Mathematical Writing

Franco Vivaldi
Queen Mary, University of London

Mathematical Reading / Writing / Translating

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Why mathematical writing?

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- MW students commented on the "unexpected depth" required of their thinking, when asked to offer verbal explanations.

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Answer this question **over the phone**.

- This task requires grasp of structure and organisation;
- it gives the students an opportunity to express their knowledge, intelligence, and individuality;
- but it also exposes logical faults, immaturity, incompetence.

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Adjoin the equation of the line to that of the circle and eliminate one of the unknowns. After a substitution, you'll end up with a quadratic equation in one unknown, whose coefficients depend on the parameter.

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Equate the discriminant of the quadratic equation to zero, to obtain an equation -also quadratic- for the slope. Its two solutions are the desired slopes of the tangent lines. Any configuration involving vertical lines (infinite slope) will require a variant of the above procedure.

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Classroom schizophrenia

Classroom schizophrenia

Teacher

Student

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Teacher

Student

definitions
theorems

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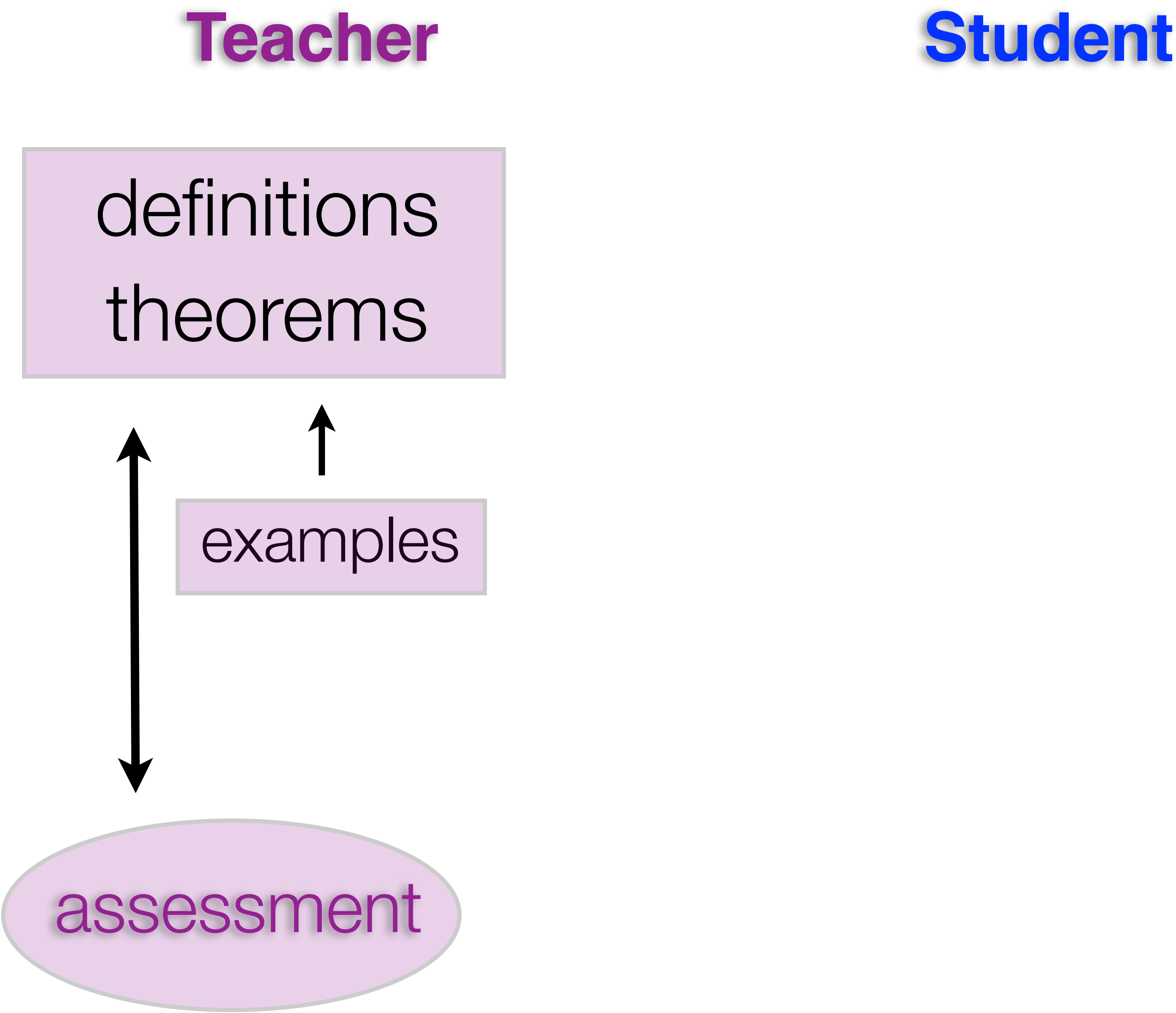
↑

examples

Classroom schizophrenia



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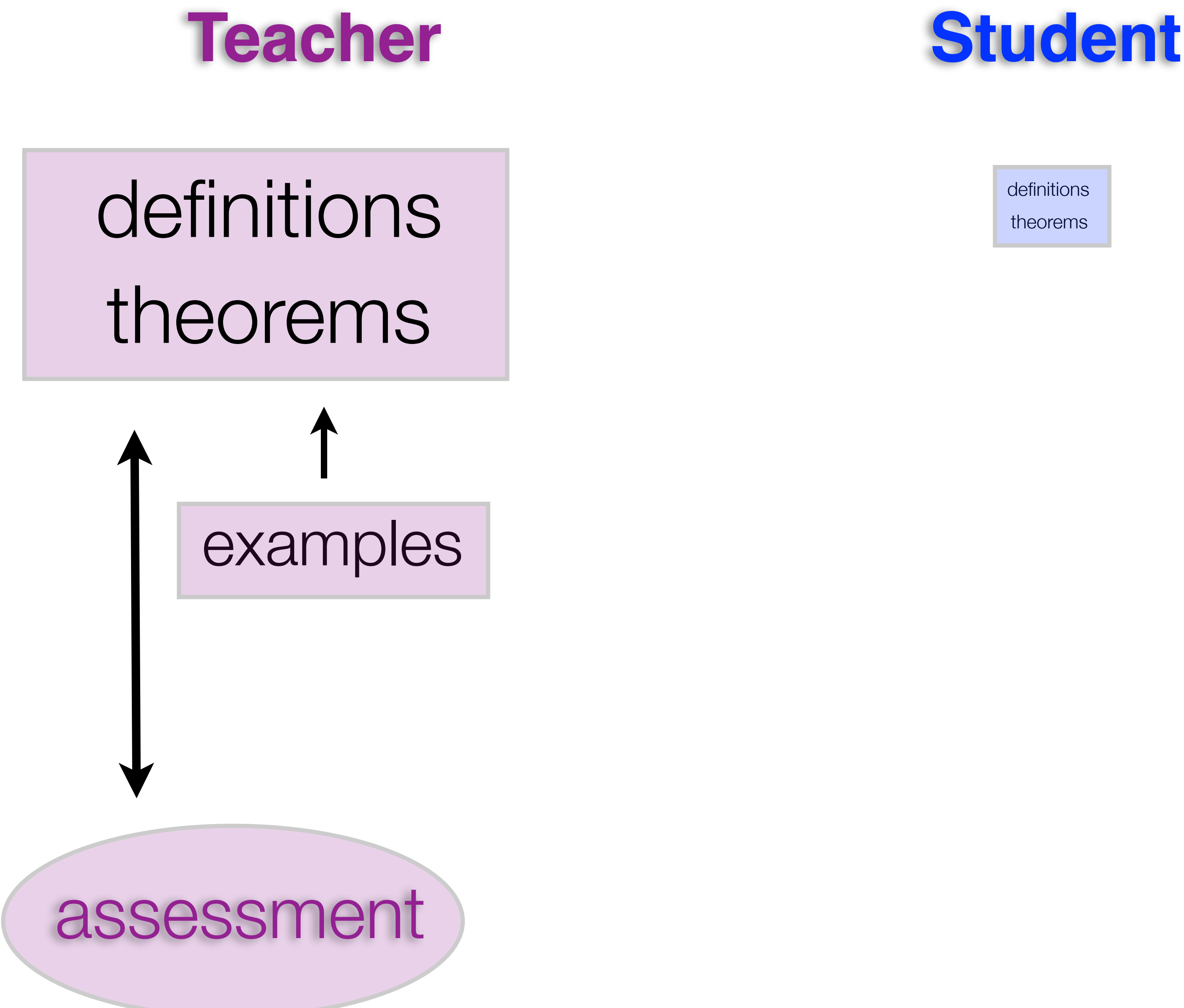

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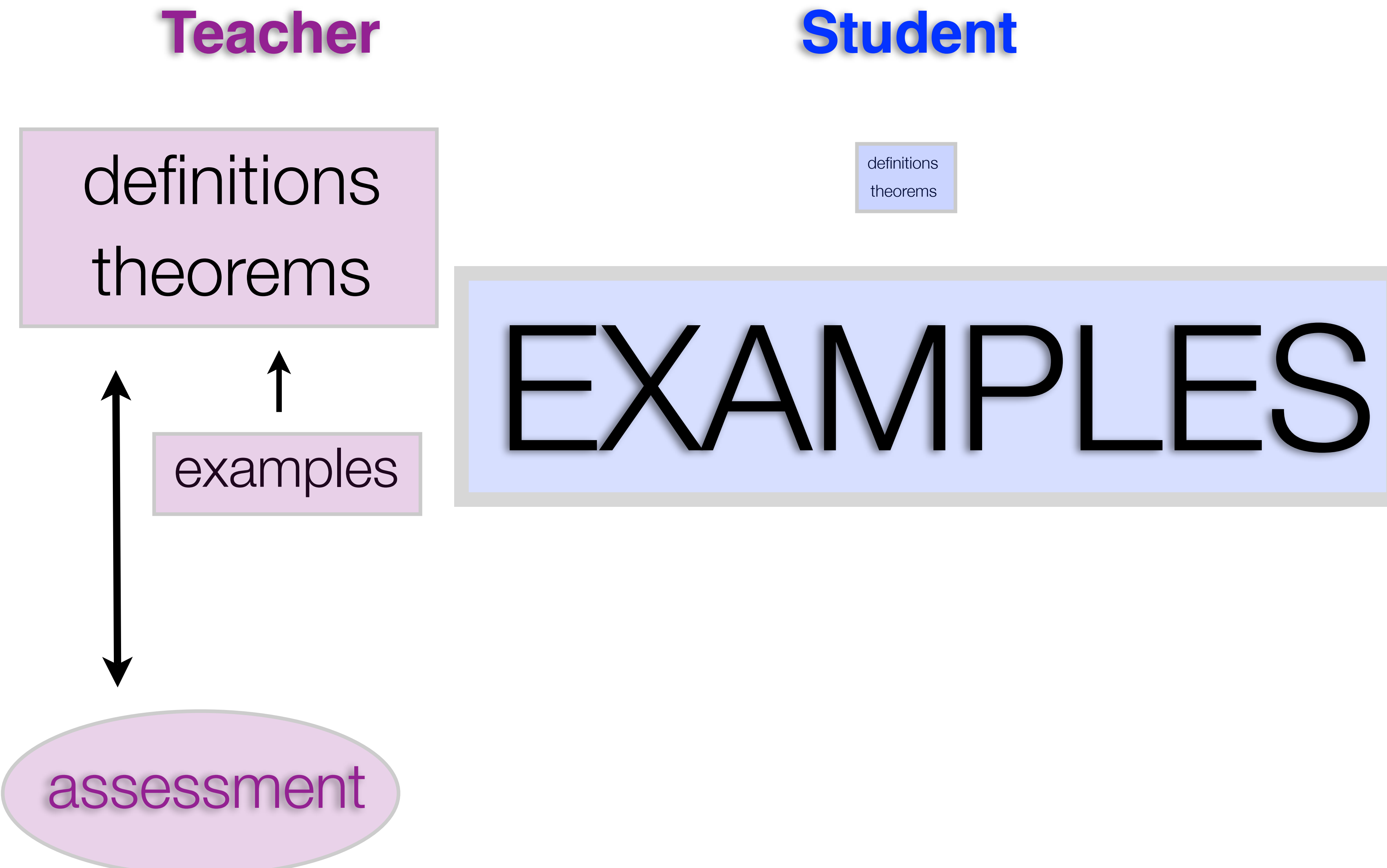

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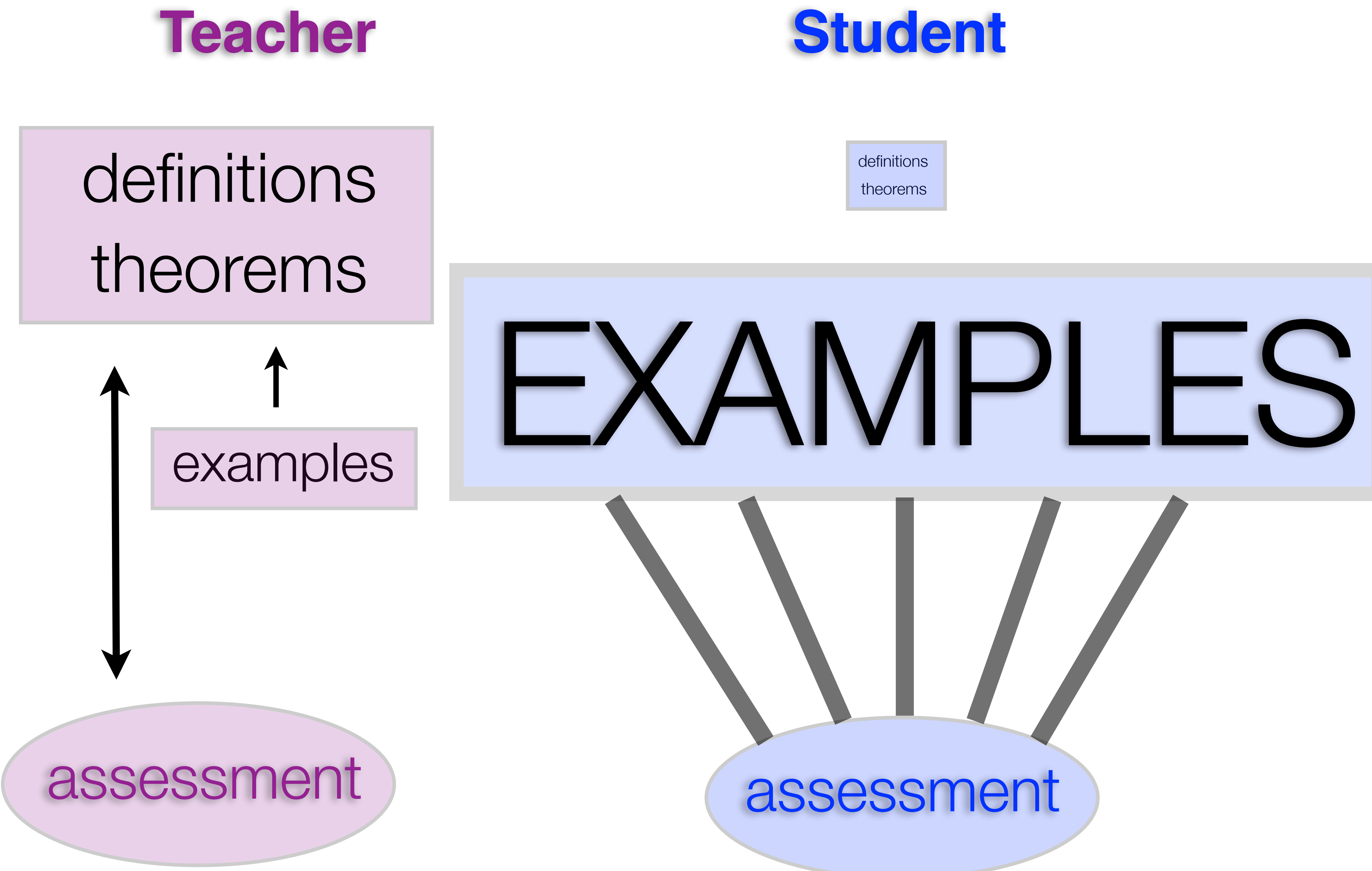

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lack of conceptual accuracy ⟶ inadequate reading
ineffective learning
difficulties with abstraction
difficulties with reasoning
poor writing

The language of processes

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Compute the value of the following expression:

$$\left\{\left(-\frac{2}{3}\right)^2+\left[\left(\frac{1}{5}-\frac{2}{25}\right)\div\left(-\frac{5}{10}+\frac{4}{5}\right)^2-2\right]^3\right\}\times\left(\frac{5}{4}+\frac{5}{8}\right)\div\left(-\frac{5}{3}\right)^2$$

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To complete the task, knowledge of the exact meaning of words and symbols is irrelevant.

The language of concepts: reading symbols

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Chinese:

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力	Power
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$f^{-1}(x)$	the value of an inverse function at a point
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... within a single expression!

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The set of even integers.
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EXAMPLES

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...and words:

By a **triangle** we mean a metric space of cardinality three.

By a **segment** we mean a maximal subpath of *P* that contains only light or only heavy edges.

By a **circle** we mean an affinoid isomorphic to max $\mathbf{C}_p(T,T\text{-}1)$.

Oblivious teaching

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Grammar and syntax should take precedence over semantics.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

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icons

metaphors

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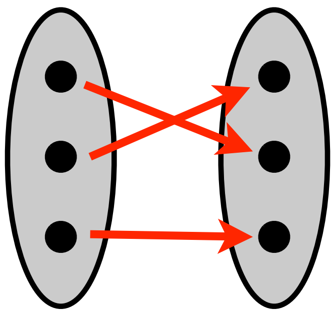
icons



metaphors

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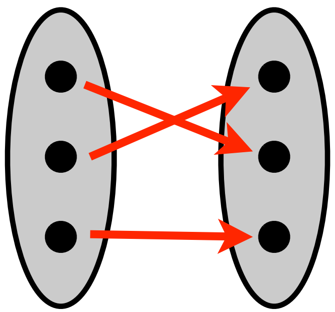

metaphors

We have a group of archers (the elements of the domain), each with one arrow (the function). If all enemies get killed, the function is surjective, if nobody is hit twice, the function is injective.

Injectivity: A function is injective if distinct elements of the domain have distinct images.

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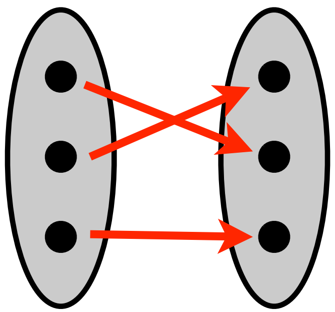

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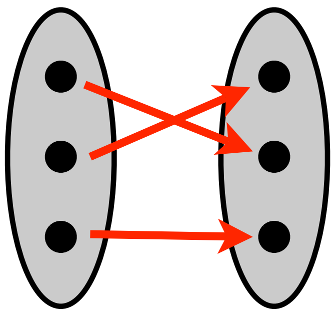

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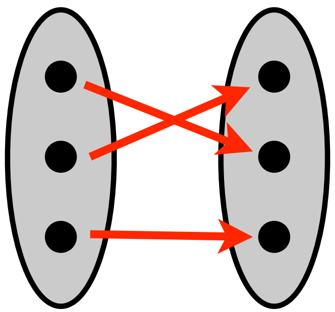

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Change words:

A diet is varied if distinct days of the week have distinct menus.

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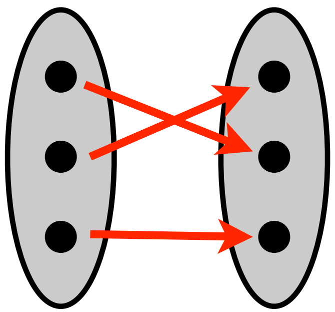

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Introduce symbols:

Let D be a diet and let x and y be two days of the week...

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(empty definition)

The Mathematical Writing course: syllabus

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The Mathematical Writing course: syllabus

- Essential dictionary: translating words into symbols, and vice-versa, with short phrases and sentences.
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- Writing effectively: choosing notation; some techniques; writing a short summary (150 words).
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- Existence and definitions: existence proofs, unique existence.

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Exercise 6. Some of these expressions are grammatically or logically incorrect. Identify them and explain what is the fault. (In what follows, $f : \mathbb{R} \to \mathbb{R}$ is a real function and $A, B, C \subset \mathbb{R}$.)

$\{1+1\}$	$\{3\} \setminus \{\{3\}\}$	$1+1 \Rightarrow 2$
$\{1,2\} \Leftrightarrow \{2,1\}$	$\sqrt{2} \Rightarrow \notin \mathbb{Q}$	$\mathbb{Z} \setminus (\mathbb{Z} \setminus \mathbb{N})$
$\mathbb{Z} \Rightarrow \mathbb{Q}$	$(\lvert x\rvert \in \mathbb{Z}) \Rightarrow (\lvert x\rvert \in \mathbb{Q})$	$(x \in A) \cup (x \in B)$
$(3 < 1) \Rightarrow \emptyset$	$A \leqslant (A \setminus B)$	$f(A) \in \{f(A)\}$
$(A \subset B) \cap C$	$A \subset (B \cap C)$	$A \subset B \subset A$
$(2,4,6,\ldots) \subset (1,2,3,\ldots)$	$\{A, \mathbb{Z}\}$	$\{\emptyset\} \cap \emptyset$
$f(1) \in \{2,3\}$	$f(\{1,2\}) \in \mathbb{N}$	$f(\mathbb{Q}) \subset \mathbb{Q}$
$\{x \in \mathbb{N} : -x\}$	$\{-x : x \in \mathbb{N}\}$	$\{x : x \Leftrightarrow 2\}$
$\{x \in \mathbb{Z} : x \notin \mathbb{Z}\}$	$\{\{x : \lvert x\rvert < 2\}\}$	$\{x \in \mathbb{Q} : 1 = 0\}$
$\{x \in \mathbb{Q} : x^2 \notin \mathbb{Z}\}$	$\{\{f(x)\} : x \in \mathbb{Q}\}$	$\{x : f(x) \in \mathbb{Q}\}$

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

Essential dictionary: from symbols to words

$$(1-x, 1+x^2, 1-x^3, \dots, 1+(-x)^n, \dots)$$

- A sequence.

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$$(1-x, 1+x^2, 1-x^3, \ldots, 1+(-x)^n, \ldots)$$

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put symbols in a context:

with integer coefficients.

with increasing degree.

with bounded coefficients.

. . .

Structure of expressions

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$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$$

the sum of the square of the reciprocal of the natural numbers

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 \quad a_n \in \mathbb{Q}$$

the square of the sum of the elements of a rational sequence,

Exercise 5. For each expression, provide two levels of description: [$\notin$]

i) a coarse description, which only identifies the object's type (set, function, equation, statement, etc.);
ii) a finer description, which defines the object in question or characterises its structure.

1. $x^3 - x - 2$
2. $x^3 - x - 2 = 0$
3. $3^3 + 4^3 + 5^3 = 6^3$
4. $x - y > 0$
5. $x = x + 1$
6. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
7. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
8. $2\mathbb{Z} \supset 4\mathbb{Z}$
9. $(\mathbb{Q} \setminus \mathbb{Z})^2$
10. $(a_1, a_3, a_5, \ldots)$
11. $((x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots)$
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2. $x^3 - x - 2 = 0$ equation
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7. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
8. $2\mathbb{Z} \supset 4\mathbb{Z}$ sentence
9. $(\mathbb{Q} \setminus \mathbb{Z})^2$ set
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12. $\sin \circ \cos$ function

From symbols to words: synthesis

Exercise 8. Let $f : \mathbb{R} \to \mathbb{R}$. Rewrite each symbolic sentence without symbols, apart from f.

1. $f(0) \in \mathbb{Q}$
2. $f(\mathbb{R}) = \mathbb{R}$
3. $\# f(\mathbb{R}) = 1$
4. $f(\mathbb{Z}) = \{0\}$
5. $0 \in f(\mathbb{Z})$
6. $f^{-1}(\{0\}) = \mathbb{Z}$
7. $f(\mathbb{R}) \subset \mathbb{Q}$
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9. $f(\mathbb{Z}) = f(\mathbb{N})$
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The function f vanishes at all integers. [Good]

From words to symbols

Exercise 4. The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers.
2. The set of natural numbers with three decimal digits.
3. The set of rational numbers which are the ratio of odd integers.
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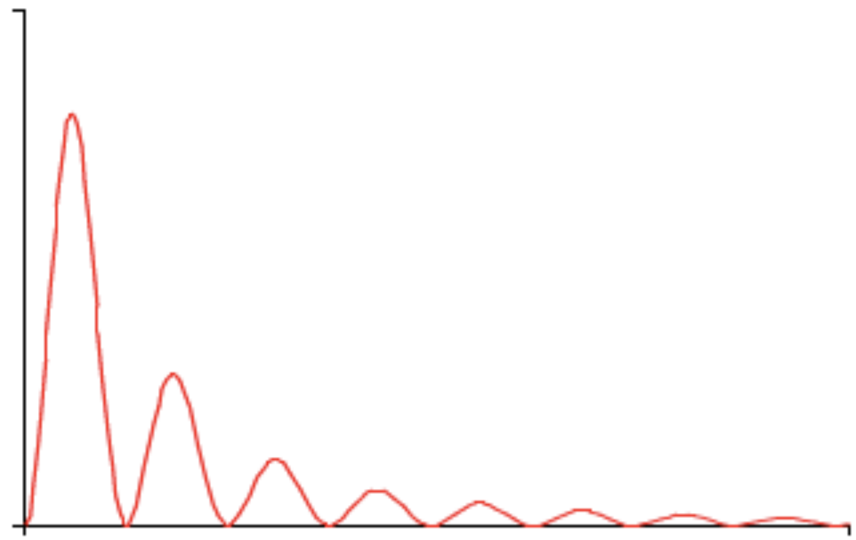
$$\{ax + by = 1 \,:\, a^2 + b^2 = 1\}$$

Describing functions

EXAMPLE. Describe the following function: [¢]



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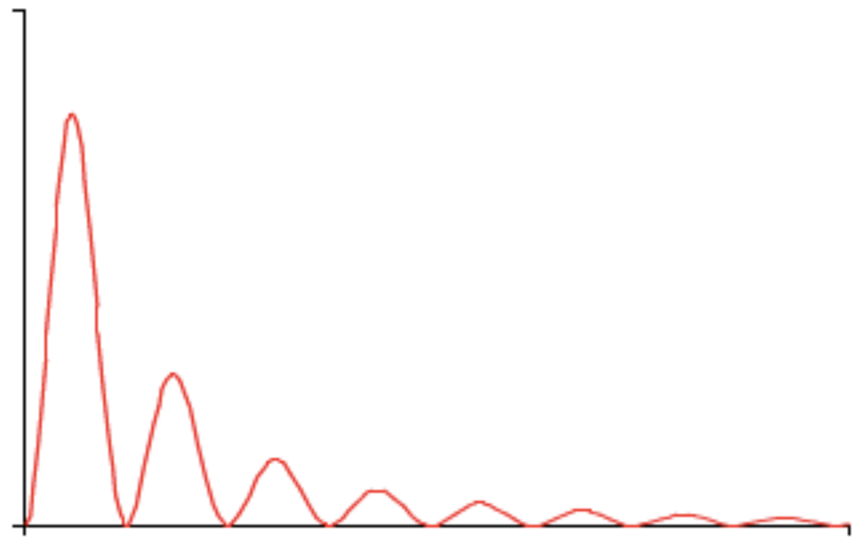

This is a smooth function, which is bounded and non-negative. It features an infinite sequence of evenly spaced local maxima, whose height decreases monotonically to zero. The function has a zero between any two consecutive maxima.

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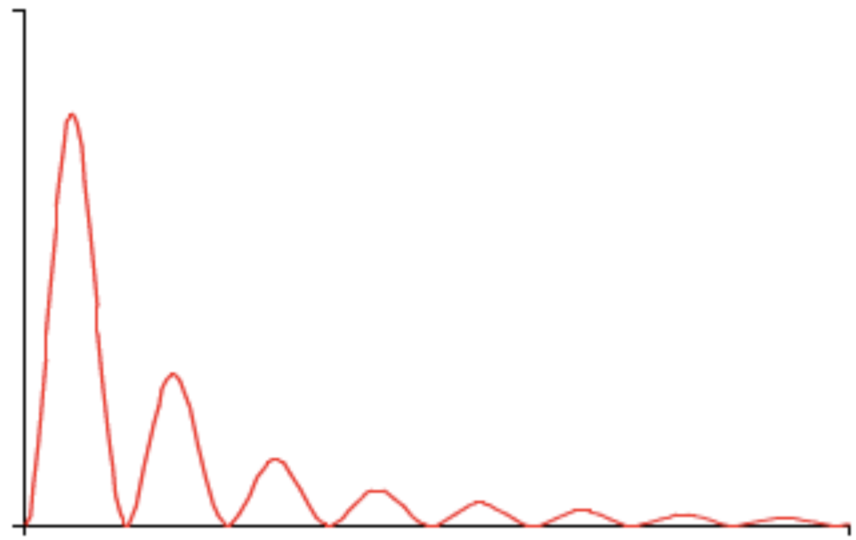

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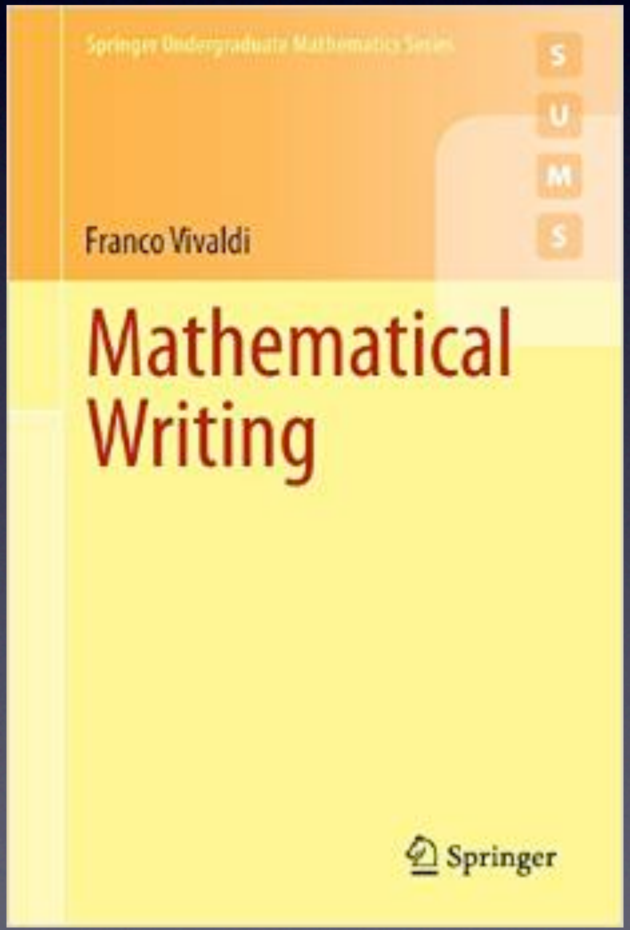
- The development of conceptual accuracy requires small-scale writing exercises (words, symbols, phrases, short sentences).
- One specialised course is insufficient: elements of writing should be embedded in most courses (as in the *Writing in the Disciplines* programme at American universities).
- Universities should develop centrally run schemes to raise the profile of writing and to support departments.

Thank you for your attention



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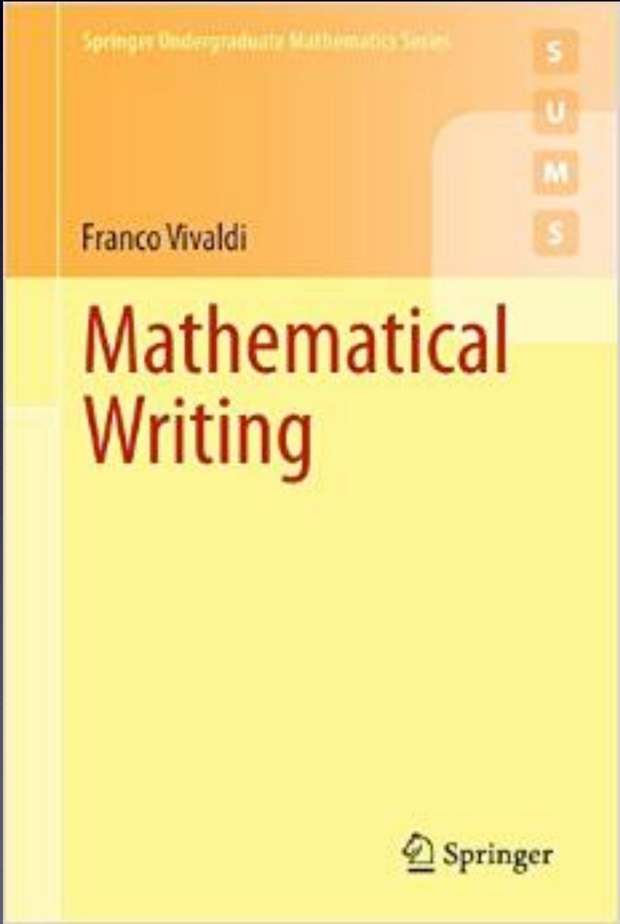
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