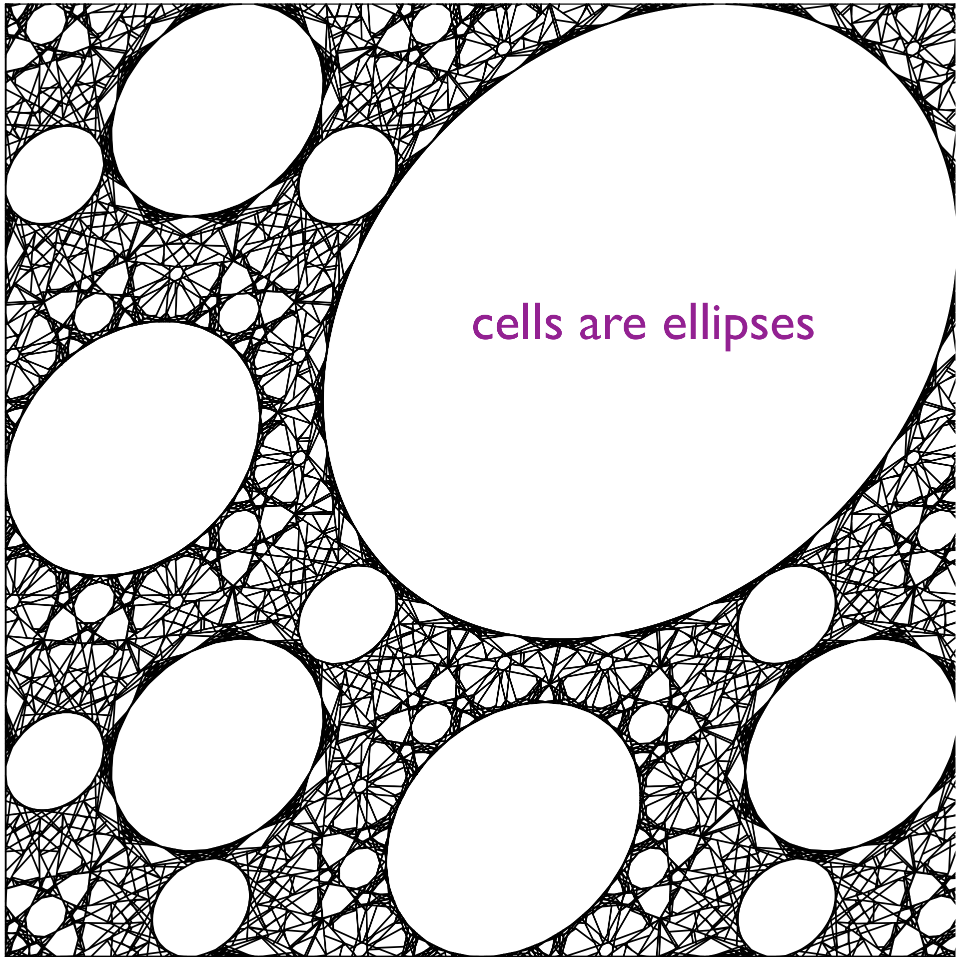

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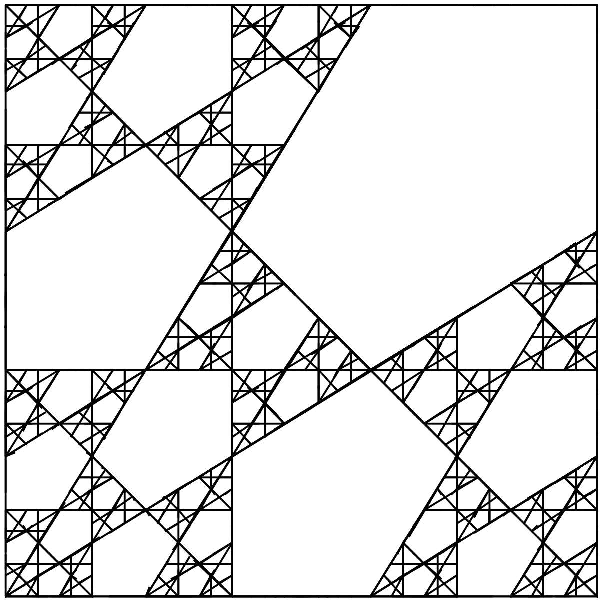
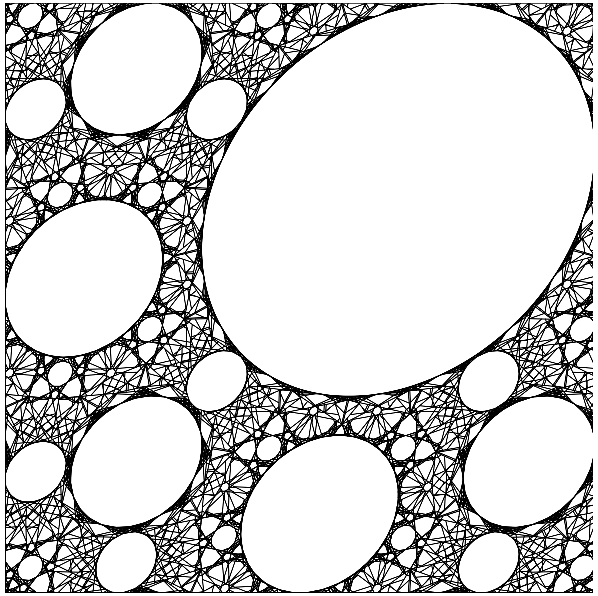
Conjecture (Ashwin 1997) *The exceptional set of a 2-D irrational piecewise isometry has positive Lebesgue measure.*

Periodic points





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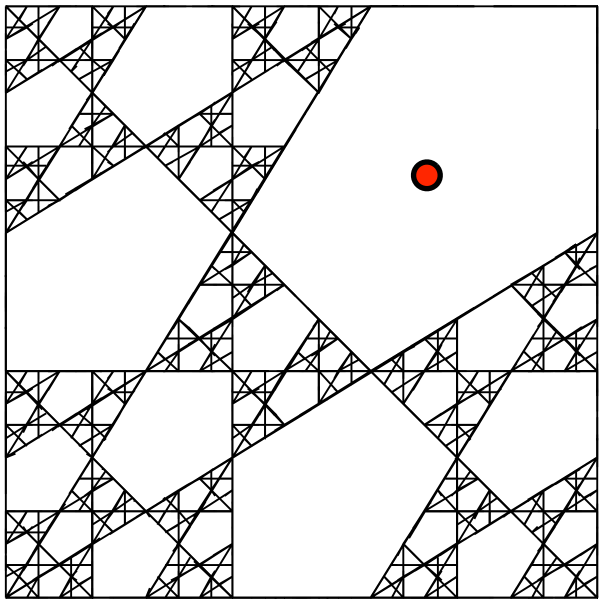

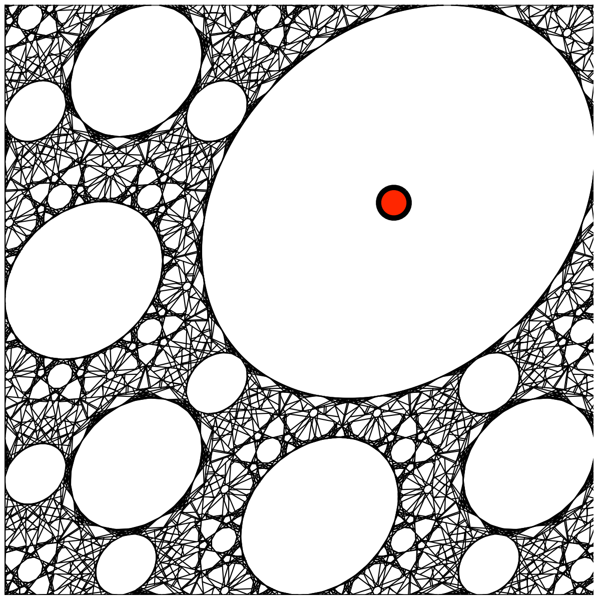

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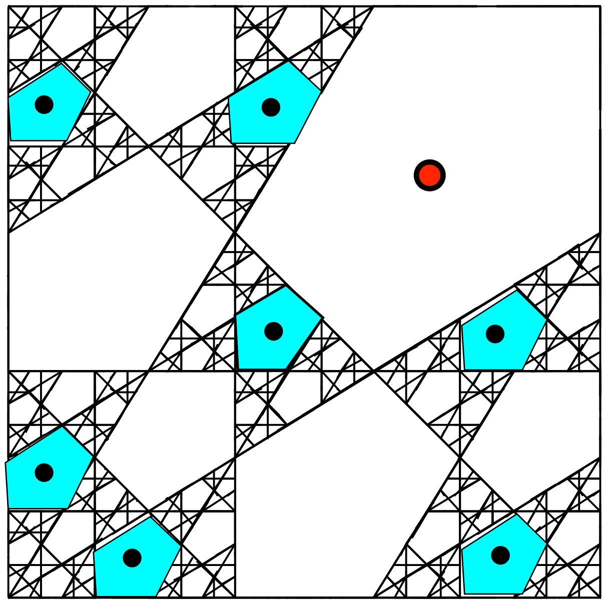

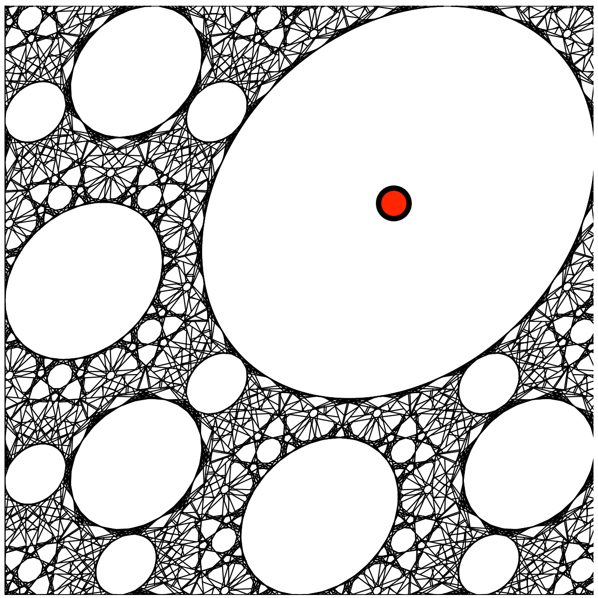

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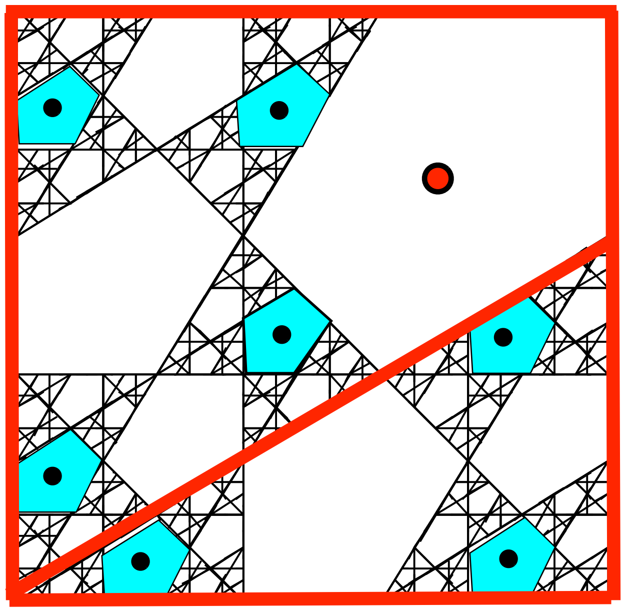

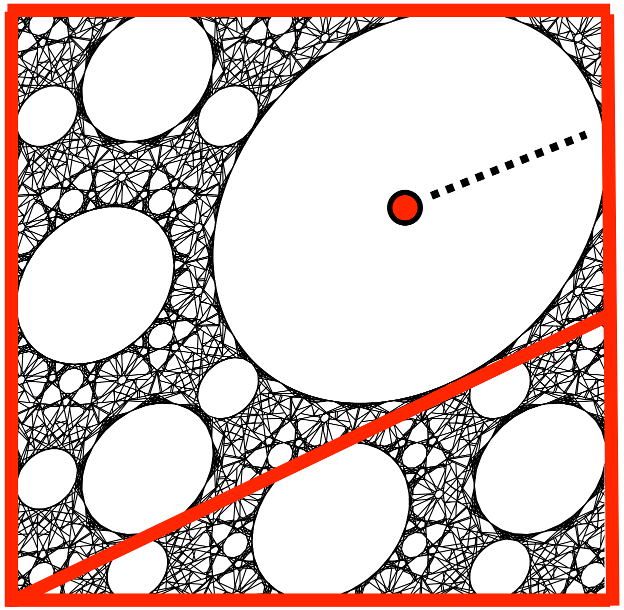

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the size of a cell is determined by the minimal distance of the periodic orbit from the boundary of the atoms

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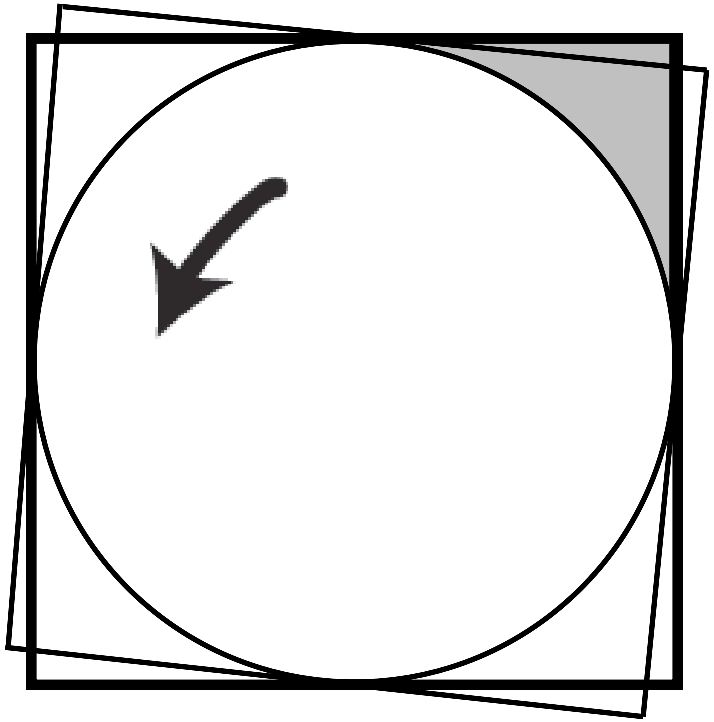

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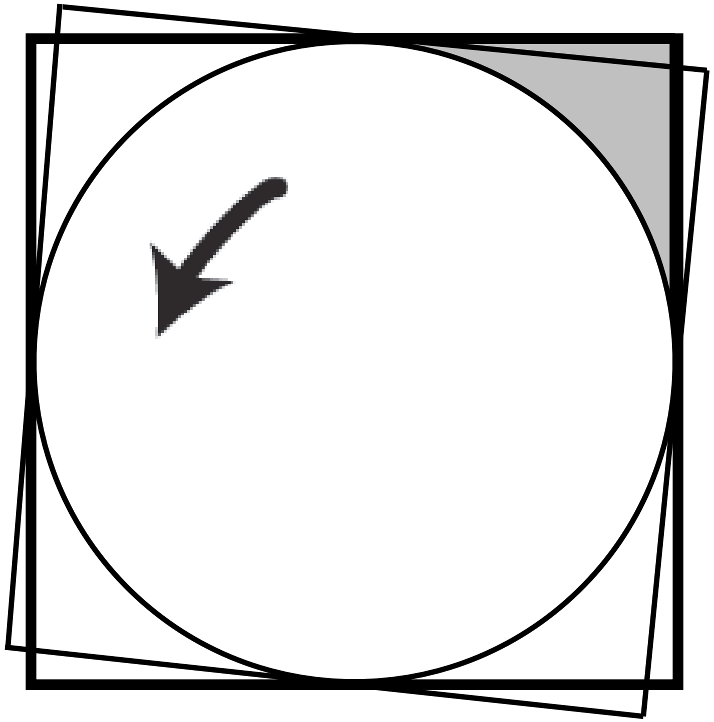

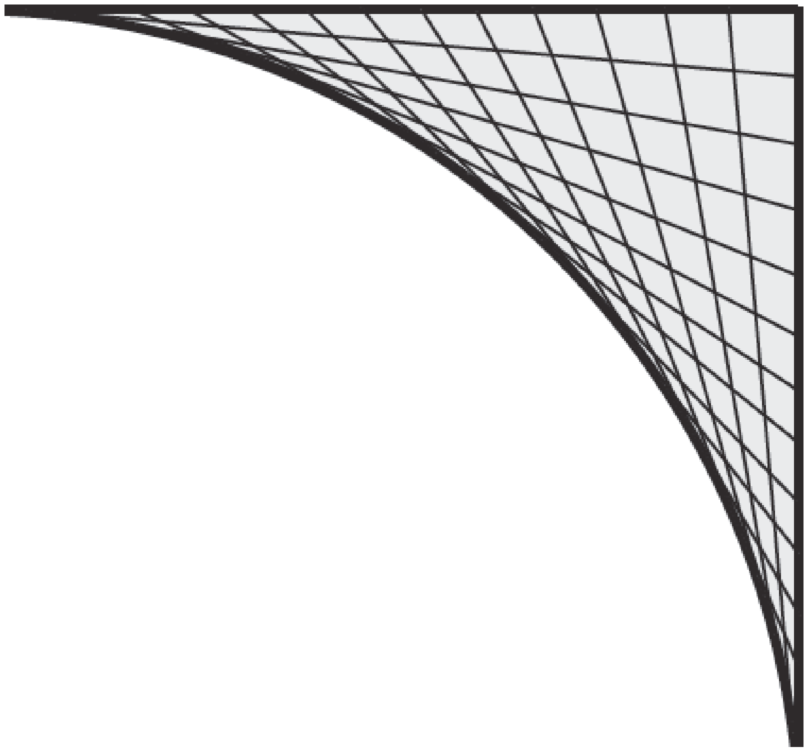

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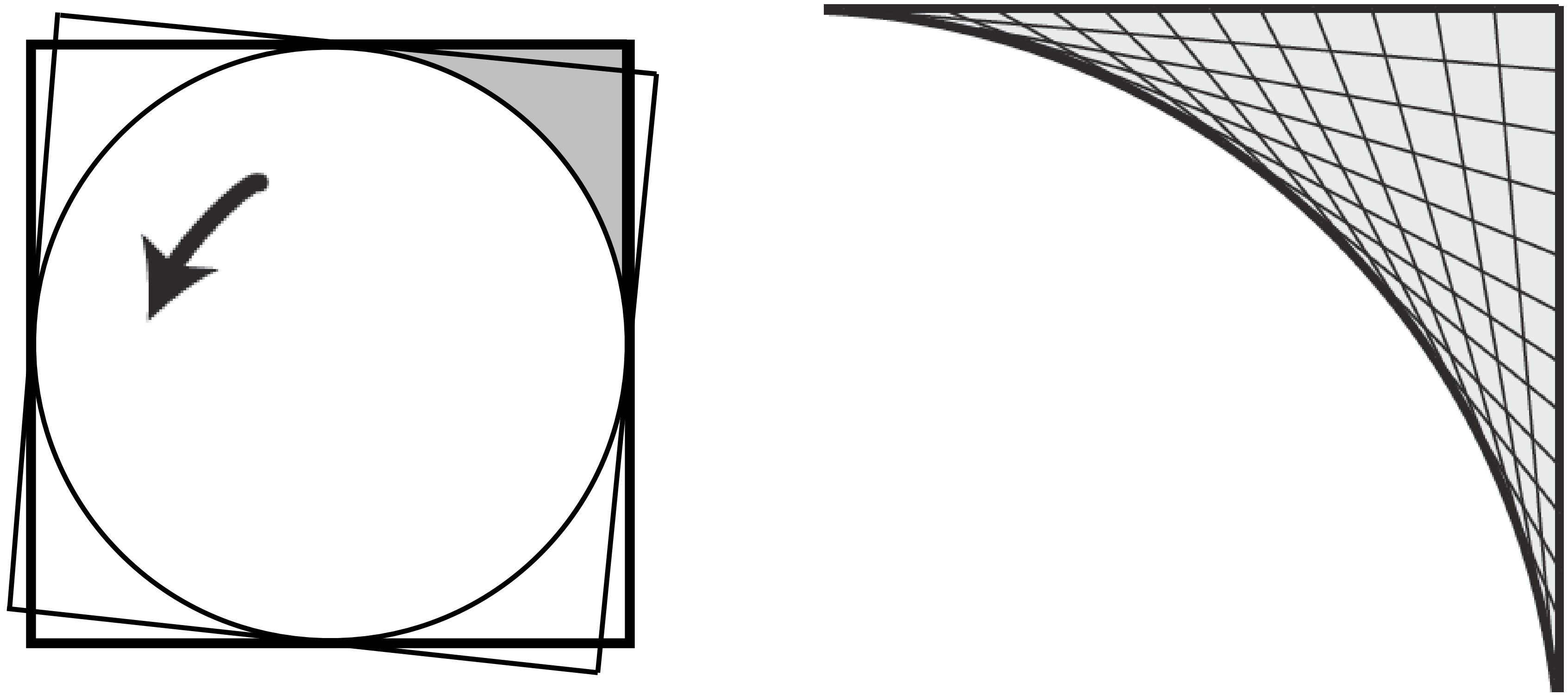

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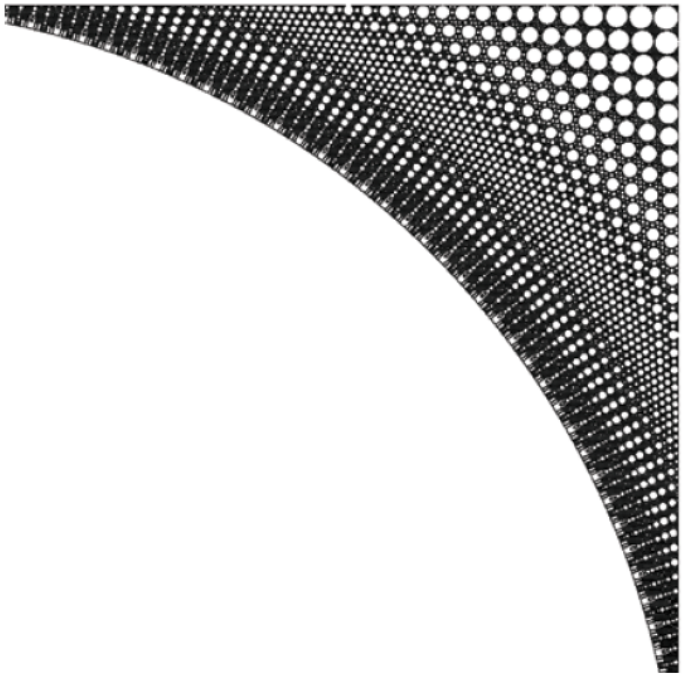
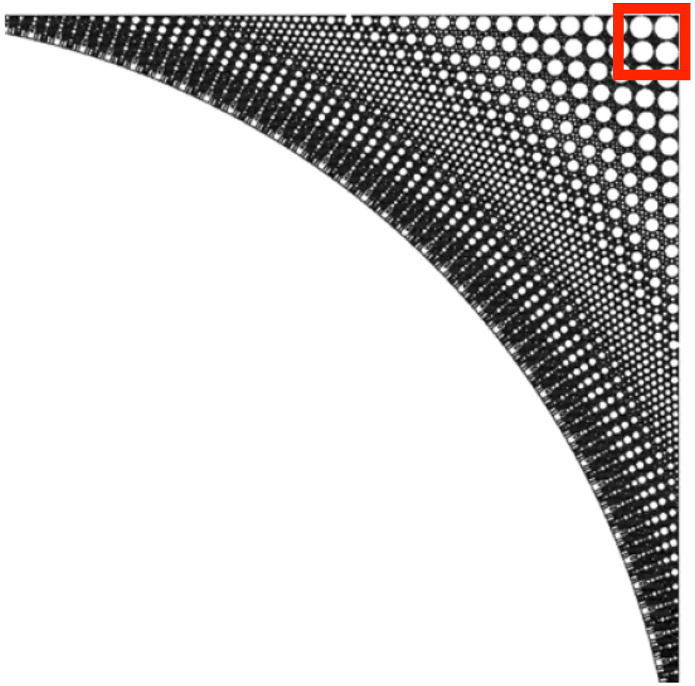
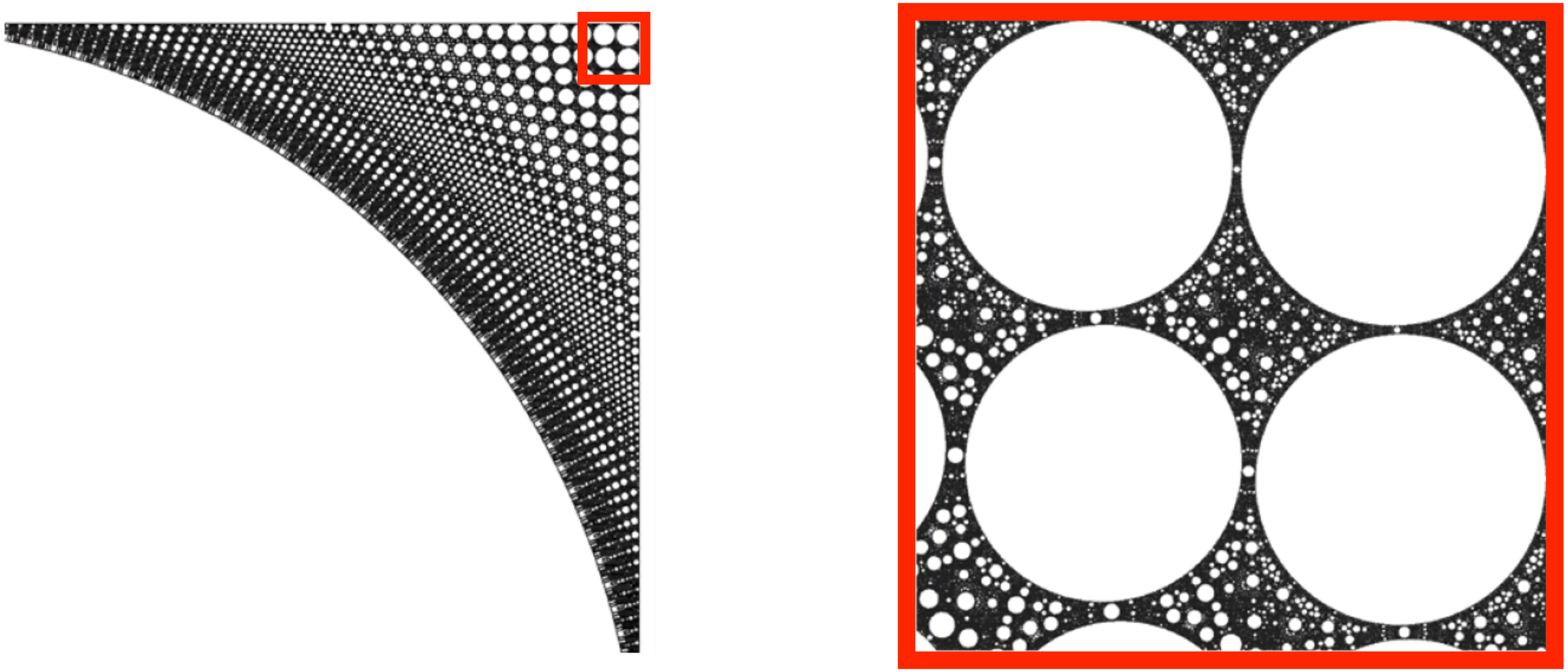
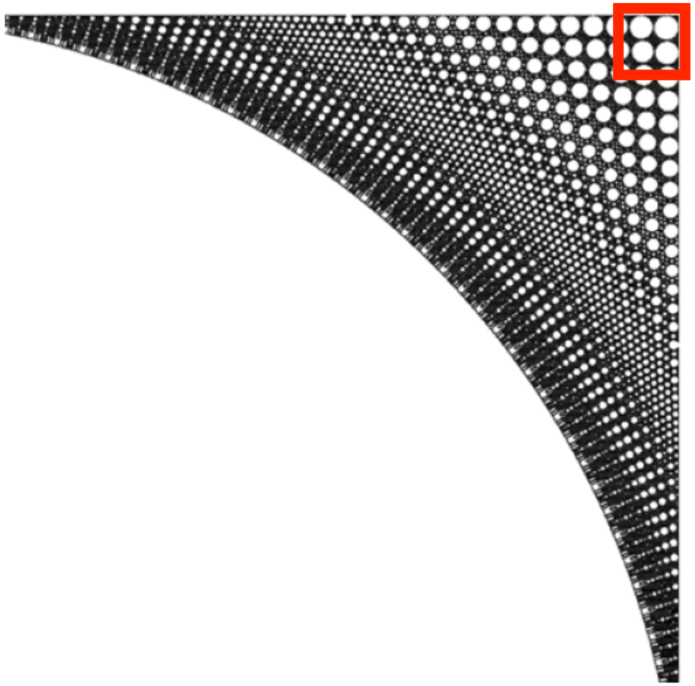
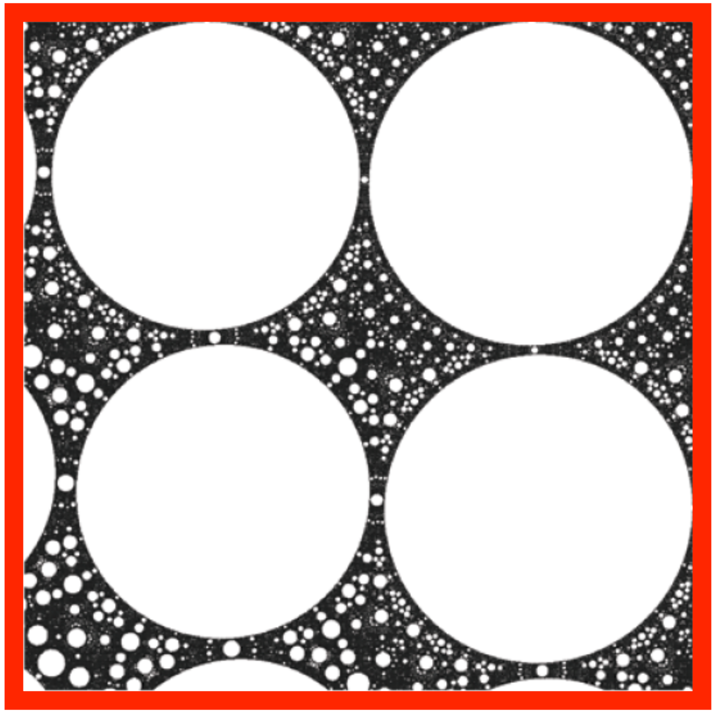
Phase space becomes very homogeneous







Theorem (Lowenstein & V.)

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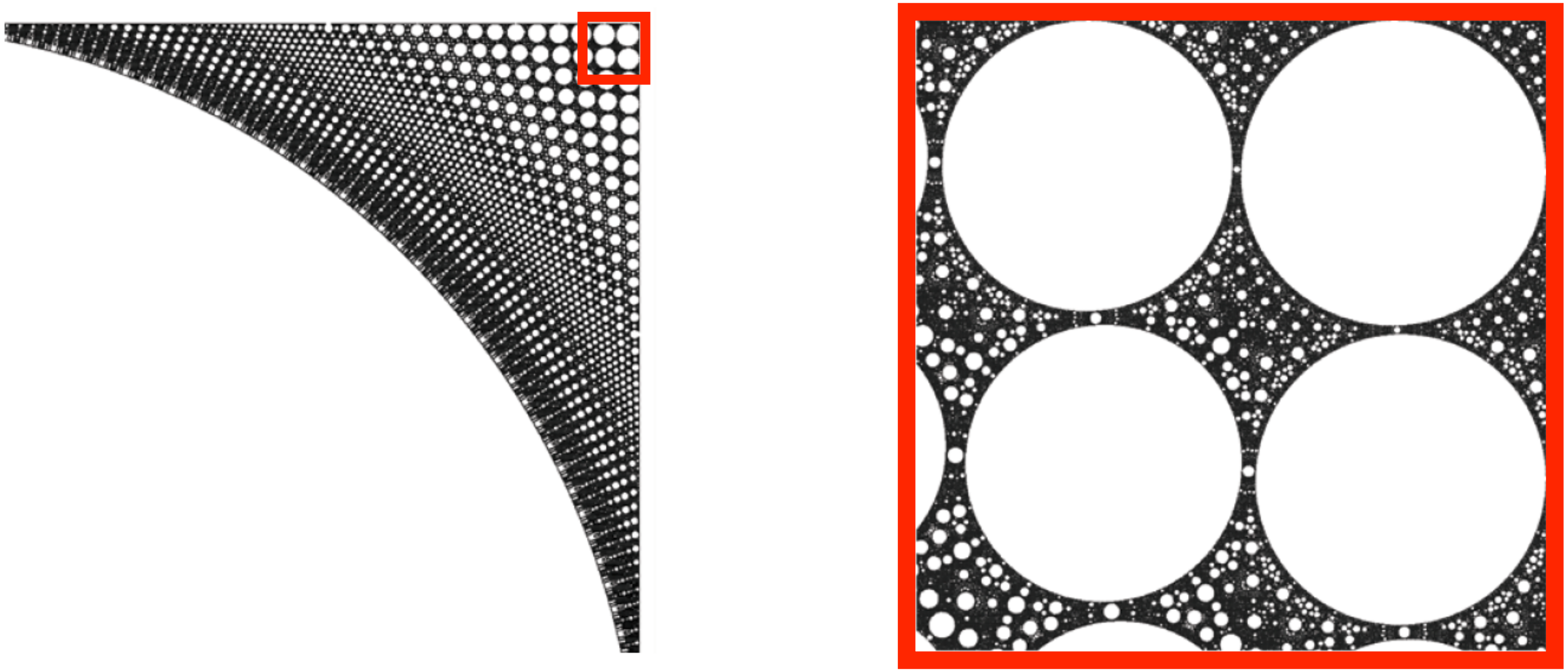

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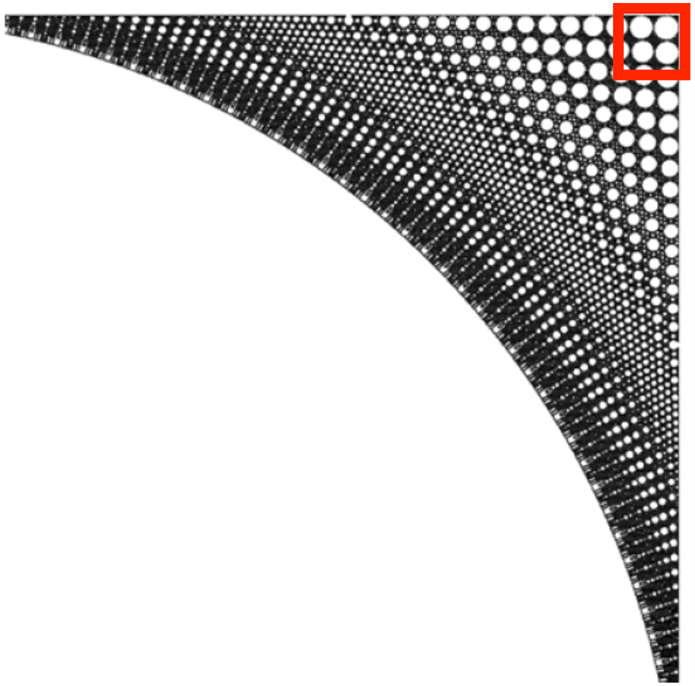

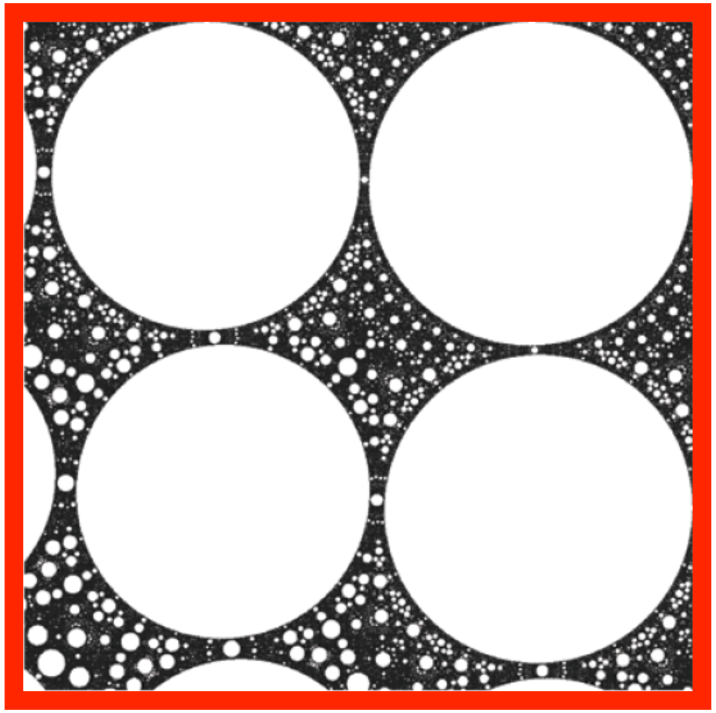

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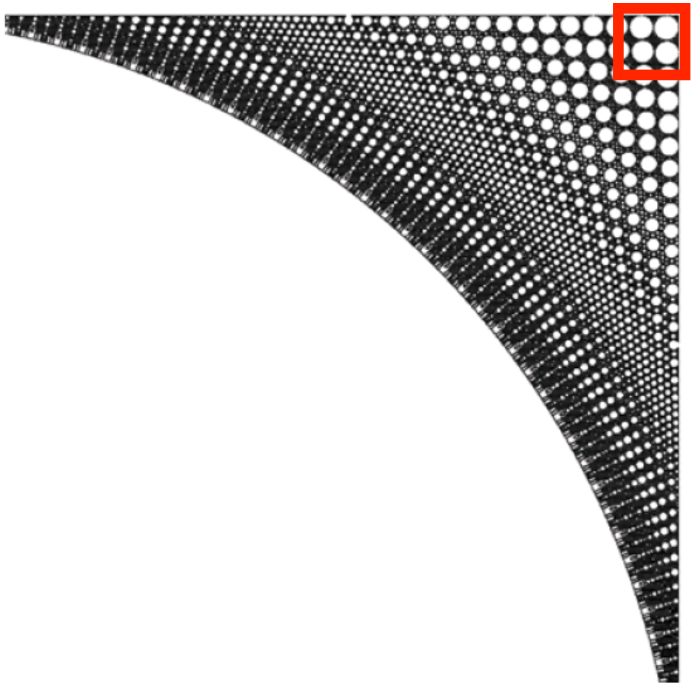

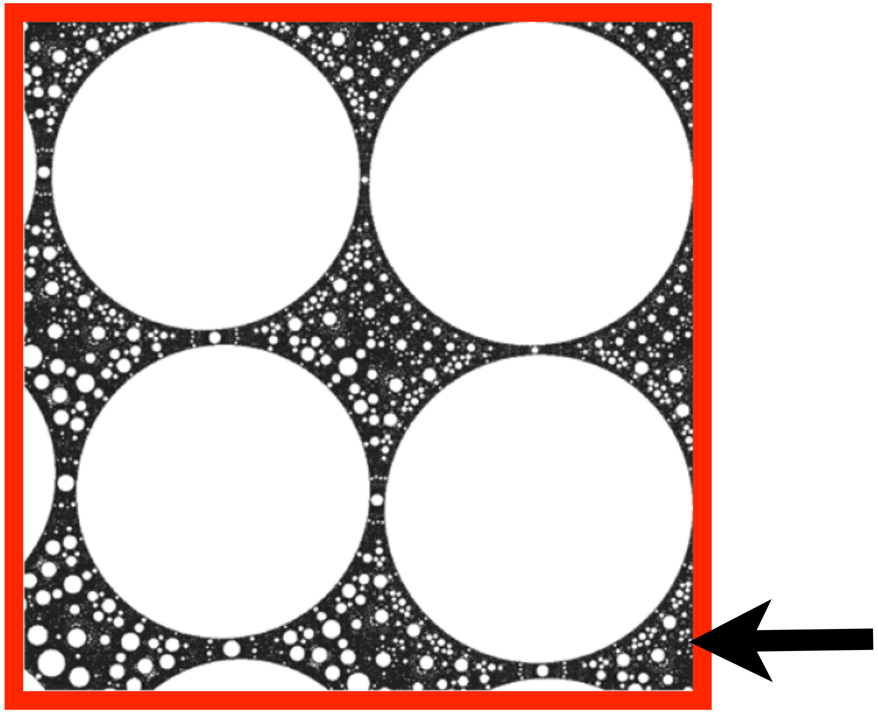

Transport?

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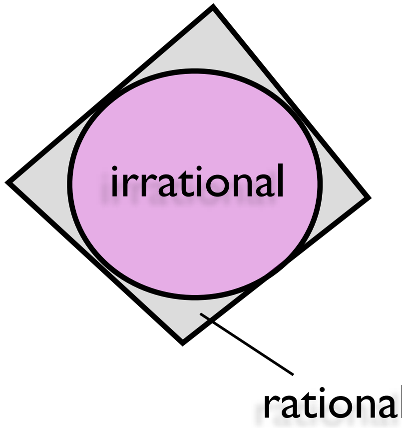
Near-rational dynamics: sector maps



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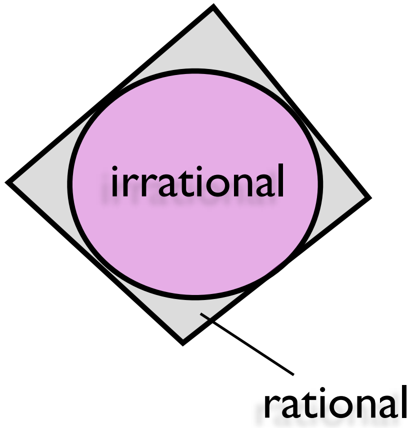

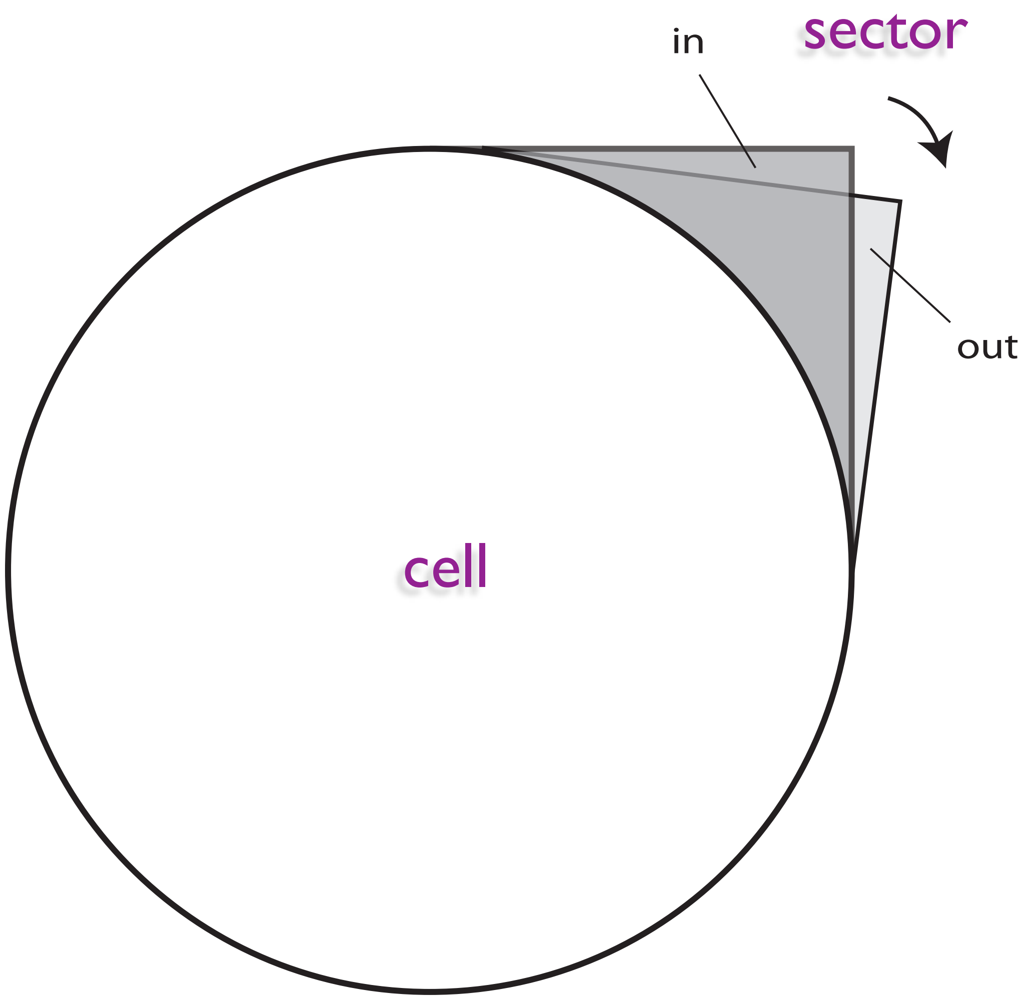

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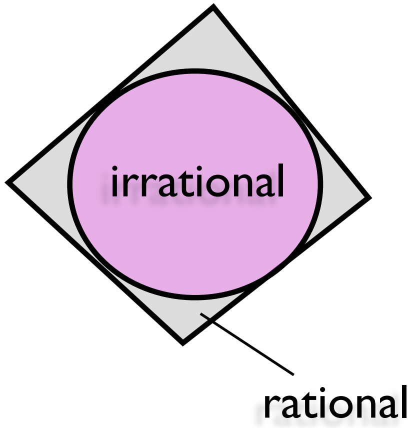
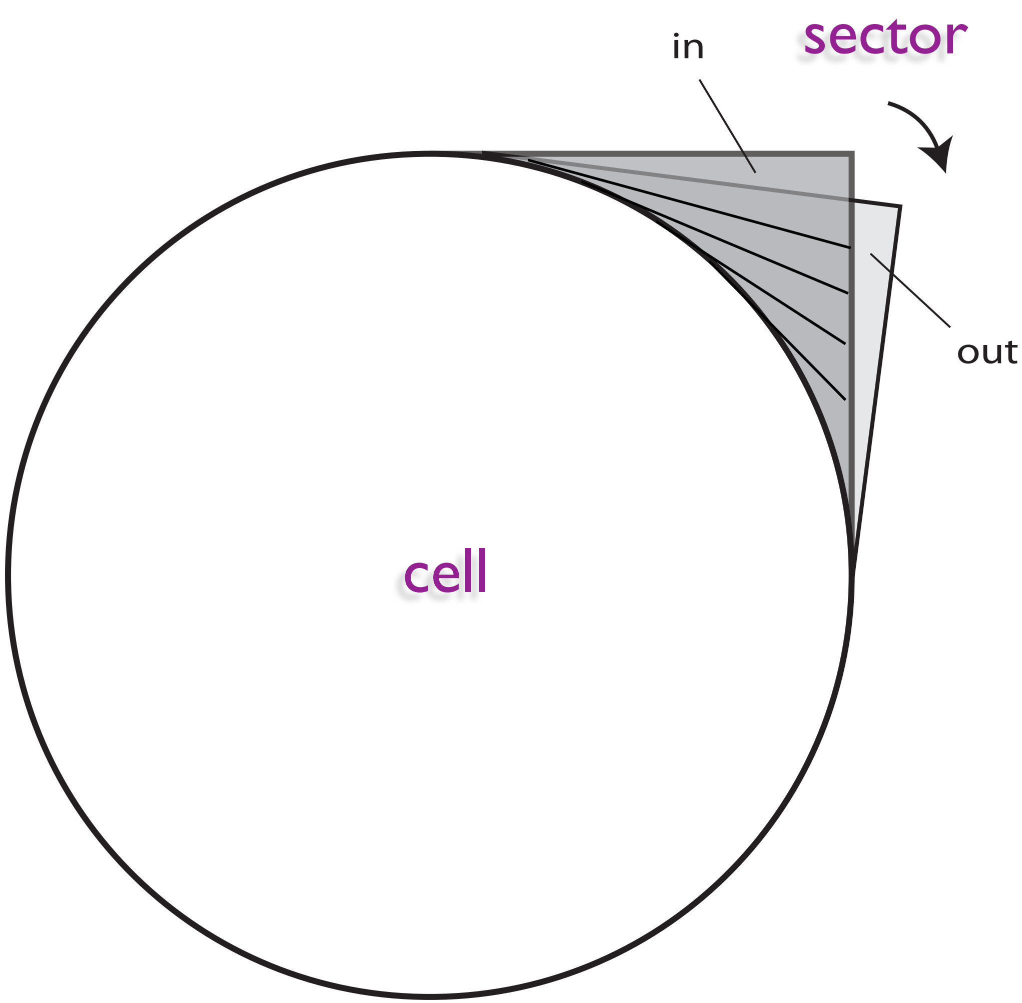
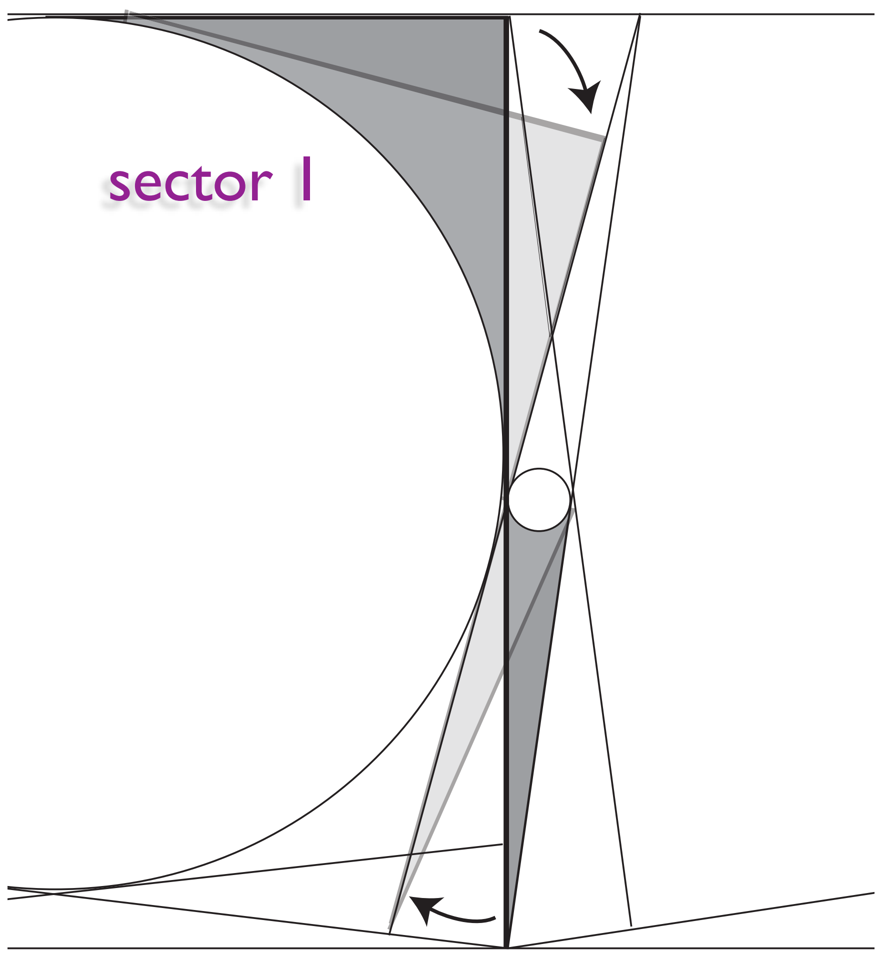




Cells interaction: linked sector maps



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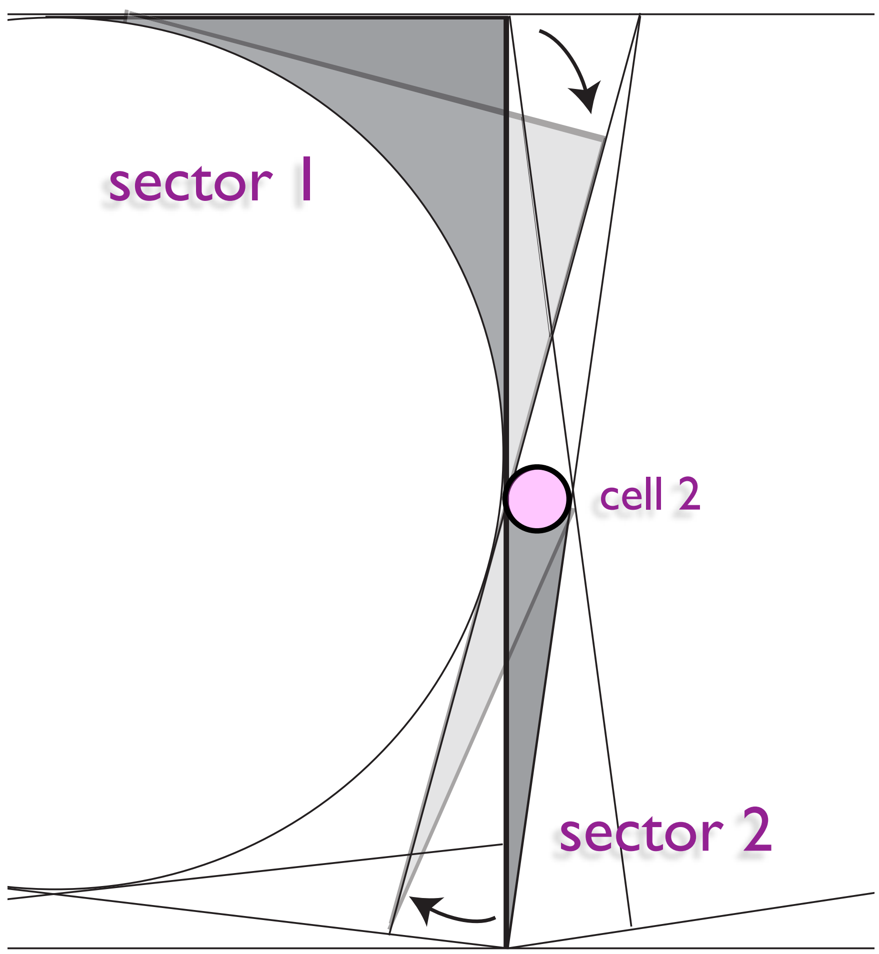

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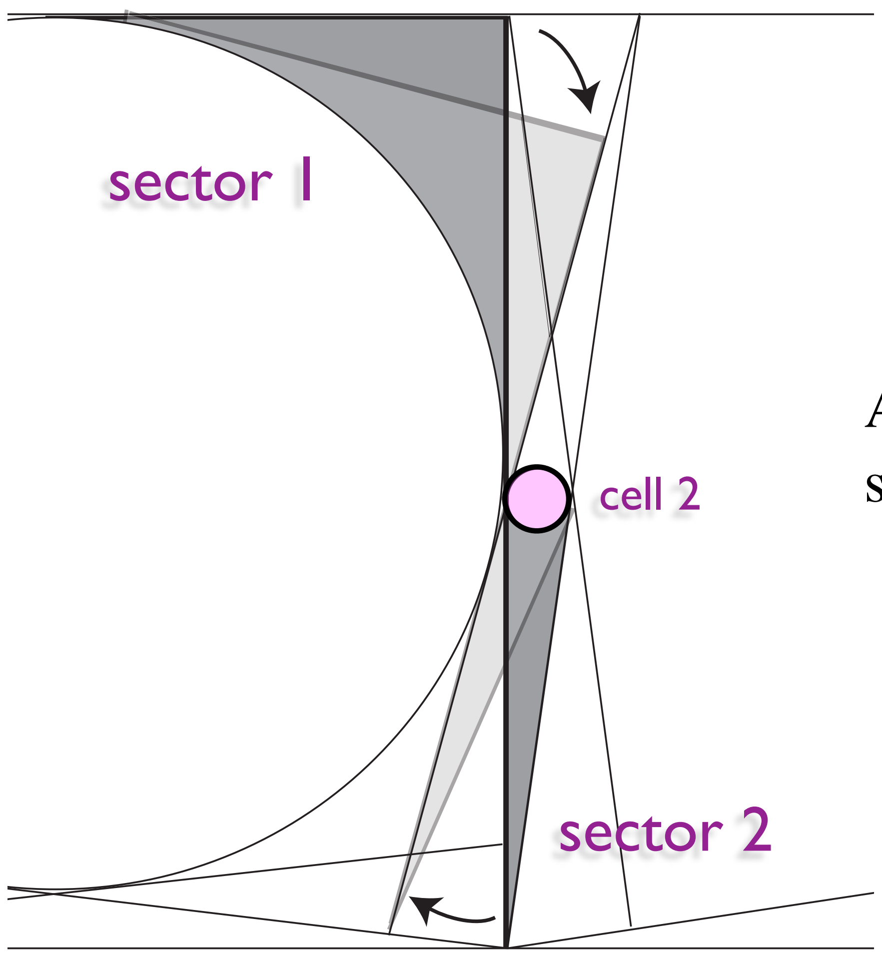
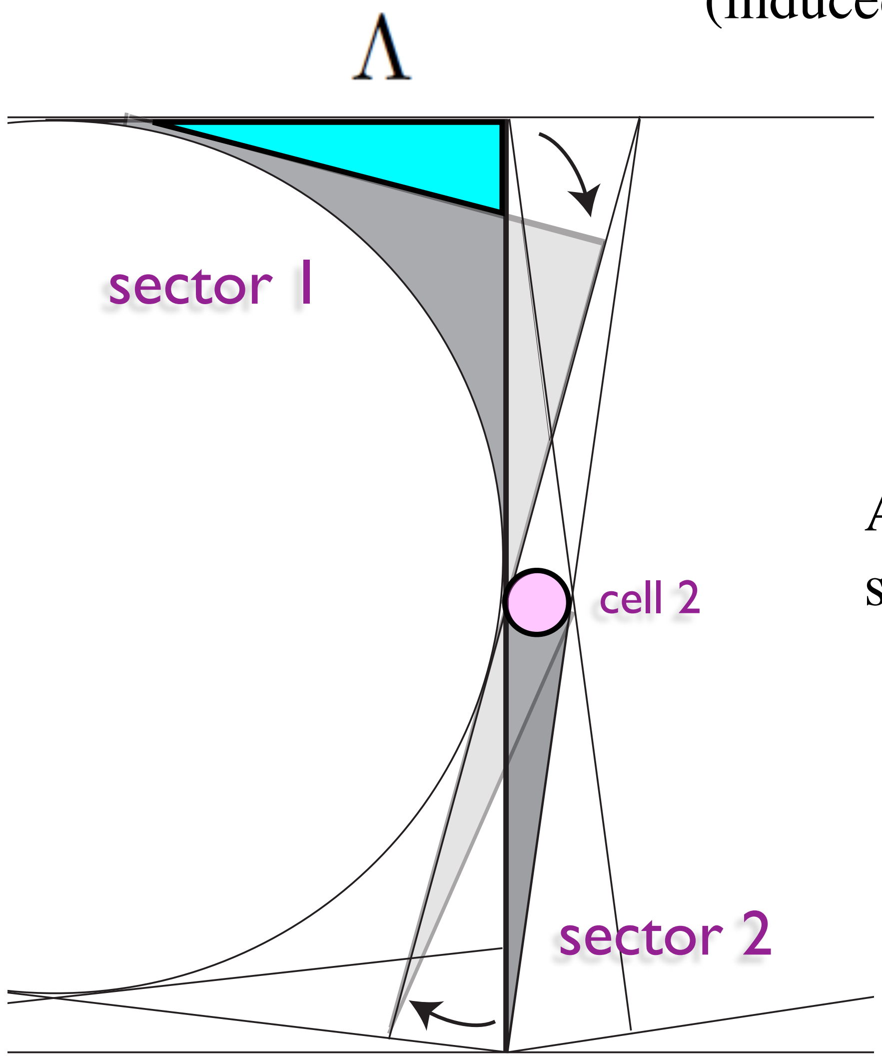


Almost all points of sector 1 return to sector 1, after passing through sector 2.

Cells interaction: linked sector maps

There is a natural surface of section (induced map).



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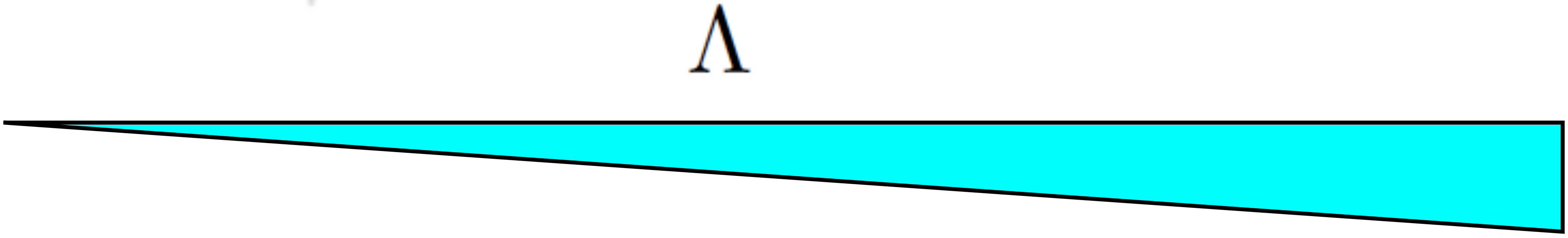
Return map



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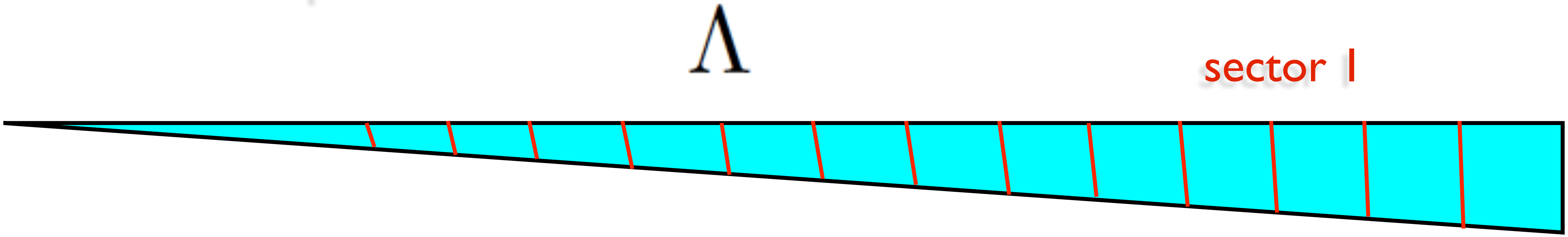

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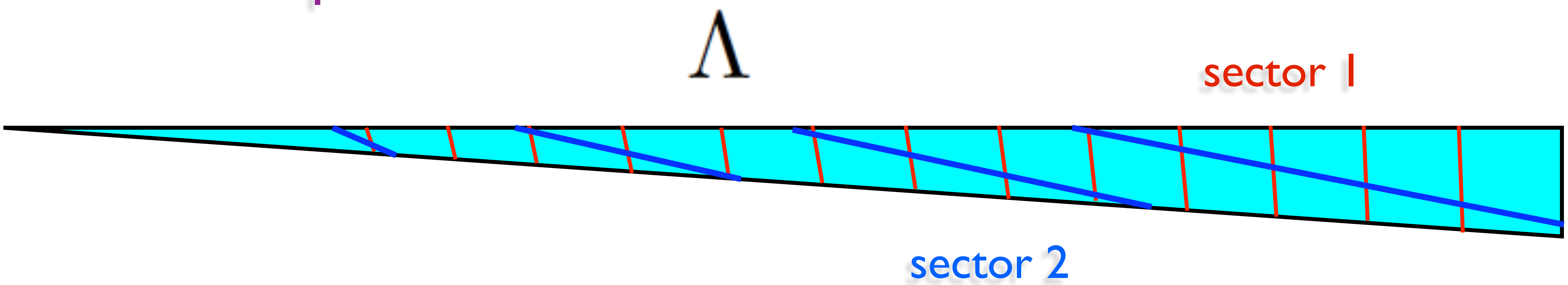

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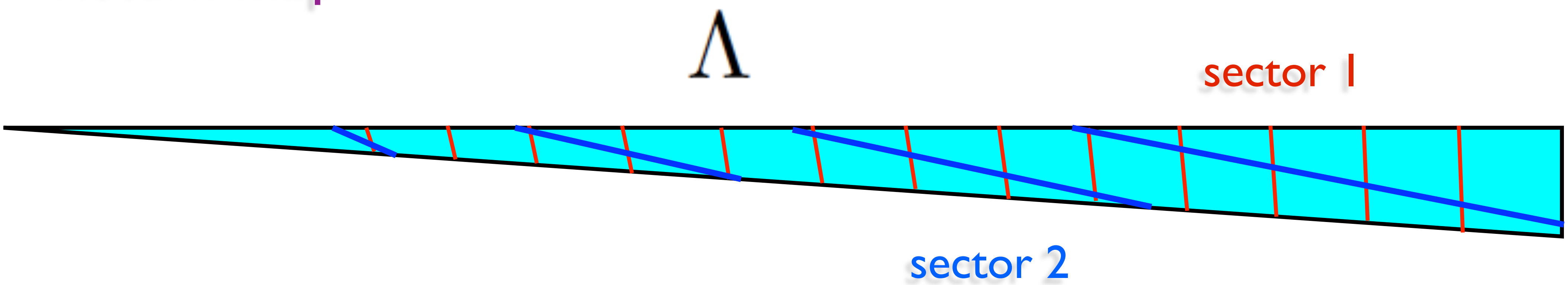

$\lambda \to 0$ (after scaling by λ in both directions)

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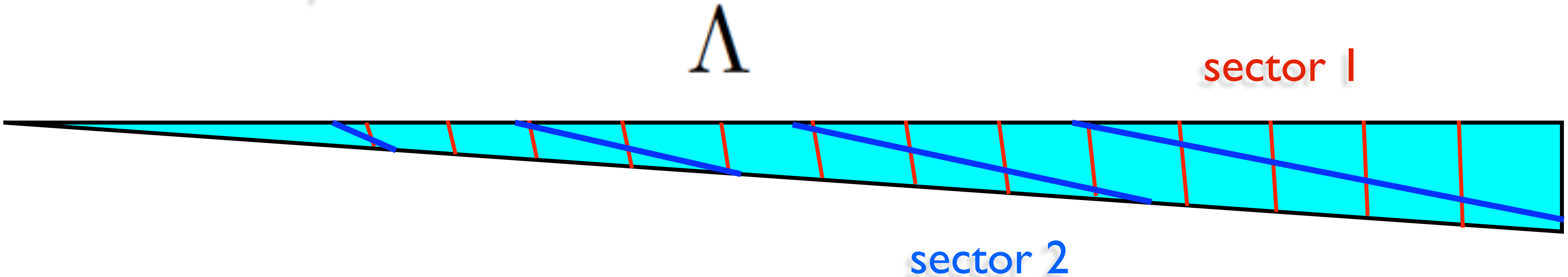

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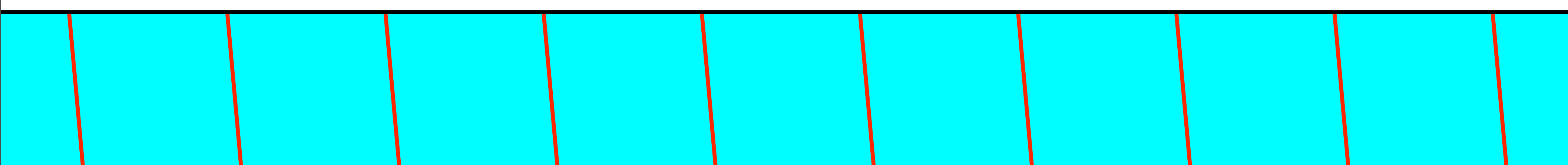

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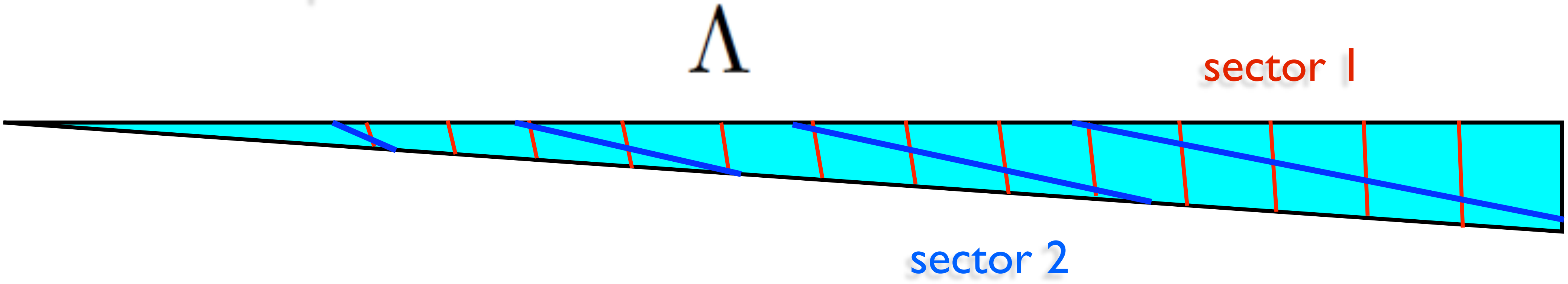

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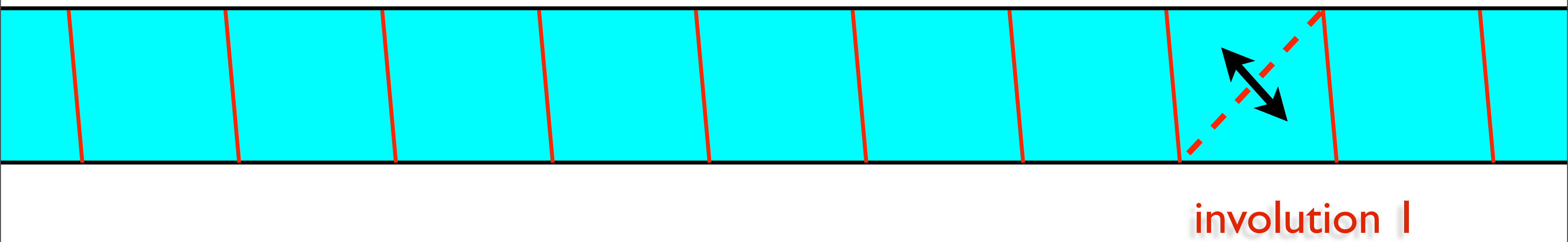

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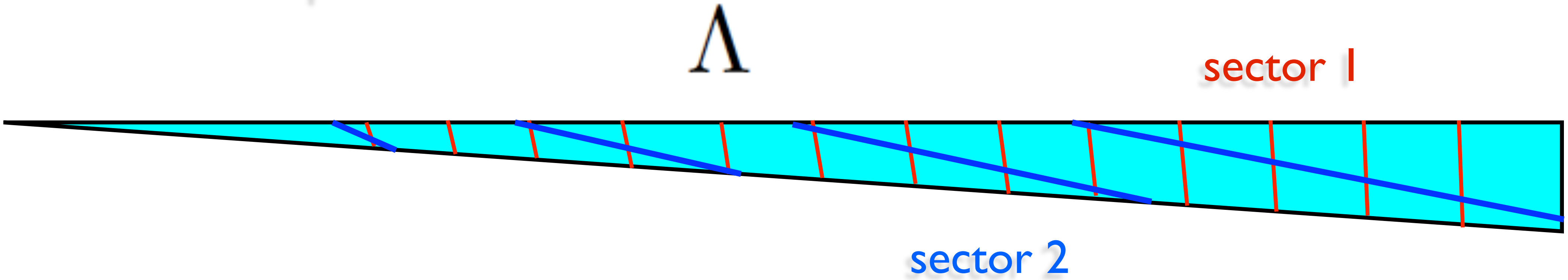

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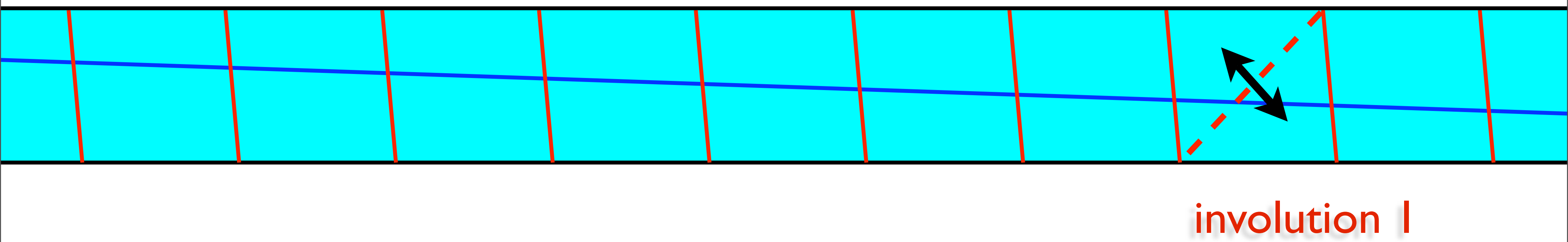

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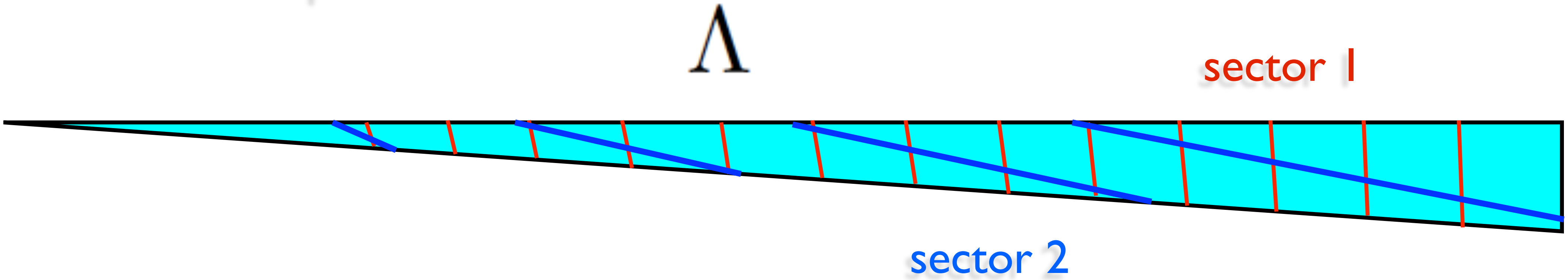

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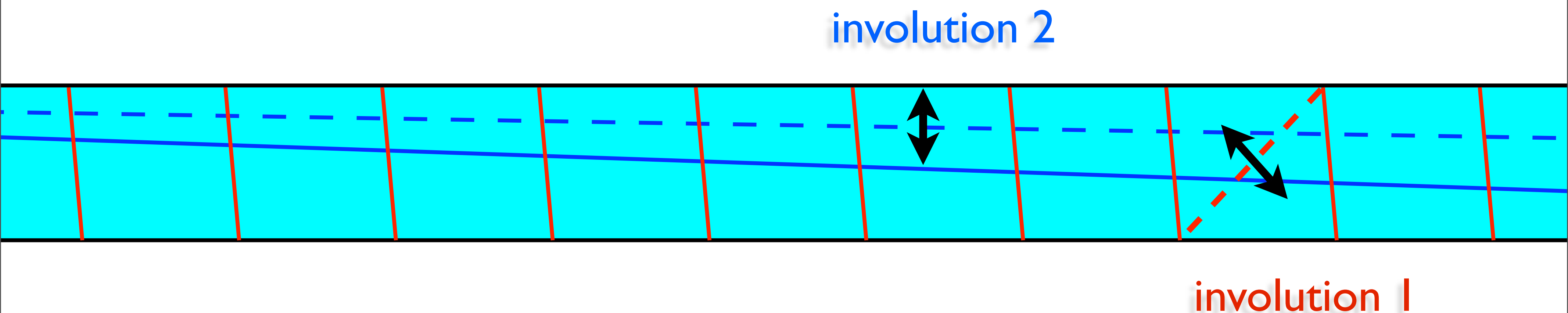

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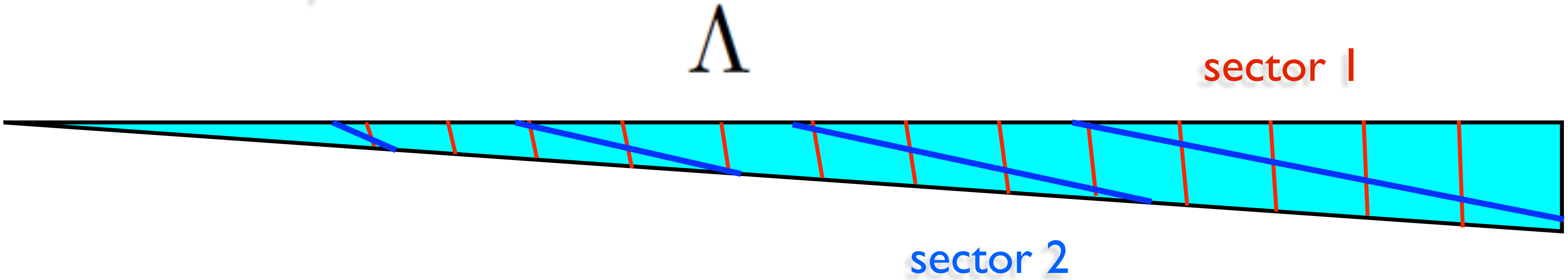

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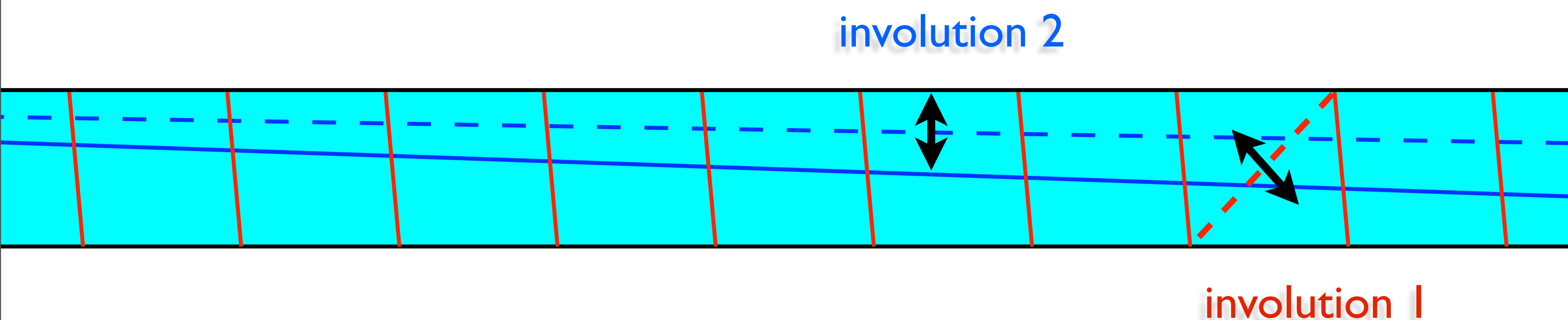

The return map is the composition of two involutions.

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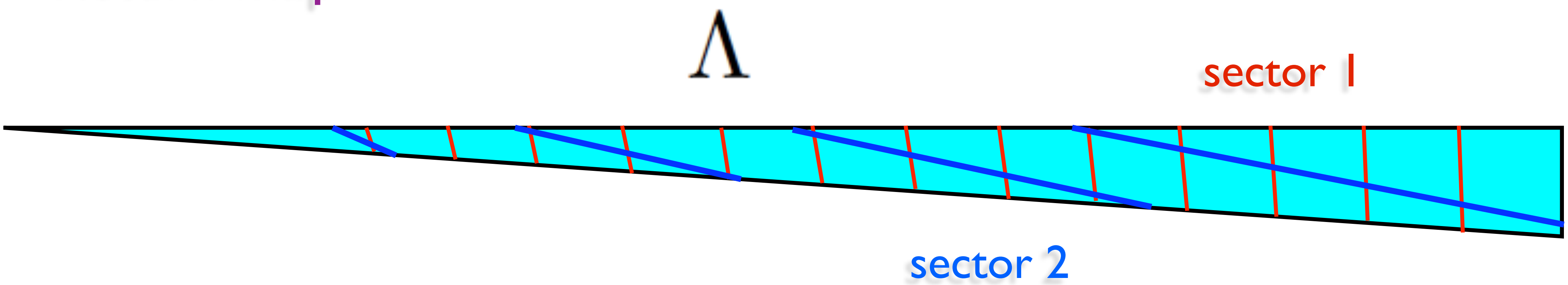

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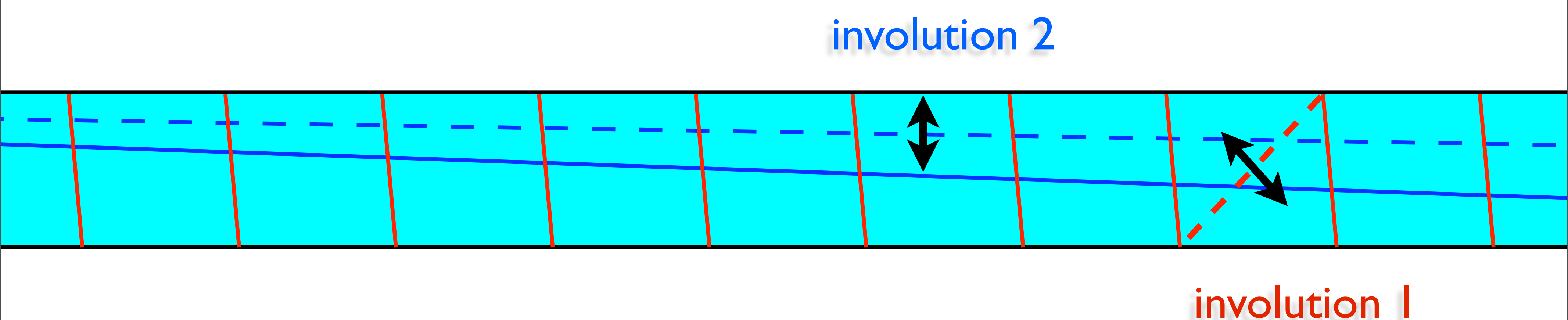

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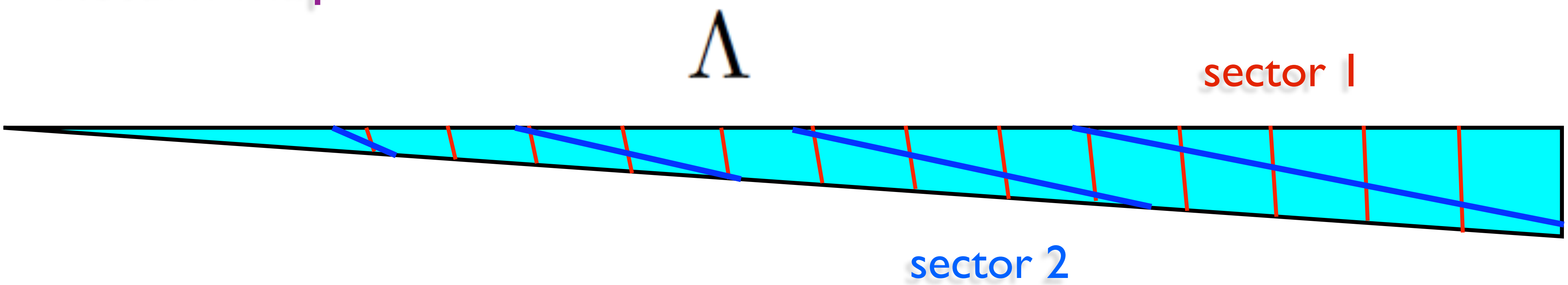
The number of atoms tends to infinity.

Return map



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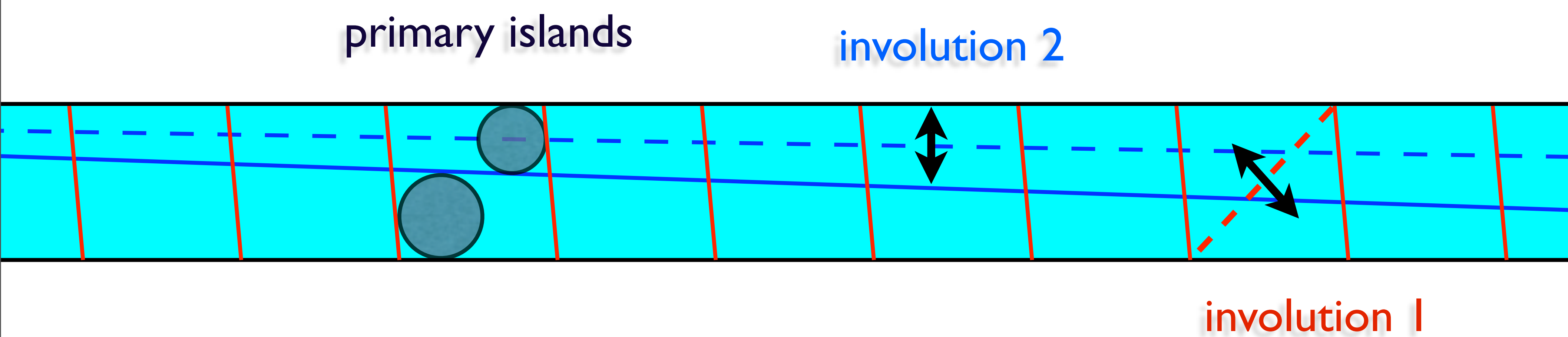

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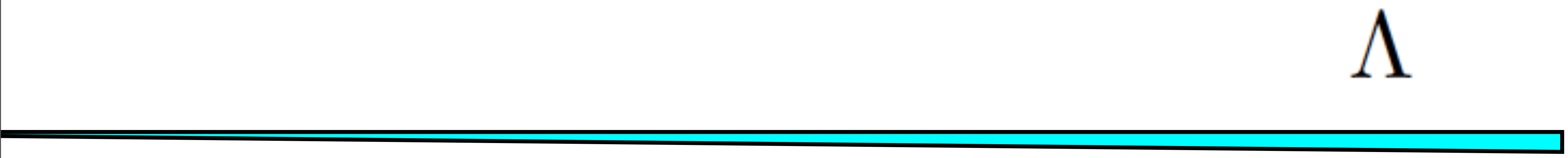
- Some estimates require computer assistance.

Limiting behaviour: the local map



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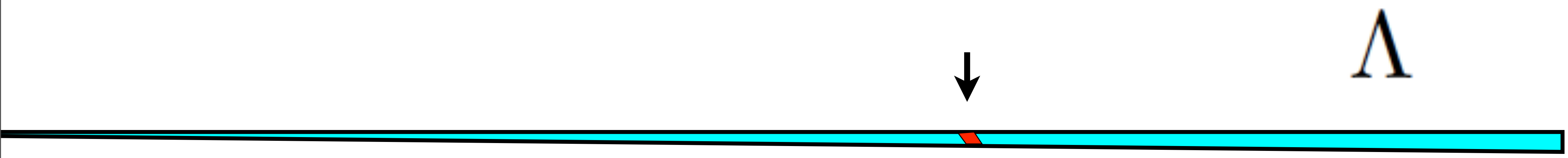

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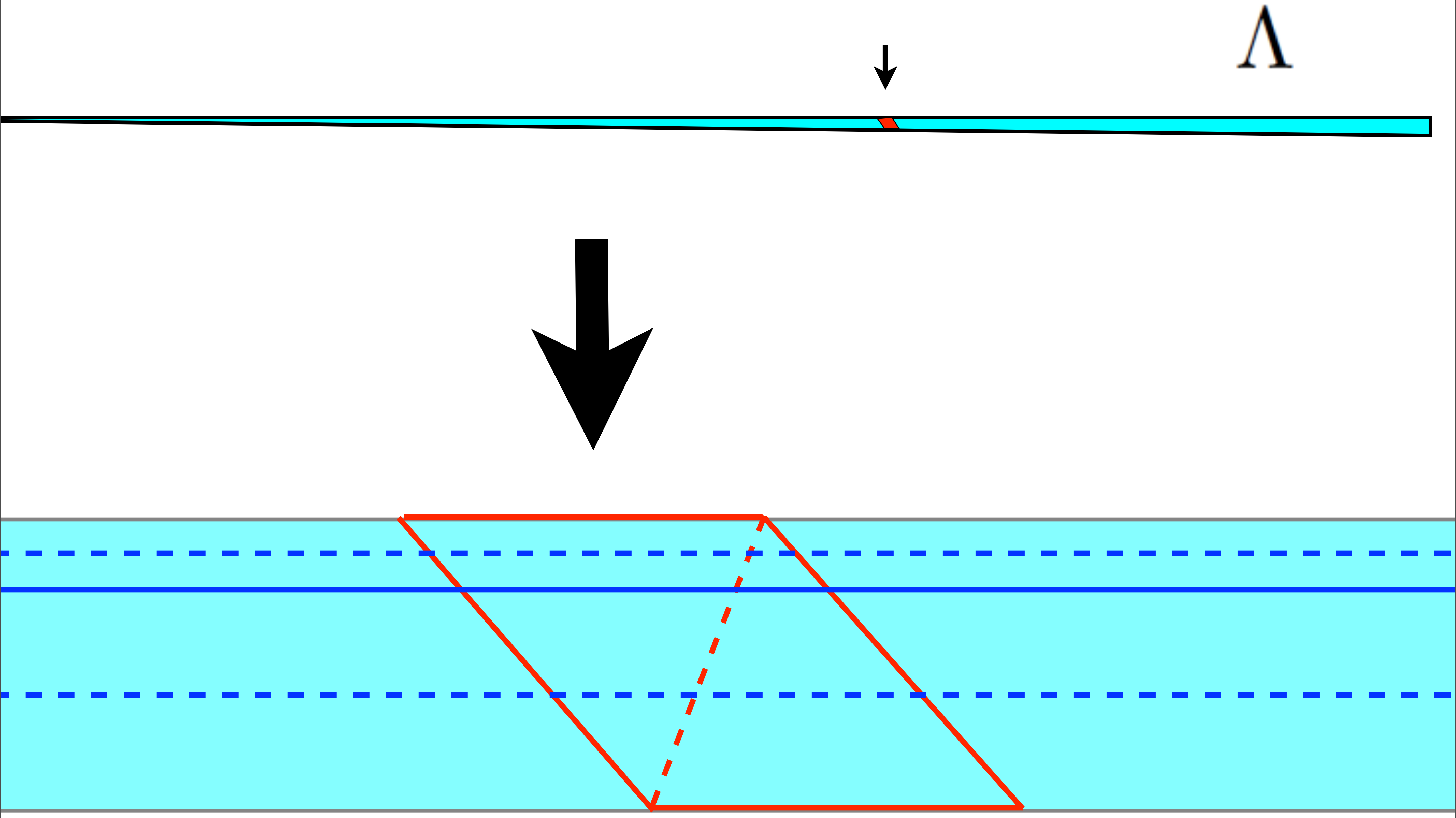

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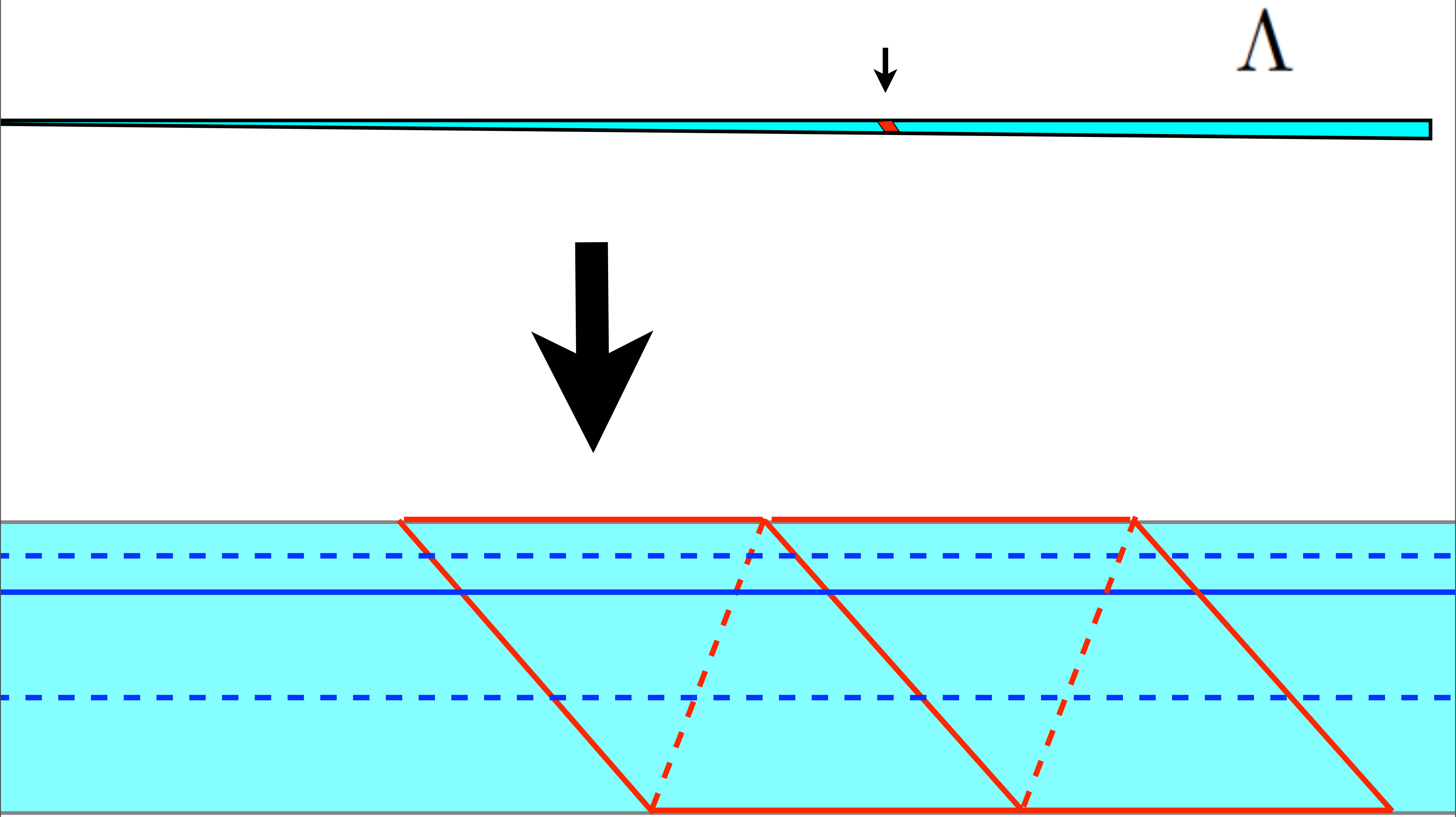
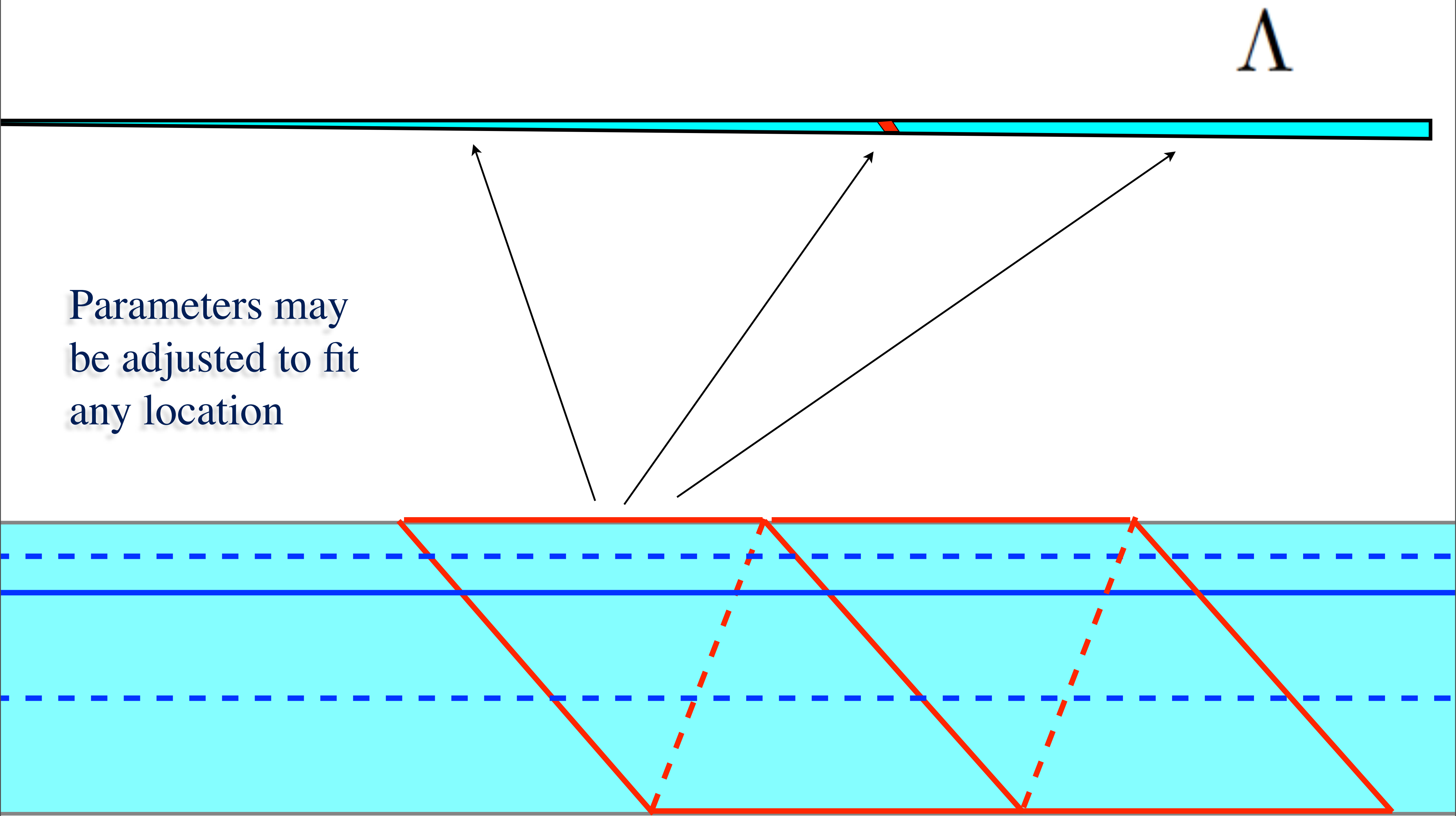


Translationally invariant: it can be put on the torus.

Limiting behaviour: the local map

Parameters may be adjusted to fit any location



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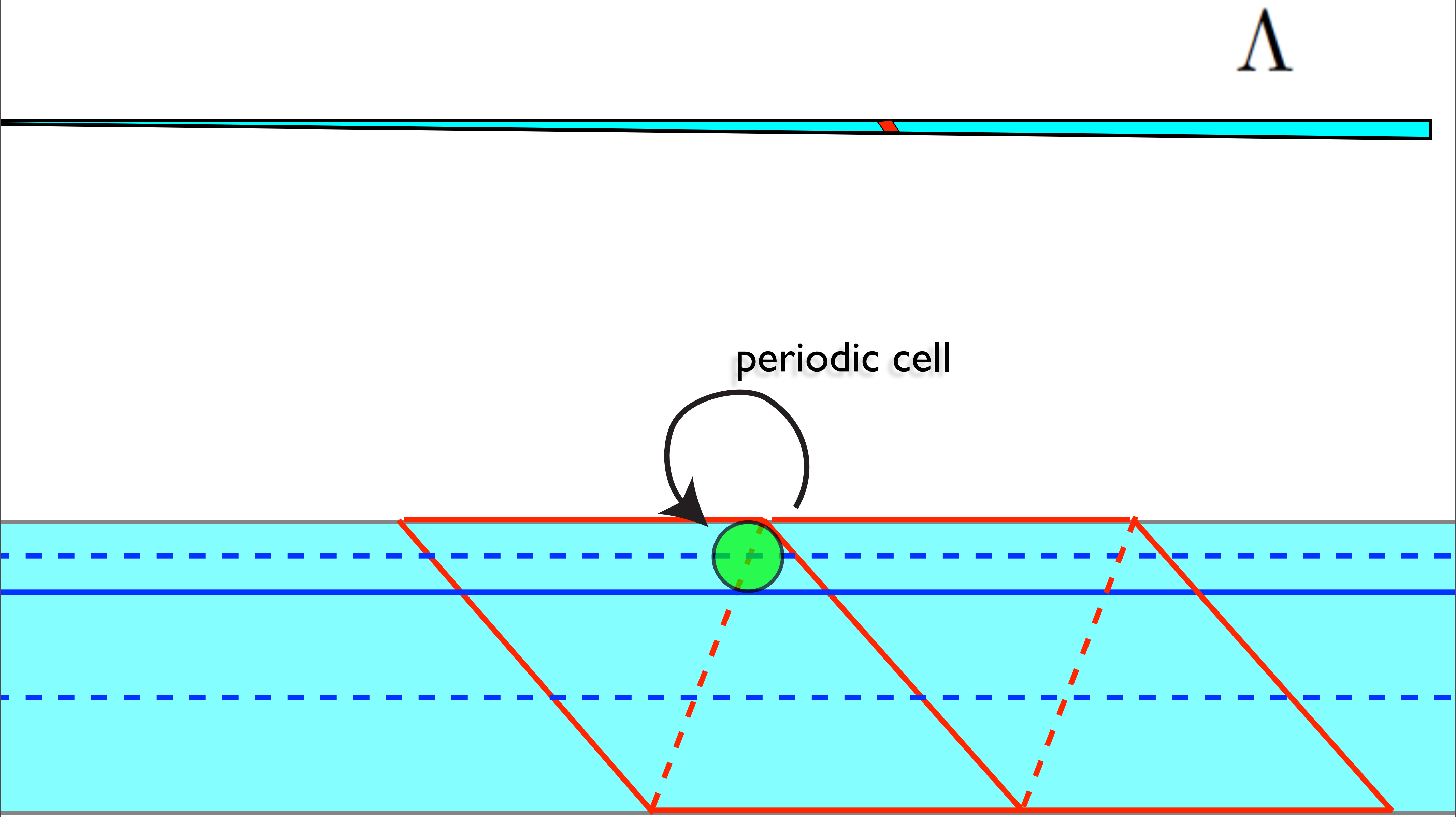

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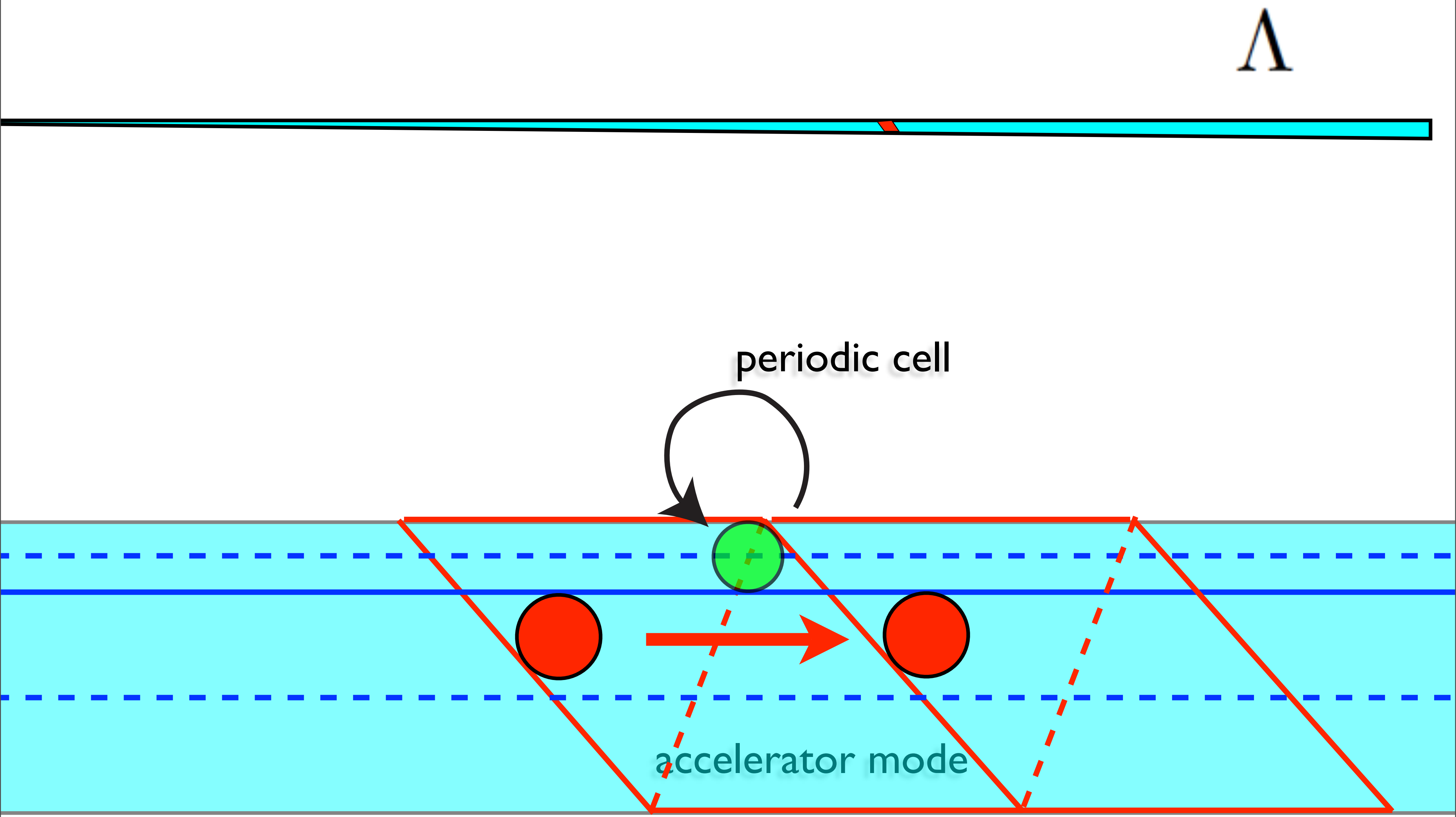

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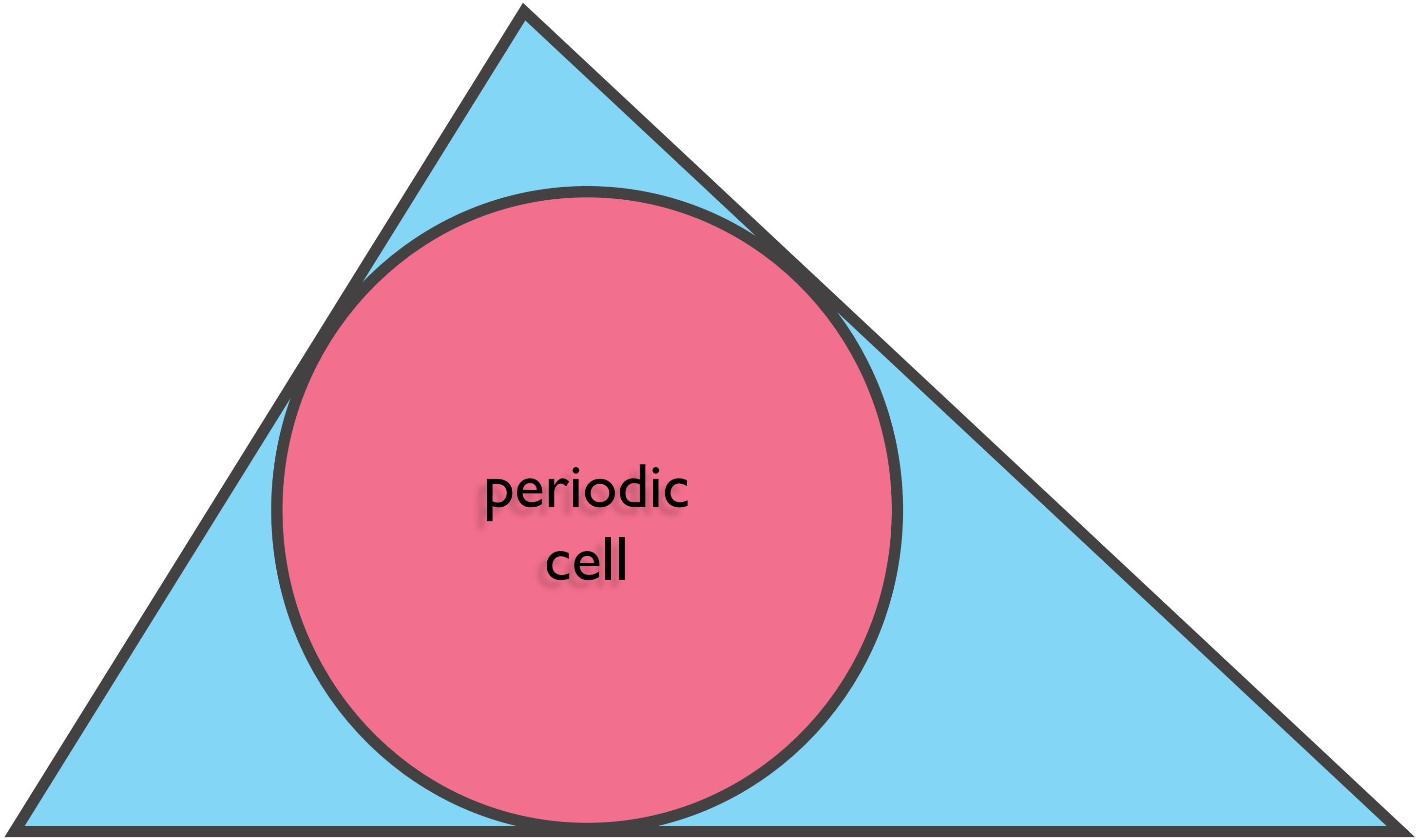

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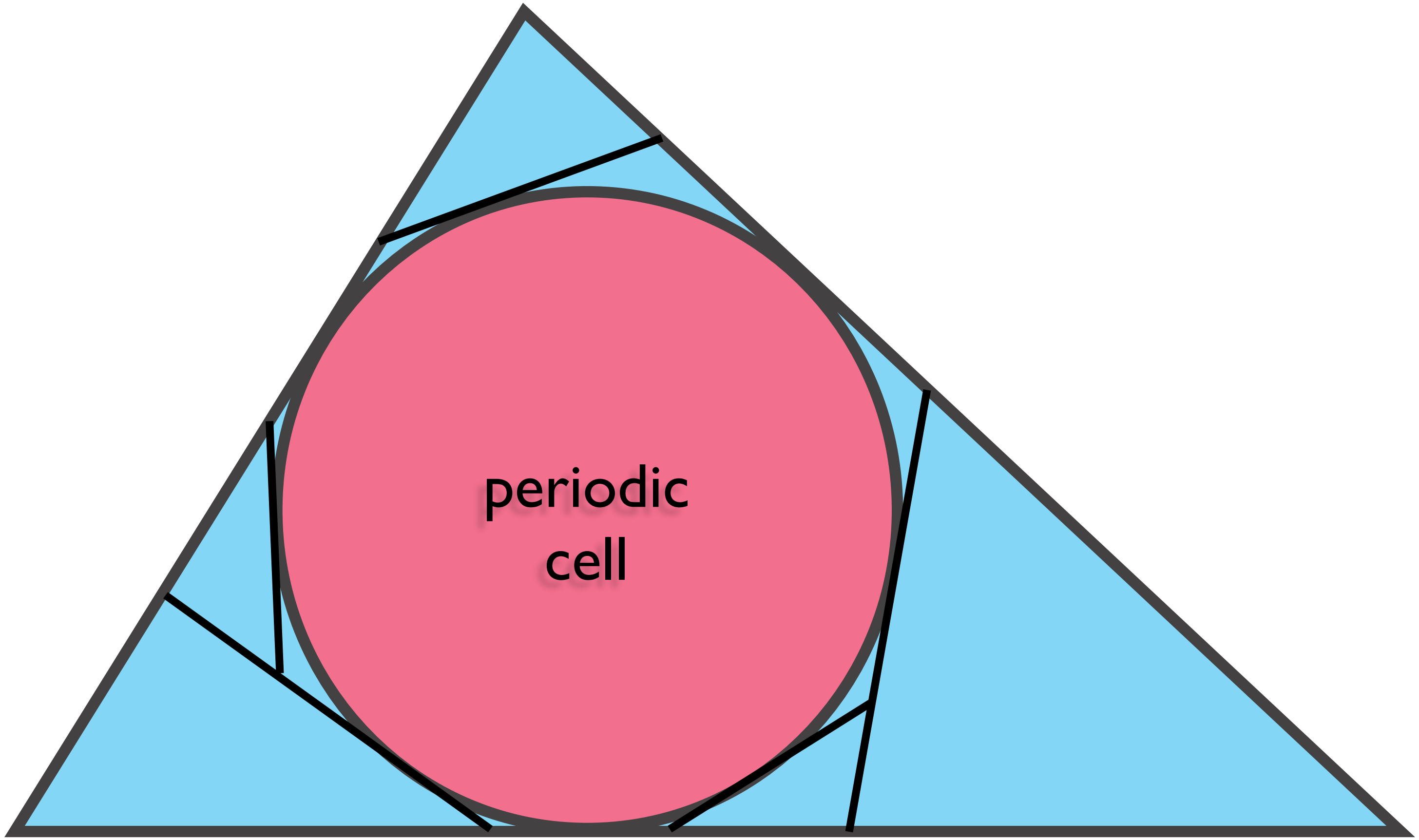

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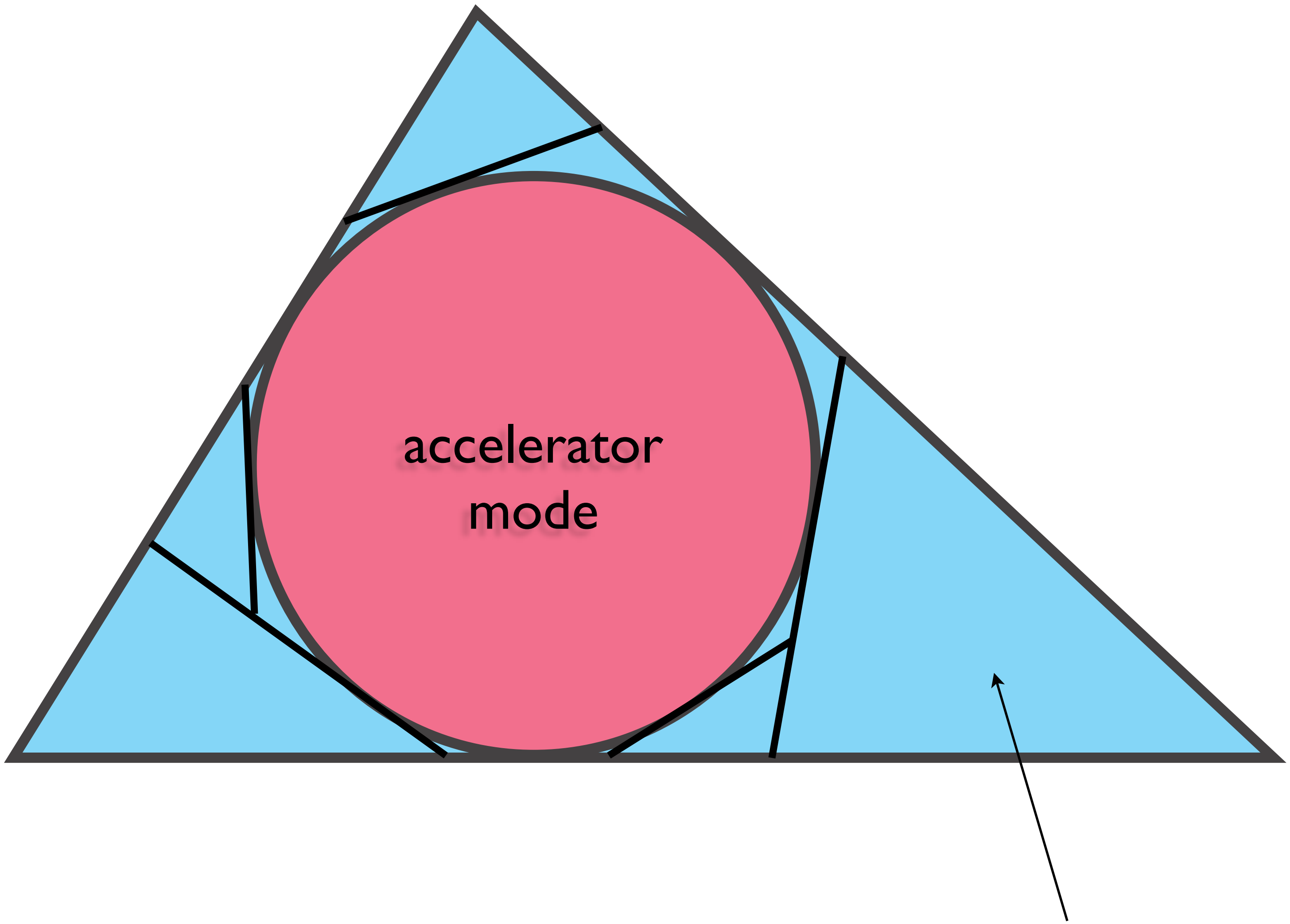
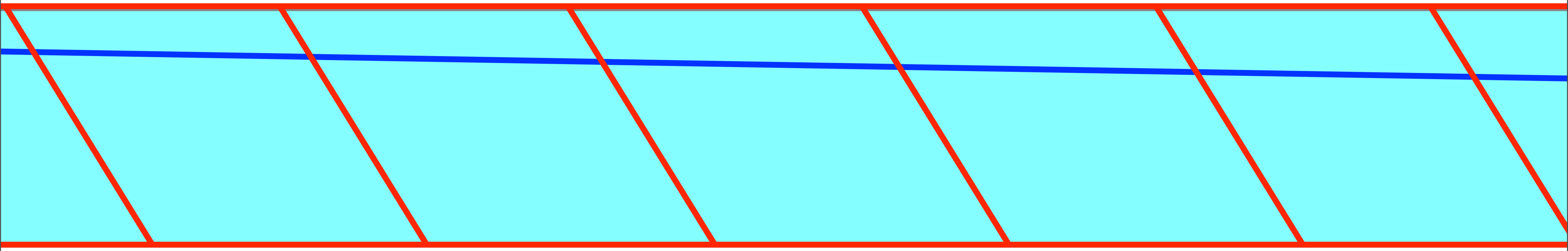


The same applies to accelerator modes, leading to **flights**.

Adiabatic perturbation



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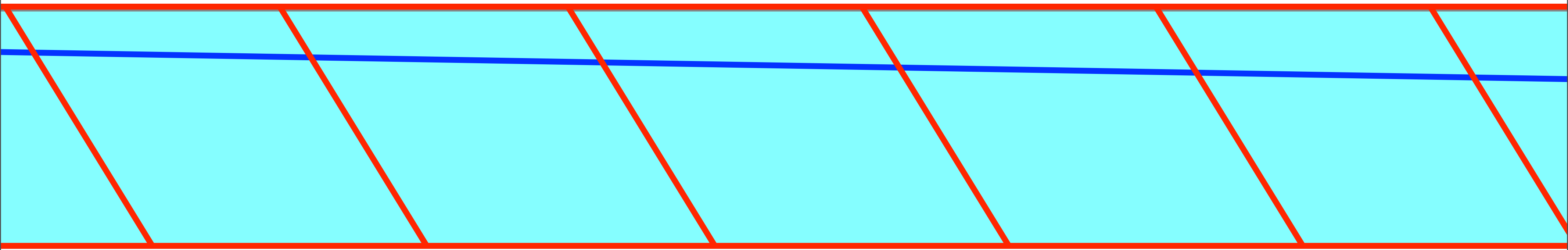

As one progresses along the chain, accelerator modes bifurcate out of existence, due to slow change of parameters.

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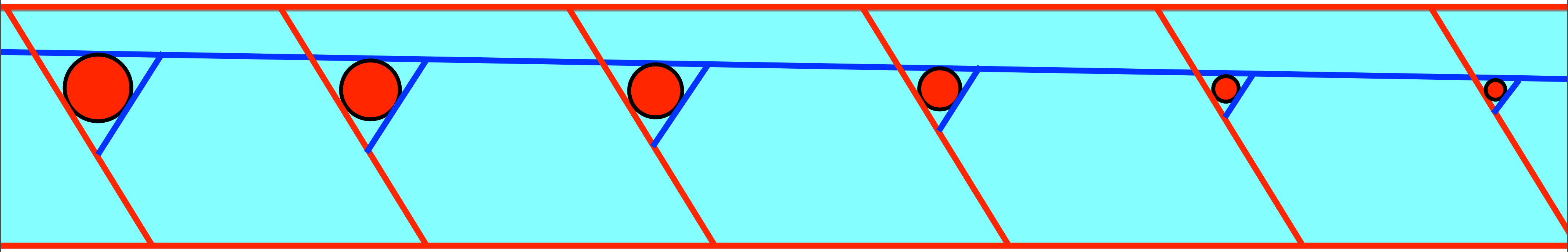

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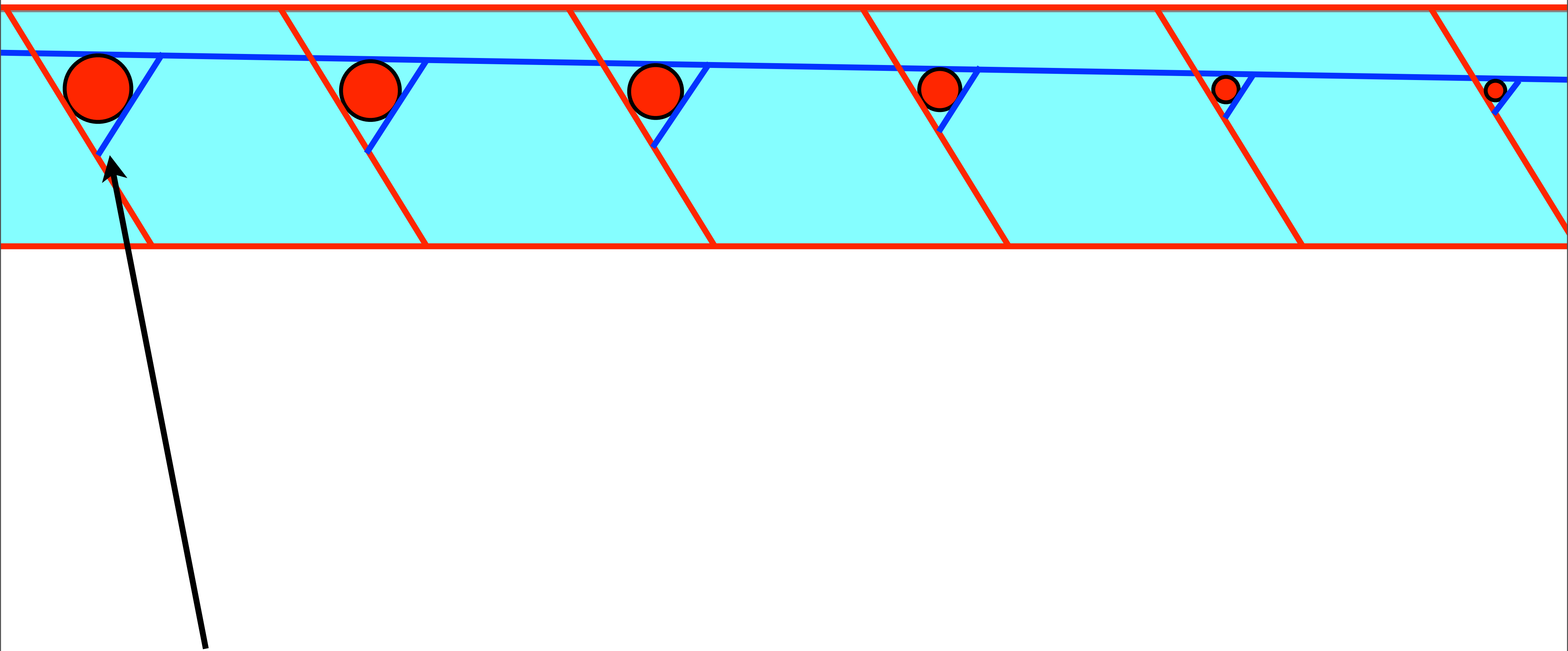

This process may be represented within the largest cell, by mapping backwards to it the relevant atom boundaries.

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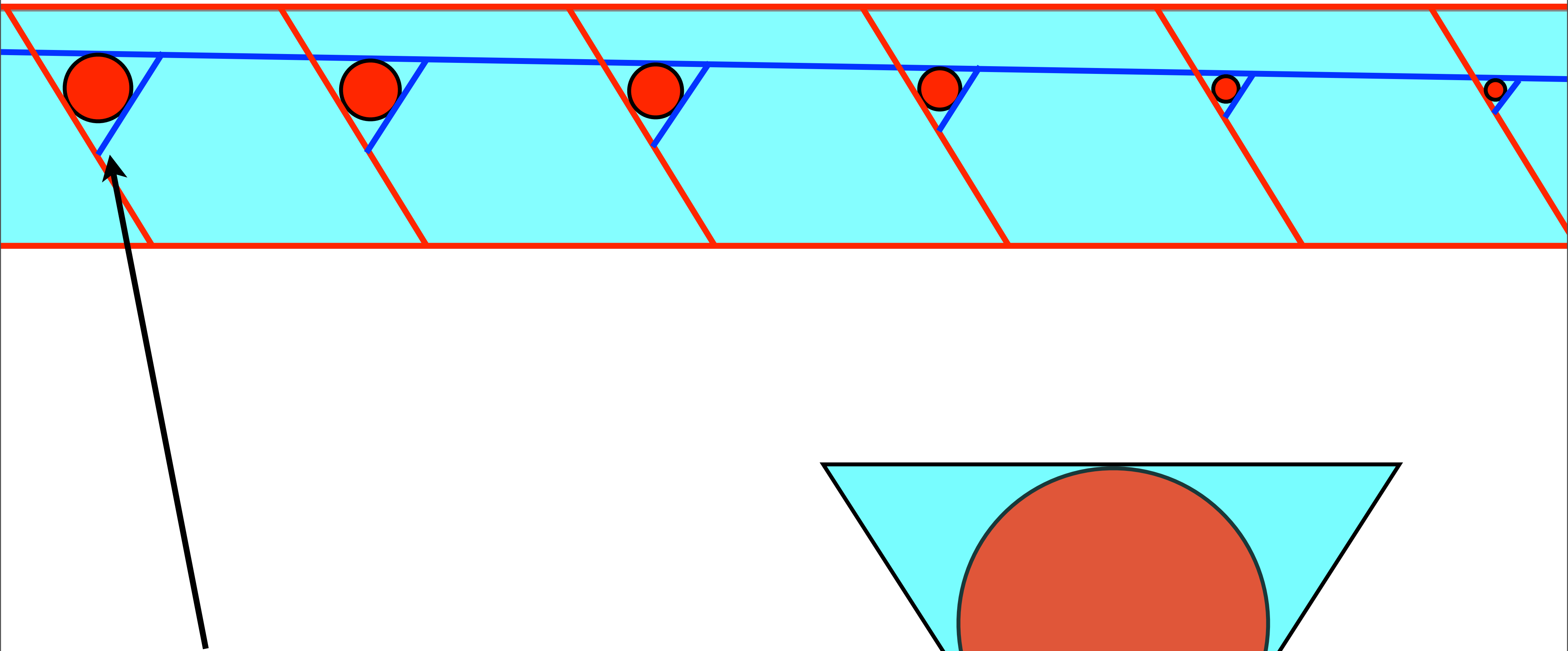

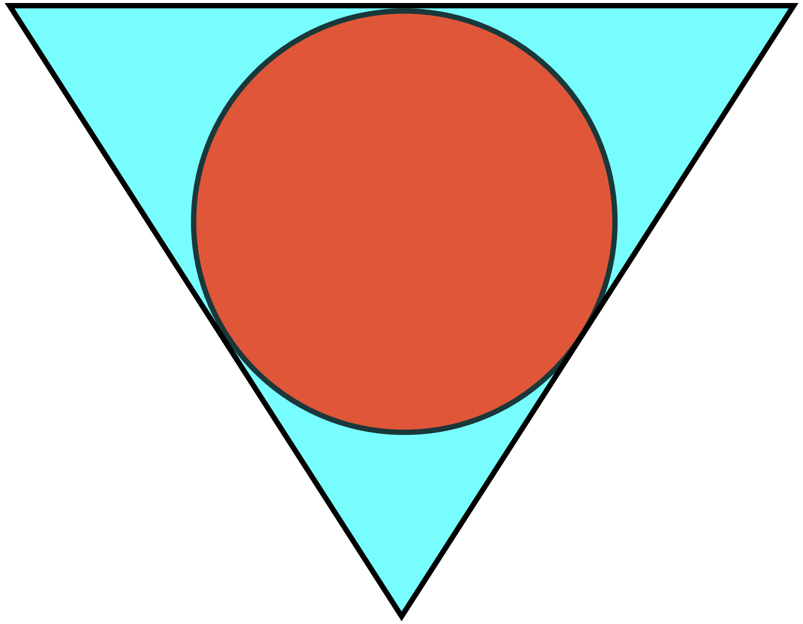

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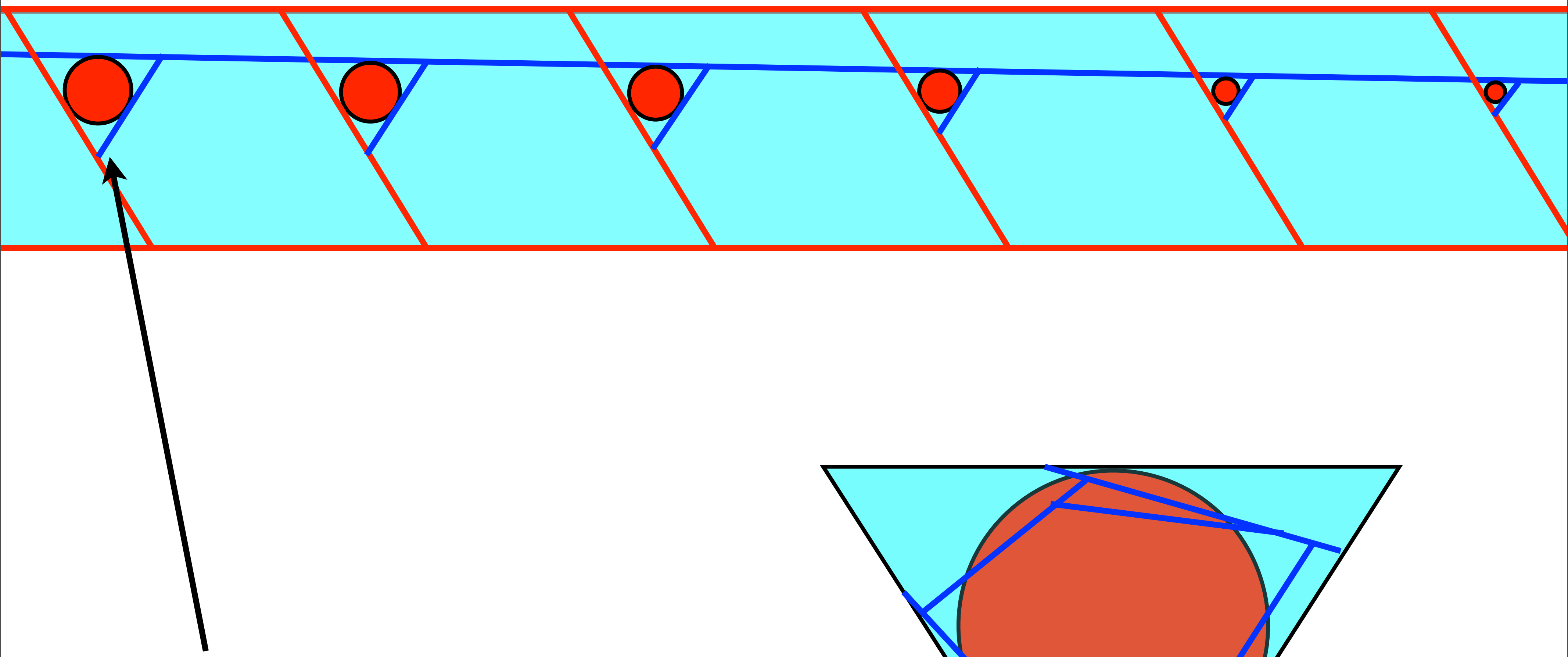

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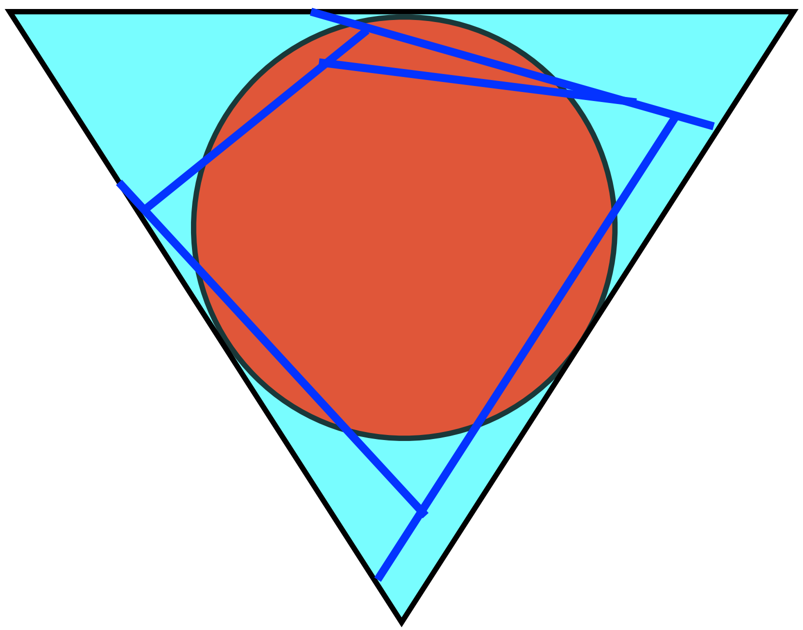
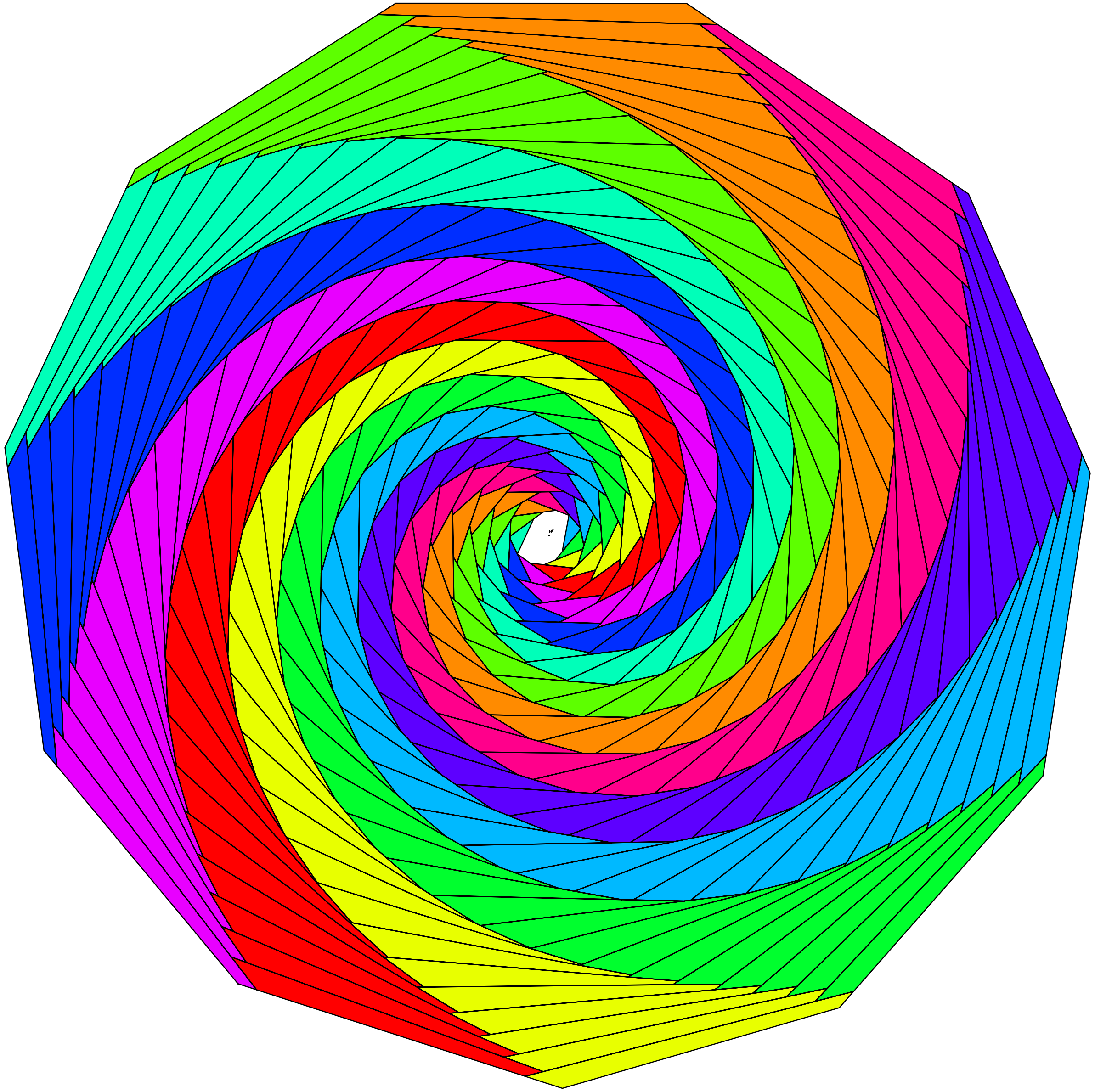
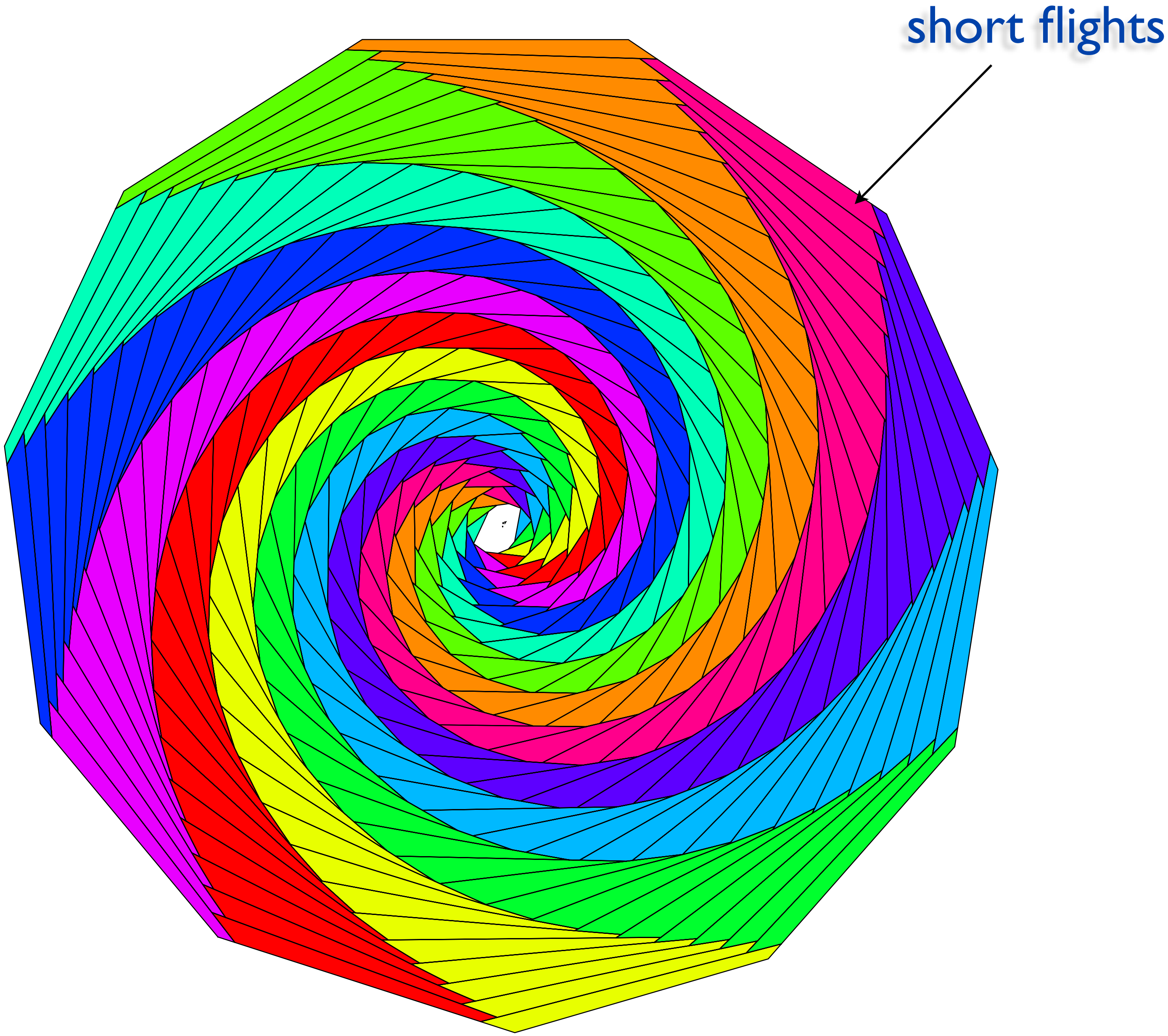
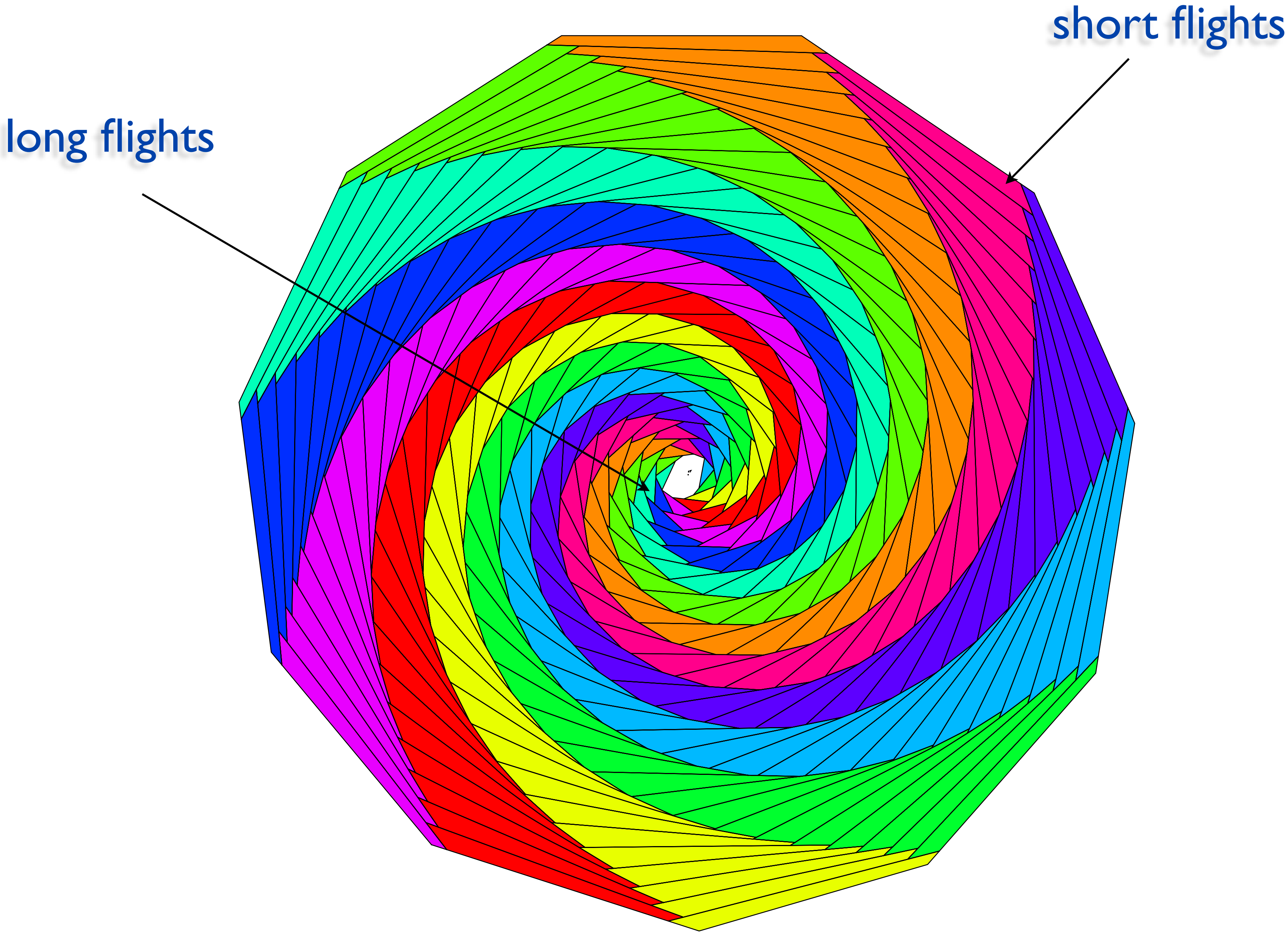





short flights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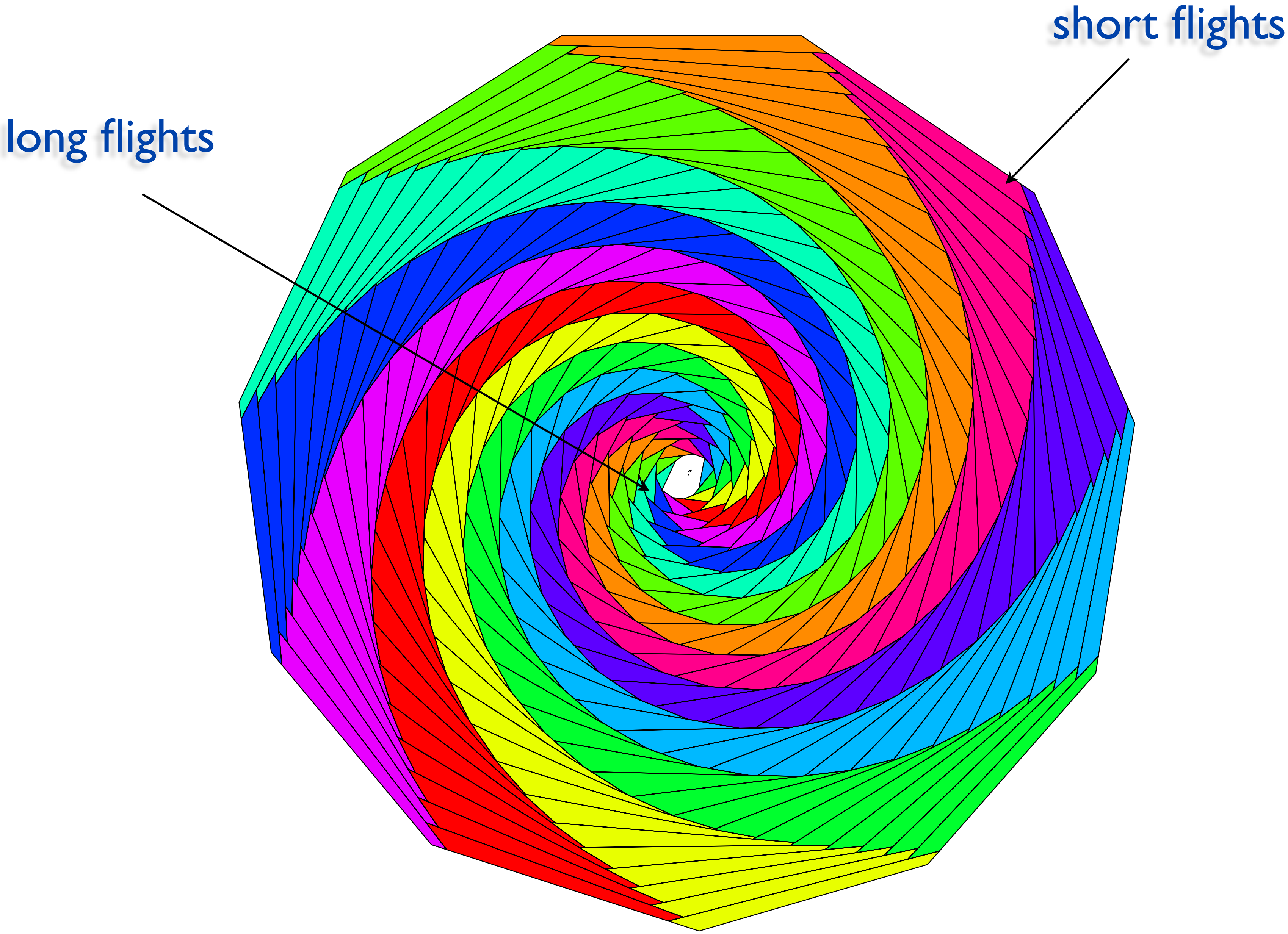
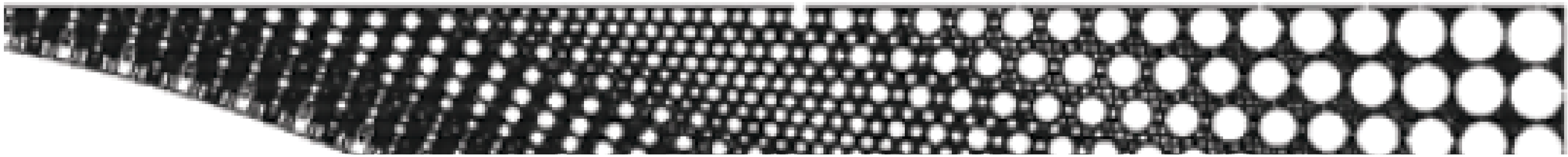
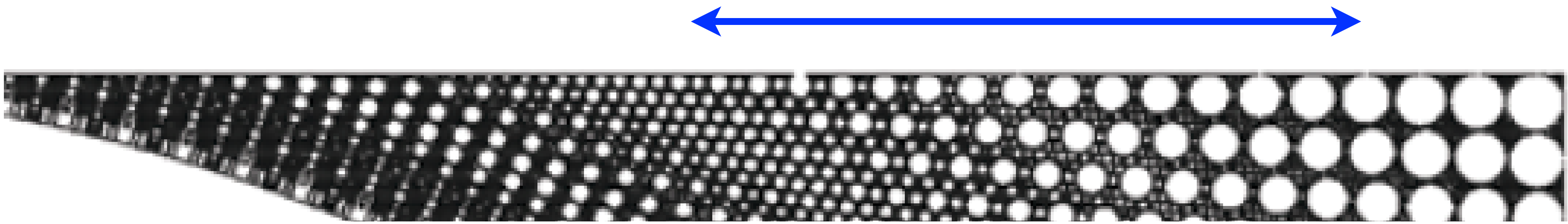


Measure of flights decays quadratically in the flight's length.



In the original system, flights travel distances of order $O(1)$.



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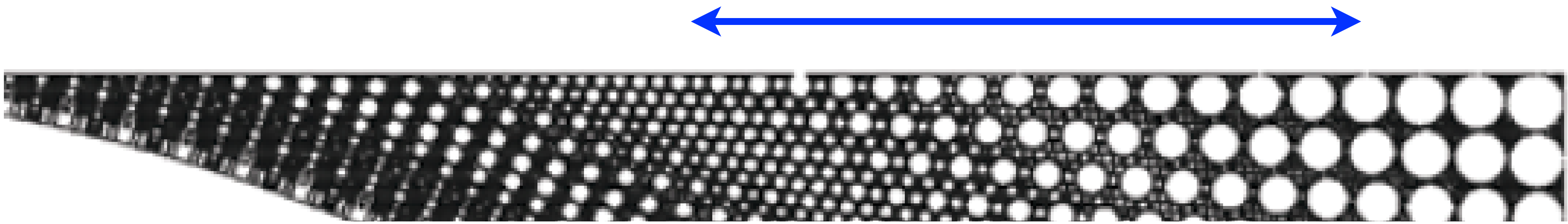
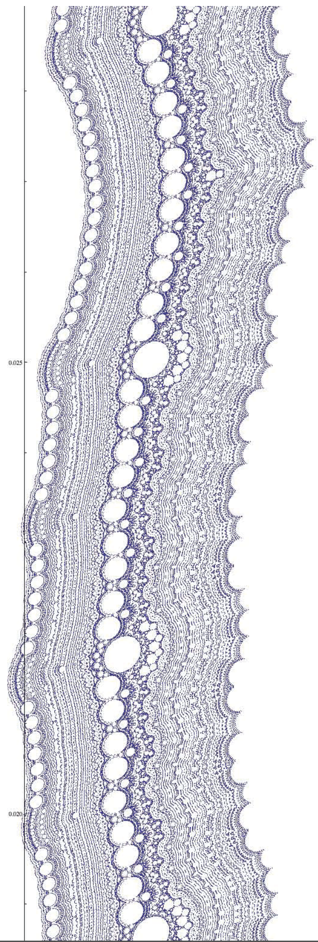


Along a flight path, we can exclude the existence of (non-smooth) invariant curves, observed in some piecewise isometric systems.



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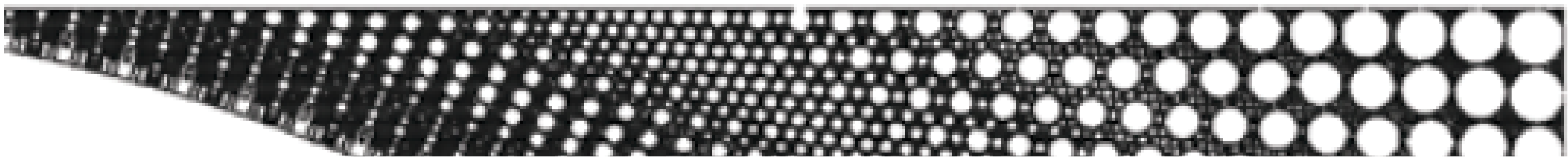

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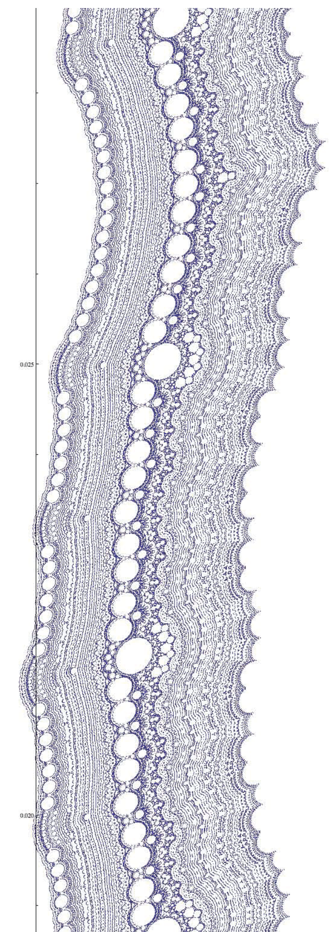

So in this model, there are no topological obstructions to transport in the region outside the primary islands.

Thank you for your attention