NEAR-INTEGRABLE BEHAVIOUR
IN A FAMILY OF DISCRETISED ROTATIONS

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ABSTRACT. We consider a one-parameter family of invertible maps of a two-dimensional lattice, obtained by discretising the space of planar rotations. We let the angle of rotation approach $\pi/2$, and show that the limit of vanishing discretisation is described by an integrable piecewise-smooth Hamiltonian flow, whereby the plane foliates into families of invariant polygons with an increasing number of sides. Considered as perturbations of the flow, the lattice maps assume a different character, described in terms of strip maps, a variant of those found in outer billiards of polygons. The perturbation introduces phenomena reminiscent of the Kolmogorov-Arnold-Moser scenario: a positive fraction of the unperturbed curves survives. We prove this for symmetric orbits, under a condition that allows us to obtain explicit values for their density, the latter being a rational number typically less than 1. This result allows us to conclude that the infimum of the density of all surviving curves —symmetric or not— is bounded away from zero.

1. INTRODUCTION

The study of near-integrable Hamiltonian dynamics on a discrete phase space presents a unique set of problems. On the continuum, the orbits of an integrable symplectic nonlinear map are rotations on invariant tori, which, generically, are quasi-periodic. According to Kolmogorov-Arnold-Moser (KAM) theory, a positive fraction of these tori, identified by their frequency, will survive a sufficiently small smooth perturbation. The complement of KAM tori consists of a hierarchical arrangement of island chains and thin stochastic layers. In low-dimensions, the tori disconnect the space and hence ensure stability.

Reproducing these structures in a discrete space is problematic, due to the lack of a framework for perturbation theory. Typically, quasi-periodic orbits do not exist. Invariant sets acting as surrogate KAM surfaces must thus be identified, and their evolution must be tracked as a perturbation parameter is varied. Even in low dimensions, these invariant sets need not disconnect the space, so their relevance to stability must be re-assessed.

There are various approaches to space discretisation. For an algebraic system it is natural to replace the real or complex coordinate fields by a finite field. Because this procedure erases all topological information, in the discrete phase space there is no near-integrable regime at all, and one witnesses a discontinuous transition from integrable to non-integrable behaviour. This transition manifests itself probabilistically via a (conjectured) abrupt change in the asymptotic (large field) distribution of the periods of the orbits [15, 24].

Round-off in computer arithmetic brings about an equally blunt discretisation of space. Here a small perturbation causes the dynamics to collapse onto a discrete set. In rare cases, round-off fluctuations act like small amplitude noise, and give rise to Gaussian transport; the latter, however, wipes out all small-scale dynamical features. More commonly, the noise model is inappropriate, but there is no general theory to rely on. (Shadowing theory is unhelpful

here because it requires hyperbolicity [16, Section 18.1].) In all, the literature devoted to the study of deterministic (as opposed to probabilistic) manifestation of near-integrability in computer arithmetic is minimal [11, 7, 31] (see also [6]).

A different kind of space discretisation occurs in piecewise isometric systems, as the combined effect of discontinuity and rationality. In isometries involving rational rotations, the space gets tessellated by polygons with algebraic number coordinates. As these polygons move rigidly, the phase space is discrete. Outer billiards of polygons are symplectic maps of this type, which feature a skeletal version of divided phase space (see [26], and references therein). Under appropriate rationality conditions, these systems support a countable family of bounding invariant sets, which prevent orbits from escaping to infinity [29, 17, 12]. These sets, which resemble more a chain of integrable resonant zones than KAM invariants, are uniformly distributed on the plane —this theme will also appear in the present work. The existence of unbounded orbits for some irrational parameters has been a significant recent advance [25] (see also [10]).

In this paper we explore discrete near-integrability in the family of invertible lattice maps

$$(1) \qquad F : \mathbb{Z}^2 \to \mathbb{Z}^2 \qquad (x, y) \mapsto (\lfloor \lambda x \rfloor - y, x) \qquad \lambda = 2\cos(2\pi\nu)$$

where $\lfloor \cdot \rfloor$ is the floor function —the largest integer not exceeding its argument. If we remove the floor function in equation (1), we obtain a one-parameter family of linear maps of the plane, $(x, y) \mapsto (\lambda x - y, x)$, which are linearly conjugate to a rotation by the angle $2\pi\nu$, where ν is the rotation number. The floor function provides the discretisation, rounding the image point $(\lambda x - y, x)$ to the nearest lattice point on the left. An essential property of this model is its invertibility, which a typical round-off scheme applied to a rotation does not have. Furthermore, rounding by the floor function —as opposed to the nearest integer— is arithmetically nicer and has fewer symmetries. In the lattice map F, the discretisation length is fixed, and the limit of vanishing discretisation corresponds to motions at infinity. This asymptotic regime is our main interest.

The deceptively simple model (1) displays a rich landscape of mathematical phenomena, connecting discrete dynamics and arithmetic. This model originated in dynamical system theory [27, 19, 20, 21, 8, 30, 18], and was subsequently studied in number theory, within the context of shift radix systems [1, 2]. The following unsolved question [1, 28] distills the difficulties encountered in the analysis of this model.[1]

Conjecture. *For all real λ with $|\lambda| < 2$, all orbits of F are periodic.*

Due to invertibility, periodicity is equivalent to boundedness. This conjecture holds trivially for $\lambda = 0, \pm 1$, where the map F is of finite order. Beyond this, the boundedness of all round-off orbits has been proved for only *eight* values of λ, which correspond to the rational values of the rotation number ν for which λ is a quadratic irrational:

$$(2) \qquad\qquad \lambda = \frac{\pm 1 \pm \sqrt{5}}{2}, \quad \pm\sqrt{2}, \quad \pm\sqrt{3}.$$

(The denominator of ν is 5, 10, 8, and 12, respectively.) In these cases the map F admits a dense and uniform embedding in a two-dimensional torus, where the round-off map extends continuously to a piecewise isometry, which has *zero entropy* (and is not ergodic). The natural density on the lattice $\mathbb{Z}^2$ is carried into the Lebesgue measure, namely the Haar measure on

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[1]A related general conjecture on the boundedness of discretised Hamiltonian rotations was first formulated in [5].

the torus. The case $\lambda = (1 - \sqrt{5})/2$ was established in [19], with computer assistance. Similar techniques were used to extend the result to the other parameter values, but only for a set of initial conditions having full density [18]. The conjecture for the eight parameters (2) was settled in [2] with an analytical proof. For any other rational value of ν, there is a similar embedding in a piecewise isometry of a higher-dimensional torus; these systems are still unexplored, even in the cubic case.

Irrational values of ν bring about a different dynamics, and a different theory. The simplest cases correspond to rational values of λ, and, in particular, to rational numbers whose denominator is the power of a prime p. In this case the map F admits a dense and uniform embedding in the ring $\mathbb{Z}_p$ of p-adic integers [8]. The embedded system extends continuously to the composition of a full shift and an isometry (which has *positive entropy*), and the natural density in $\mathbb{Z}^2$ is now carried into the Haar measure on $\mathbb{Z}_p$. This construct was later used to prove a central limit theorem for the departure of the round-off orbits from the unperturbed ones [30]. This phenomenon injects a probabilistic element in the determination of the period of the lattice orbits, highlighting the nature of the difficulties that surround conjecture 1. Very recently, Akiyama and Pethő [3] proved that (1) has infinitely many periodic orbits for any parameter value.

In this work we consider a new regime, namely the limit $\lambda \to 0$ of equation (1), corresponding to the rotation number $\nu \to 1/4$. This is one of five limits (the other limits being $\lambda \to \pm 1, \pm 2$) where the dynamics at the limit is trivial because there is no round-off. After scaling, we embed the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$, and show that there is a non-smooth integrable Hamiltonian *flow* (not a rotation), which represents the limiting unperturbed dynamics. This integrable system is non-linear, namely its time-advance map satisfies a twist condition. Thus the limit $\lambda \to 0$ is singular. The parameter λ acts as a perturbation parameter, and a discrete version of near-integrable symplectic dynamics emerges on the lattice when the perturbation is switched on.
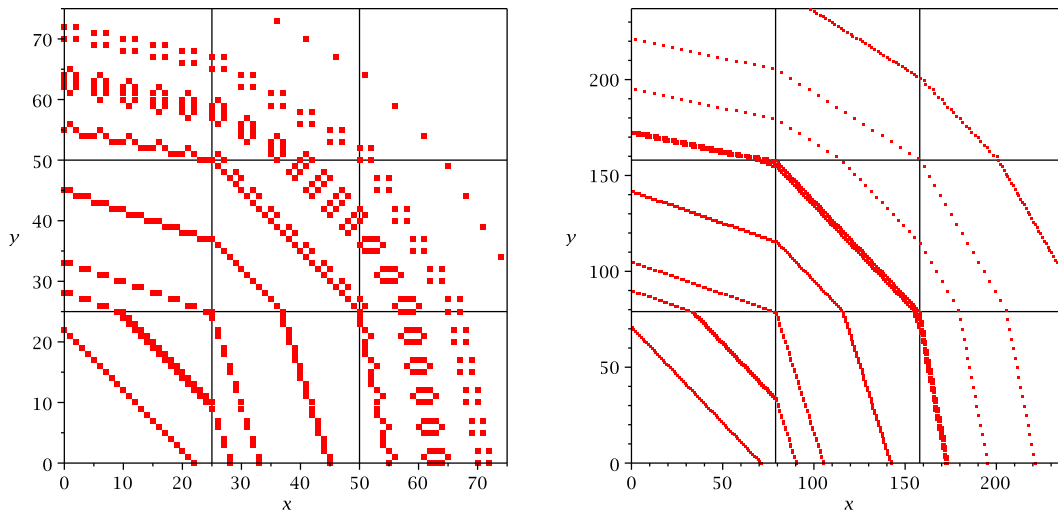

FIGURE 1. Some periodic orbits of the map F, for small values of the parameter λ. Left: $\lambda = 1/25$; right: $\lambda = 1/79$. The boxes have side λ^{-1}.

More precisely, if λ is small, then the orbits lie approximately on convex polygons; the smaller λ, the closer the approximation (figure 1). The number of sides of these polygons

increases with the distance from the origin; near the origin they are squares, while at infinity they approach circles.

In the spirit of Takens' theorem [4, section 6.2.2], we introduce a piecewise smooth integrable Hamiltonian function (equation (11)), whose invariant curves are polygons, representing the limit foliation of the plane for the system (1). To match Hamiltonian flow and lattice map, we exploit the fact that, for small λ, the composite map F^4 is close to the identity. After scaling, it is possible to identify the action of F^4 with the unit time-advance map of the flow. The two actions agree along the sides of the polygons, but they differ in vanishingly small regions near the vertices. This discrepancy provides the perturbation mechanism.

All integrable orbits must close after one revolution around the origin. By contrast, the lattice orbits need not do so, leading to a non-trivial clustering of the periods around integer multiples of a basic rotational period —see figure 2. The lowest branch of the period function
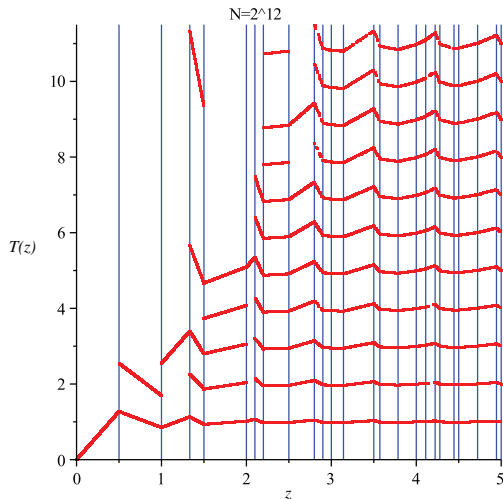

FIGURE 2. The normalised period function $T_\lambda(z)$ for points $z = (x, x)$, and $\lambda = 2^{-12}$. The vertical lines mark the location of critical polygons, which pass through lattice points.

comprises the *minimal orbits,* which shadow the unperturbed orbits, and close after one revolution. For these orbits, the effects of the perturbation cancel out. (The role of cancellation of singularities in the existence of invariant tori of non-smooth systems was noted long ago [14].) These are the orbits of the integrable system that survive the perturbation. The other orbits mimic the divided phase space structure of a near-integrable area-preserving map, in embryonic form near the origin, and increasing in complexity for larger amplitudes.

The set of invariant polygons is partitioned by *critical polygons*, which contain $\mathbb{Z}^2$ points, into infinitely many *polygon classes*, which can be characterised arithmetically in terms of sums of squares. The main result of this work is that for infinitely many classes, a positive fraction of the minimal orbits having time-reversal symmetry survive. The restriction to infinitely many classes —as opposed to all classes— stems from a coprimality condition we impose in order to achieve convergence of the density. This density turns out to be a rational number smaller than unity, which depends only on the family of polygons being considered. As the number of sides of the polygons increases to infinity, the density tends to zero. As

a corollary, we obtain a positive rational lower bound for the density of all minimal orbits —symmetric or not. These results appear as theorems A and B, stated in section 5 after a somewhat lengthy preparation.

Our analysis is based on the study of the first-return map to a thin strip placed along the symmetry axis of the map (1). The minimal orbits are fixed points of this map, and we study those having time reversal symmetry. The analysis of the return map requires tracking the return orbits, and this is done through repeated applications of a *strip map*, an acceleration device which exploits local integrability. This is a variant of a construct introduced for outer billiards of polygons (see [26, chapter 7], and references therein), although in our case the strip map has an increasing number of components, providing a dynamics of increasing complexity. There is a symbolic dynamics associated with the strip map; its cylinder sets in the return domain are congruence classes modulo a nested sequence of two-dimensional lattices. The key result is that, within a polygon class, this lattice structure becomes independent of λ, provided that λ is small enough. This fact gives a 'non archimedean' character to the dynamics; the rationality of the density of minimal orbits, and their locally uniform distribution then follow.

The plan of this paper is the following. In section 2 we introduce the integrable Hamiltonian, characterise its invariant curves via a symbolic dynamics, and connect them to the arithmetical problem of sums of two squares (theorem 2). In section 3 we switch on the round-off perturbation, and show that all orbits recur to a small neighbourhood of the symmetry axis. Accordingly, we construct a return map of this neighbourhood, and show that the return orbits shadow the integrable orbits (theorem 5). Most proofs for this section are postponed to section 4. Matching the symbolic dynamics of integrable and perturbed orbits is more delicate, requiring the exclusion of certain anomalous domains, and establishing that the size of these domains is negligible in the limit. This is done in section 5, where we also state the main results of this work, theorems A and B. The first theorem states that, for all sufficiently small λ, the return map commutes with translations by the elements of a two-dimensional lattice, which depends only on the polygonal class being considered. The second theorem states that, if the symbolic dynamics of a polygonal class satisfies certain coprimality conditions, then, as $\lambda \to 0$ the density of symmetric minimal orbits among all symmetric orbits converges to a positive rational number, which is computed explicitly. An immediate corollary of theorem B is the existence of a positive rational lower bound for the density of minimal orbits —symmetric or otherwise— among all orbits (corollary 8). At the end of section 5 we also briefly discuss some experiments on the conditions of the statement of theorem B. In section 6 we introduce the strip map, and establish some of its properties (propositions 9 and 10). In the final section we demonstrate the link between the symbolic dynamics of the perturbed orbits and the aforementioned group of lattice translations. This result leads to the conclusion of the proof of the main theorems.

There are many issues we haven't considered. The nature of the phase portrait at infinity, stability, the role played by non-symmetric orbits, the distribution of periods among the branches of the period function. More generally, one may consider the construct of strip maps for the purpose of developing a Hamiltonian perturbation theory over discrete spaces.

These questions deserve further investigation.

2. The integrable limit

Figure 1 suggests that the analysis of the limit $\lambda \to 0$ requires some scaling; equation (1) suggests that the quantity to be held constant should be λx. Accordingly, we normalise the metric by introducing the **scaled lattice map** F_λ, which is conjugate to F, and acts on points $z = \lambda(x, y)$ of the scaled lattice $(\lambda\mathbb{Z})^2$:

$$F_\lambda : (\lambda\mathbb{Z})^2 \to (\lambda\mathbb{Z})^2 \qquad F_\lambda(z) = \lambda F(z/\lambda).$$

The discretisation length of F_λ is λ. Then we define the **discrete vector field**, which measures the deviation of F_λ^4 from the identity:

$$(3) \qquad \mathbf{v} : (\lambda\mathbb{Z})^2 \to (\lambda\mathbb{Z})^2 \qquad \mathbf{v}(z) = F_\lambda^4(z) - z.$$

To capture the main features of $\mathbf{v}$ on the scaled lattice, we introduce an **auxiliary vector field** $\mathbf{w}$ on the plane, given by

$$(4) \qquad \mathbf{w} : \mathbb{R}^2 \to (\lambda\mathbb{Z})^2 \qquad \mathbf{w}(x, y) = \lambda(2\lfloor y \rfloor + 1, -(2\lfloor x \rfloor + 1)).$$

The field $\mathbf{w}$ is constant on every translated unit square (called a *box*)

$$(5) \qquad B_{m,n} = \{(x, y) \in \mathbb{R}^2 : \lfloor x \rfloor = m, \lfloor y \rfloor = n\}, \quad m, n \in \mathbb{Z}$$

and we denote the value of $\mathbf{w}$ on $B_{m,n}$ as

$$(6) \qquad \mathbf{w}_{m,n} = \lambda(2n + 1, -(2m + 1)).$$

The following proposition, whose proof we defer until section 4, states that if we ignore a set of points of zero density, then the functions $\mathbf{v}$ and $\mathbf{w}$ agree on the lattice $(\lambda\mathbb{Z})^2$.

Proposition 1. *Let r be a positive real number. We define the set*

$$(7) \qquad A(r, \lambda) = \{z \in (\lambda\mathbb{Z})^2 : \|z\|_\infty < r\},$$

(with $\|(u, v)\|_\infty = \max(|u|, |v|)$), and the ratio

$$\mu_1(r, \lambda) = \frac{\#\{z \in A(r, \lambda) : \mathbf{v}(z) = \mathbf{w}(z)\}}{\#A(r, \lambda)}.$$

Then we have

$$\lim_{\lambda \to 0} \mu_1(r, \lambda) = 1.$$

The asymptotic regime that results from replacing $\mathbf{v}$ by $\mathbf{w}$ will be referred to as the **integrable limit** of the system (1), as $\lambda \to 0$. The points where the two vector fields differ have the property that λx or λy is close to an integer. The perturbation of the integrable orbits will take place in these small domains.

2.1. The integrable Hamiltonian.
We define the real function

$$(8) \qquad P : \mathbb{R} \to \mathbb{R} \qquad P(x) = \lfloor x \rfloor^2 + (2\lfloor x \rfloor + 1)\{x\}$$

where $\{x\}$ denotes the fractional part of x. The function P is piecewise affine, and coincides with the function $x \mapsto x^2$ on the integers. Thus:

$$(9) \qquad P(\lfloor x \rfloor) = \lfloor x \rfloor^2 \qquad \lfloor \sqrt{P(x)} \rfloor = \lfloor x \rfloor.$$

Using this fact, we can invert P up to sign by defining

$$(10) \qquad P^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \qquad P^{-1}(x) = \frac{x + \lfloor \sqrt{x} \rfloor(1 + \lfloor \sqrt{x} \rfloor)}{2\lfloor \sqrt{x} \rfloor + 1},$$

so that $(P^{-1} \circ P)(x) = |x|$.

We define the following Hamiltonian

$$(11) \qquad \mathscr{P} : \mathbb{R}^2 \to \mathbb{R} \qquad \mathscr{P}(x, y) = P(x) + P(y).$$

The function $\mathscr{P}$ is continuous and piecewise affine. It is differentiable in $\mathbb{R}^2 \setminus \Delta$, where Δ is a set of orthogonal lines given by

$$(12) \qquad \Delta = \{(x, y) \in \mathbb{R}^2 \ : \ (x - \lfloor x \rfloor)(y - \lfloor y \rfloor) = 0\}.$$

The set Δ is the boundary of the boxes $B_{m,n}$, defined in (5). The associated (scaled) Hamiltonian vector field, defined for all points $(x, y) \in \mathbb{R}^2 \setminus \Delta$, is parallel to the vector field $\mathbf{w}$ given in (4):

$$(13) \qquad \lambda \left(\frac{\partial \mathscr{P}(x, y)}{\partial y}, -\frac{\partial \mathscr{P}(x, y)}{\partial x} \right) = \mathbf{w}(x, y) \qquad (x, y) \in \mathbb{R}^2 \setminus \Delta.$$

The parameter λ merely rescales the time.

For a point $z \in \mathbb{R}^2$, we write $\Pi(z)$ for the level set of $\mathscr{P}$ passing through z:

$$\Pi(z) = \{w \in \mathbb{R}^2 \ : \ \mathscr{P}(w) = \mathscr{P}(z)\}.$$

Below (theorem 2) we shall see that these sets are polygons. The **value** of a polygon $\Pi(z)$ is the real number $\mathscr{P}(z)$, and if $\Pi(z)$ contains a lattice point, then we speak of a **critical polygon**. The critical polygons form a distinguished subset of the plane:

$$\Gamma := \bigcup_{z \in \mathbb{Z}^2} \Pi(z).$$

All topological information concerning the Hamiltonian $\mathscr{P}$ is encoded in the partition of the plane generated by $\Gamma \cup \Delta$. The elements of Γ act as separatrices, whose vertices belong to Δ.

To characterise $\mathscr{P}$ arithmetically, we consider the Hamiltonian

$$\mathscr{Q}(x, y) = x^2 + y^2$$

which represents the unperturbed rotations (no round-off) in the limit $\lambda \to 0$. Its level sets are circles, and the circles containing lattice points will be called **critical circles**. By construction, the functions $\mathscr{P}$ and $\mathscr{Q}$ coincide over $\mathbb{Z}^2$, and hence the value of every critical polygon belongs to $\mathscr{Q}(\mathbb{Z}^2)$, the set of non-negative integers which are representable as the sum of two squares. We denote this set by $\mathscr{E}$.

A classical result, due to Fermat and Euler, states that a natural number n is a sum of two squares if and only if any prime congruent to 3 modulo 4 which divides n occurs with an even exponent in the prime factorisation of n [13, theorem 366]). We refer to $\mathscr{E}$ as the set of **critical numbers**, and use the notation

$$\mathscr{E} = \{e_i : i \geq 0\} = \{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, \dots\}.$$

There is an associated family of **critical intervals**, defined as

$$(14) \qquad I^{e_i} = (e_i, e_{i+1}).$$

Let us define

$$\mathscr{E}(x) = \#\{e \in \mathscr{E} \ : \ e \leq x\}.$$

The following result, due to Landau and Ramanujan, gives the asymptotic behaviour of $\mathscr{E}(x)$ (see, e.g., [22])

$$(15) \qquad \lim_{x \to \infty} \frac{\sqrt{\ln x}}{x} \mathscr{E}(x) = K,$$

where K is the Landau-Ramanujan constant

$$K = \frac{1}{\sqrt{2}} \prod_{\substack{p \text{ prime} \\ p \equiv 3 \,(\text{mod } 4)}} \left(1 - \frac{1}{p^2}\right)^{-1/2} = 0.764\ldots.$$

Furthermore, let $r(n)$ be the number of representations of the integer n as a sum of two squares. To compute $r(n)$, we first factor n as follows

$$n = 2^a \prod p^b \prod q^c$$

where p and q are primes congruent to 1 and 3 modulo 4, respectively. (Each product is equal to 1 if there are no prime divisors of the corresponding type.) Then we have [13, theorem 278]

$$(16) \qquad r(n) = 4 \prod (b+1) \prod \left(\frac{1 + (-1)^c}{2}\right).$$

Note that this product is zero whenever n is not a critical number, i.e., $r(n) = 0$ if $n \notin \mathscr{E}$.

We now have the following characterisation of the invariant curves of the Hamiltonian $\mathscr{P}$.

Theorem 2. *The level sets $\Pi(z)$ of $\mathscr{P}$ are convex polygons, invariant under the dihedral group D_4, generated by the two orientation-reversing involutions*

$$(17) \qquad G: \quad (x,y) \mapsto (y,x) \qquad G': \quad (x,y) \mapsto (x,-y).$$

The polygon $\Pi(z)$ is critical if and only if $\mathscr{P}(z) \in \mathscr{E}$. The number of sides of $\Pi(z)$ is equal to

$$(18) \qquad 4\left(2 \left\lfloor \sqrt{\mathscr{P}(z)} \right\rfloor + 1\right) - r(\mathscr{P}(z))$$

where the function r is given in (16). For every $e \in \mathscr{E}$, the critical polygon with value e intersects one and only one critical circle, namely that with the same value. The intersection consists of $r(e)$ lattice points, and the polygon lies inside the circle.

PROOF. The symmetry properties follow from the fact that the Hamiltonian $\mathscr{P}$ is invariant under the interchange of its arguments, and the function P is even:

$$\begin{aligned}
P(-x) &= \lfloor -x \rfloor^2 + \{-x\}(2\lfloor -x \rfloor + 1) \\
&= \begin{cases} (-\lfloor x \rfloor - 1)^2 - (1 - \{x\})(2\lfloor x \rfloor + 1) & x \notin \mathbb{Z} \\ (-\lfloor x \rfloor)^2 & x \in \mathbb{Z} \end{cases} \\
&= \lfloor x \rfloor^2 + \{x\}(2\lfloor x \rfloor + 1) = P(x).
\end{aligned}$$

The vector field (13) is piecewise-constant, and equal to $\mathbf{w}_{m,n}$ in the box $B_{m,n}$ (cf. equations (5) and (6)). Hence a level set $\Pi(z)$ is the union of segments. It is easy to verify that no three segments can have an end-point in common (considering end-points in the first octant will suffice). Thus $\Pi(z)$ is a polygonal curve. Equally, segments cannot intersect inside boxes, because they are parallel there. But a non self-intersecting symmetric polygonal must be a polygon.

Next we prove convexity. Due to dihedral symmetry, if $\Pi(z)$ is convex within the open first octant $0 < y < x$, then it is piecewise convex. Thus we suppose that $\Pi(z)$ has an edge in the box $B_{m,n}$, where $0 < n \leq m$. The adjacent edge in the direction of the flow must be in one of the boxes

$$B_{m,n-1}, \quad B_{m+1,n-1}, \quad B_{m+1,n}.$$

Using (6) one verifies that the three determinants

$$\det(\mathbf{w}_{m,n}, \mathbf{w}_{m,n-1}) \qquad \det(\mathbf{w}_{m,n}, \mathbf{w}_{m+1,n-1}) \qquad \det(\mathbf{w}_{m,n}, \mathbf{w}_{m+1,n})$$

are negative. This means that, in each case, at the boundary between adjacent boxes, the integral curve turns clockwise. So $\Pi(z)$ is piecewise convex. It remains to prove that convexity is preserved across the boundaries of the first octant, which belong to the fixed sets $\mathrm{Fix}\,G$ (the line $x = y$) and $\mathrm{Fix}\,G'$ (the line $y = 0$) of the involutions (17). Indeed, $\Pi(z)$ is either orthogonal to $\mathrm{Fix}\,G$ (in which case convexity is clearly preserved), or has a vertex (m,m) on it; in the latter case, the relevant determinant is $\det(\mathbf{w}_{m-1,m}, \mathbf{w}_{m,m-1}) = -8m < 0$. The preservation of convexity across $\mathrm{Fix}\,G'$ is proved similarly, and thus $\Pi(z)$ is convex.

The statement on the criticality of $\mathscr{P}(z)$ follows from the fact that, on $\mathbb{Z}^2$, we have $\mathscr{P} = \mathscr{Q}$.

Consider now the edges of $\Pi(z)$. The intersections of $\Pi(z)$ with the x-axis have abscissas $\pm P^{-1}(\mathscr{P}(z))$. Using (9) we have that there are $2\lfloor\sqrt{\mathscr{P}(z)}\rfloor + 1$ integer points between them, hence as many lines orthogonal to the x-axis with integer abscissa. The same holds for the y-axis. If $\Pi(z)$ is non-critical, it follows that $\Pi(z)$ intersects Δ in exactly $4(2\lfloor\sqrt{\mathscr{P}(z)}\rfloor + 1)$ points, each line being intersected twice. Because the vector field changes across each line, the polygon has $4(2\lfloor\sqrt{\mathscr{P}(z)}\rfloor + 1)$ vertices. If the polygon is critical, then we have $\mathscr{P}(z) = e \in \mathscr{E}$. At each of the $r(e)$ vertices that belong to $\mathbb{Z}^2$, two lines in Δ intersect, resulting in one fewer vertex. So $r(e)$ vertices must be removed from the count.

Next we deal with intersections of critical curves. Let us consider two arbitrary critical curves

$$\mathscr{P}(x,y) = e \qquad \mathscr{Q}(x,y) = e + f \qquad e, e + f \in \mathscr{E}.$$

This system of equations yields

$$\{x\}^2 + \{y\}^2 - \{x\} - \{y\} = f$$

which is a circle with centre at $(1/2, 1/2)$, and radius ρ, where

(19)
$$\rho^2 = f + \frac{1}{2}.$$

Since we must have $0 \leq \{x\}, \{y\} < 1$, we find $\rho^2 \leq 1/2$, and since f is an integer, we obtain $\{x\} = \{y\} = f = 0$. So critical polygons and circles intersect only if they have the same value, and their intersection consists of lattice points. Then the number of these lattice points is necessarily equal to $r(e)$.

Finally, let the lattice point (m,n) belong to the intersection of two critical curves, and let $\mathbf{w}'_{m,n}$ be the vector field of the Hamiltonian $\mathscr{Q}$ at that point. Without loss of generality, we assume that (m,n) lies within the first octant. If $n > 0$, then the vector field of $\mathscr{P}$ before and after the vertex in the direction of the flow is equal to $\mathbf{w}_{m-1,n}$ and $\mathbf{w}_{m,n-1}$, respectively. One verifies that the determinants

$$\det(\mathbf{w}_{m-1,n}, \mathbf{w}'_{m,n}) \qquad \det(\mathbf{w}'_{m,n}, \mathbf{w}_{m,n-1})$$

are negative. This means that, near this vertex, the polygon lies inside the circle. If $n = 0$, the field after the vertex is $\mathbf{w}_{m-1,-1}$, and the same result holds.

The proof is complete. □

From this theorem it follows that the set Γ of critical polygons partitions the plane into concentric domains, which we call **polygon classes**. Each domain contains a single critical circle, and has no lattice points in its interior. The values of all the polygons in a class is a critical interval (14). There is a dual arrangement for critical circles. Because counting critical polygons is the same as counting critical circles, the number of critical polygons (or, equivalently, of polygon classes) contained in a circle of radius $\sqrt{x}$ is equal to $\mathscr{E}(x)$, with asymptotic formula (15). From equation (19), one can show that the total variation $\Delta\mathscr{Q}(a)$ of $\mathscr{Q}$ along the polygon $\mathscr{P}(z) = a$ satisfies the bound

$$\Delta\mathscr{Q}(a) \leq \frac{1}{2}$$

which is strict (e.g., for $a = 1$).

2.2. **Symbolic dynamics of polygon classes.** In theorem 2 we classified the invariant curves of the Hamiltonian $\mathscr{P}$ in terms of critical numbers. We found that the set Γ of critical polygons partitions the plane into concentric annular domains —the polygon classes. In this section we define a symbolic dynamics on the set of classes, which specifies the common itinerary of all orbits in a class, taken with respect to the lattice $\mathbb{Z}^2$.

Suppose that the polygon $\Pi(z)$ is non-critical. Then all vertices of $\Pi(z)$ belong to $\Delta \setminus \mathbb{Z}^2$, where Δ was defined in (12). Let ξ be a vertex. Then ξ has one integer and one non-integer coordinate, and we let u be the value of the non-integer coordinate. We say that the vertex ξ is of **type** v if $\lfloor |u| \rfloor = v$. Then we write v_j for the type of the jth vertex, where the vertices of $\Pi(z)$ are enumerated according to their position in the plane, starting from the positive half of symmetry line $\operatorname{Fix} G$ and proceeding clockwise.

As the type of a vertex is defined using the modulus of the non-integer coordinate, the sequence of vertex types v_j reflects the eight-fold symmetry of $\Pi(z)$. Hence if the kth vertex lies on the x-axis, then there are $2k-1$ vertices belonging to each quarter-turn, and the vertex types satisfy

(20) $$v_j = v_{2k-j} = v_{(2k-1)i+j}, \qquad 1 \leq j \leq k, \quad 0 \leq i \leq 3.$$

Thus it suffices to consider the vertices in the first octant, and the **vertex list** of $\Pi(z)$ is the sequence of vertex types

$$V = (v_1, \ldots, v_k).$$

We note that the vertex list can be decomposed into two disjoint subsequences; those entries belonging to a vertex with integer x-coordinate and those belonging to a vertex with integer y-coordinate. These subsequences are non-decreasing and non-increasing, respectively.

From theorem 2, it follows that for every $e \in \mathscr{E}$, the set of polygons $\Pi(z)$ with $\mathscr{P}(z) \in I^e$ have the same vertex list. Let k be the number of entries in the vertex list. Since the polygon $\Pi(z)$ is non-critical, equation (18) gives us that $4(2\lfloor\sqrt{e}\rfloor + 1) = 4(2k - 1)$, and hence

$$k = \#V = \lfloor\sqrt{e}\rfloor + 1.$$

Any two polygons with the same vertex list have not only the same number of edges, but intersect the same collection of boxes, and have the same collection of tangent vectors. The critical polygons which intersect the lattice $\mathbb{Z}^2$, where the vertex list is multiply defined, form the boundaries between classes. The symbolic dynamics of these polygons is ambiguous, but this item will not be required in our analysis.

Thus the vertex list is a function on classes, hence on $\mathscr{E}$. For example, the polygon class identified with the interval $I^9 = (9, 10)$ (see figure 3) has vertex list

$$V(9) = (2, 2, 0, 3).$$

See figure 4 for further examples of V(e). For each class, there are two vertex types which we can calculate explicitly; the first and the last. If $\mathscr{P}(z) \in I^e$, and the polygon $\Pi(z)$ intersects the symmetry line $\mathrm{Fix}\, G$ at some point $(x, x) \in \mathbb{R}^2$, then by the definition (11) of the Hamiltonian $\mathscr{P}$

$$\mathscr{P}(z) = 2P(x).$$

Thus inverting P and using (9), it is straightforward to show that the first vertex type is given by

(21) $$v_1 = \lfloor |x| \rfloor = \lfloor P^{-1}(\mathscr{P}(z)/2) \rfloor = \lfloor \sqrt{e/2} \rfloor.$$

Similarly the last vertex type, corresponding to the vertex on the x-axis is given by

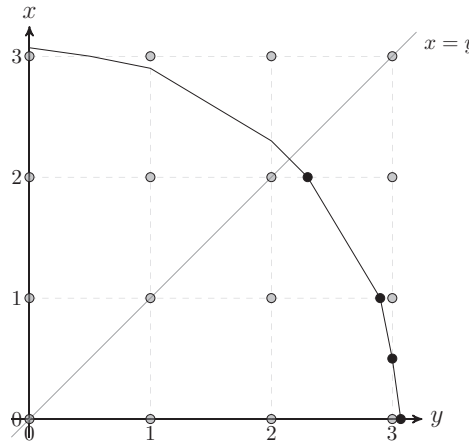(22) $$v_k = \lfloor P^{-1}(\mathscr{P}(z)) \rfloor = \lfloor \sqrt{e} \rfloor.$$

FIGURE 3. A polygon with $\mathscr{P}(z)$ in the interval $(9, 10)$ and its vertices in the first octant.

e	V(e)
9	$(2, 2, 0, 3)$
10	$(2, 1, 3, 3)$
18	$(3, 3, 1, 4, 4)$
29	$(3, 4, 2, 5, 5, 5)$
49	$(4, 5, 3, 6, 6, 6, 0, 7)$
52	$(5, 4, 6, 6, 6, 1, 7, 7)$

FIGURE 4. A table showing the vertex list $V(e)$ for a selection of critical numbers e. Notice that the last entry in the vertex list is always $\lfloor \sqrt{e} \rfloor$, and the number of entries in the list is $k = \lfloor \sqrt{e} \rfloor + 1$.

3. Recurrence and return map

(The proofs of all statements in this section are deferred until section 4.)

The lattice map F has fewer symmetries than the Hamiltonian $\mathscr{P}$, but it is easy to verify that F is still reversible, being conjugate to its inverse via the involution G given in equation (17):

$$F^{-1} = G \circ F \circ G \qquad\qquad G^2 = \mathrm{Id}.$$

The scaled map F_λ has the same property, and all orbits of F_λ return repeatedly to a neighbourhood of the symmetry line $\mathrm{Fix}\,G$. In this section we consider the return map associated to this recurrence, and identify some asymptotic properties of the first-return orbits.

From equation (1), the rotation number ν has the asymptotic form

$$\nu = \frac{1}{2\pi}\arccos\left(\frac{\lambda}{2}\right) = \frac{1}{4} - \frac{\lambda}{4\pi} + O(\lambda^3) \qquad\qquad \lambda \to 0.$$

The integer $t = 4$ is the **zeroth-order recurrence time** of orbits under F_λ, that is, the number of iterations needed for a point to return to an $O(\lambda)$-neighbourhood of its starting point. It turns out (see proof of proposition 1) that the field $\mathbf{v}(z)$ (equation (3)) is non-zero for all non-zero points z, so no orbit has period four. Accordingly, for small $\lambda > 0$, we define the **first-order recurrence time** t^* of the rotation to be the next time of closest approach:

$$(23) \qquad t^*(\lambda) = \inf\left\{k \in \mathbb{N} \;:\; d_H(k\nu, \mathbb{N}) < d_H(4\nu, \mathbb{N})\right\} = \frac{\pi}{\lambda} + O(1) \qquad\qquad \lambda \to 0,$$

where d_H is the Hausdorff distance, and the expression $d_H(x, A)$, with $x \in \mathbb{R}$, is to be understood as the Hausdorff distance between the sets $\{x\}$ and A. (Throughout this paper we use $\mathbb{N}$ to denote the set of positive integers.)

The integer t^* provides a natural recurrence timescale for F_λ. Let $T(z)$ be the minimal period of the orbit under F of the point $z \in \mathbb{Z}^2$, so that $T(z/\lambda)$ is the corresponding function for points $z \in (\lambda\mathbb{Z})^2$ under F_λ. (In accordance with the periodicity conjecture, we assume that this function is well-defined.) Since, as $\lambda \to 0$, the recurrence time t^* diverges, the periods of the orbits will cluster around integer multiples of t^*, giving rise to branches of the period function. The lowest branch corresponds to orbits that perform a single revolution around the origin, and their period is approximately equal to t^*. The period function T has a normalised counterpart, given by (cf. (23))

$$T_\lambda : (\lambda\mathbb{Z})^2 \to \frac{\lambda}{\pi}\mathbb{N} \qquad\qquad T_\lambda(z) = \frac{\lambda}{\pi}T(z/\lambda).$$

The values of T_λ oscillate about the integers (figure 2).

We construct a Poincaré return map Φ on a neighbourhood of the positive half of the symmetry line $\mathrm{Fix}\,G$. Let $d(z)$ be the perpendicular distance between a point z and $\mathrm{Fix}\,G$:

$$d(z) = d_H(z, \mathrm{Fix}\,G).$$

We define the domain X of the return map Φ to be the set of points $z \in (\lambda\mathbb{Z}_{\geq 0})^2$ which are closer to $\mathrm{Fix}\,G$ than their pre-images under F_λ^4, and at least as close as their images:

$$(24) \qquad\qquad X = \{z \in (\lambda\mathbb{Z}_{\geq 0})^2 : d(z) \leq d(F_\lambda^4(z)),\ d(z) < d(F_\lambda^{-4}(z))\}.$$

Asymptotically, for every non-negative integer m, the set X has non-empty intersection with the boxes $B_{m,m}$, $B_{m+1,m}$, $B_{m,m+1}$. The main component is in $B_{m,m}$, a thin strip of width

$\|\mathbf{w}_{m,m}\|$ lying parallel to the symmetry line $\mathrm{Fix}\,G$ (figure 5). To ensure that this component is non-empty, we require $\lambda < \lambda_m$, where the critical parameter λ_m is given by

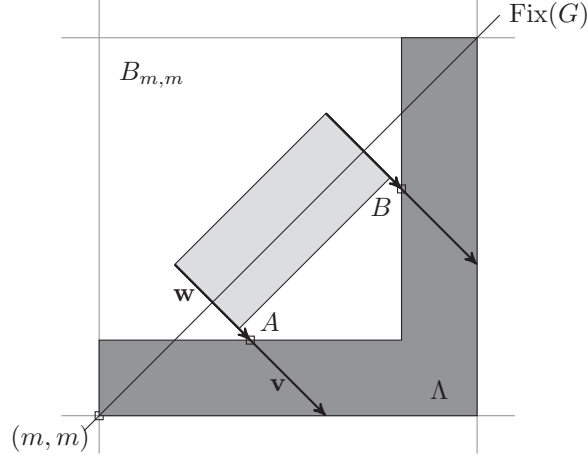$$(25) \qquad \lambda_m = \frac{1}{6(m+1)}.$$

FIGURE 5. Structure of the box $B_{m,m}$. The the strip containing the main component of X is shown in light grey; the intersection of $B_{m,m}$ with the set Λ of equation (27) is shown in dark grey. The set X comprises the lattice points which are closer to $\mathrm{Fix}\,G$ than their pre-images under translation by $\mathbf{v}$, and at least as close as their images. In $B_{m,m} \setminus \Lambda$ we have $\mathbf{v} = \mathbf{w}_{m,m}$ (lemma 4), so that the width of the strip is $\|\mathbf{w}_{m,m}\|$, and the points A and B lie on the line $y = x - 2m\lambda$. If $A \in \Lambda$, then the y-coordinate of A cannot be larger than $m + \lambda(2m + 3)$, and if $B \in \Lambda$, then the x-coordinate of B cannot be smaller than $m + 1 - \lambda(2m + 3)$ (see proof of lemma 4). Thus the strip is non-empty as long as $m + \lambda(2m + 3) < m + 1 - \lambda(2m + 3) - 2m\lambda$, giving the critical parameter λ_m of equation (25).

The transit time τ to the set X is well-defined for all $z \in (\lambda\mathbb{Z})^2$:

$$(26) \qquad \tau : (\lambda\mathbb{Z})^2 \to \mathbb{N} \qquad \tau(z) = \min\{k \in \mathbb{N} : F_\lambda^k(z) \in X\}.$$

Thus the first return map Φ is the function

$$\Phi : X \to X \qquad \Phi(z) = F_\lambda^{\tau(z)}(z).$$

We refer to the orbit of $z \in X$ up to the return time $\tau(z)$ as the **return orbit** of z:

$$\mathcal{O}_\tau(z) = \{F_\lambda^k(z) : 0 \le k \le \tau(z)\}, \qquad z \in X.$$

We let τ_- be the transit time to X under F_λ^{-1}:

$$\tau_- : (\lambda\mathbb{Z})^2 \to \mathbb{Z}_{\ge 0} \qquad \tau_-(z) = \min\{k \in \mathbb{Z}_{\ge 0} : F_\lambda^{-k}(z) \in X\},$$

so that the return orbit for a general $z \in (\lambda\mathbb{Z})^2$ is given by:

$$\mathcal{O}_\tau(z) = \{F_\lambda^k(z) : \tau_-(z) \le k \le \tau(z)\}, \qquad z \in (\lambda\mathbb{Z})^2.$$

To associate a unique return orbit to an integrable orbit, we define the rescaled round-off function R_λ, which rounds points on the plane down to the next lattice point:

$$R_\lambda : \mathbb{R}^2 \to (\lambda\mathbb{Z})^2 \qquad R_\lambda(w) = \lambda R(w/\lambda),$$

where R is the integer round-off function

$$R : \mathbb{R}^2 \to \mathbb{Z}^2 \qquad R(u,v) = (\lfloor u \rfloor, \lfloor v \rfloor).$$

For every point $w \in \mathbb{R}^2$ and every $\delta > 0$, the set of points

$$\{z \in (\lambda\mathbb{Z})^2 : z = R_\lambda(w), \, 0 < \lambda < \delta\}$$

that represent w on the lattice as $\lambda \to 0$ is countably infinite. The corresponding set of points on $\mathbb{Z}^2$, before rescaling, is unbounded.

In the rest of this section, we state some metric properties of the return orbits, and then show that the return orbits shadow the integrable orbits. The proofs will be found in section 4.

According to proposition 1 on page 6, the points of the scaled lattice $(\lambda\mathbb{Z})^2$ at which integrable and discrete vector fields have different values are rare, as a proportion of lattice points. The following result shows that these points are also rare within each return orbit.

Proposition 3. *For any $w \in \mathbb{R}^2$, if we define the ratio*

$$\mu_2(w, \lambda) = \frac{\#\{z \in \mathcal{O}_\tau(R_\lambda(w)) : \mathbf{v}(z) = \mathbf{w}(z)\}}{\#\mathcal{O}_\tau(R_\lambda(w))},$$

then we have

$$\lim_{\lambda \to 0} \mu_2(w, \lambda) = 1.$$

To establish propositions 1 and 3, we seek to isolate the lattice points $z \in (\lambda\mathbb{Z})^2$ where the discrete vector field $\mathbf{v}(z)$ deviates from the integrable vector field $\mathbf{w}(z)$. We say that a point $z \in (\lambda\mathbb{Z})^2$ is a **transition point** if z and its image under F_λ^4 do not belong to the same box, namely if

$$R(F_\lambda^4(z)) \neq R(z).$$

Let Λ be the set of transition points. Then

(27) $$\Lambda = \bigcup_{m,n \in \mathbb{Z}} \Lambda_{m,n},$$

where

$$\Lambda_{m,n} = F_\lambda^{-4}(B_{m,n} \cap (\lambda\mathbb{Z})^2) \setminus B_{m,n}.$$

For small λ, the set of transition points Λ consists of thin strips of lattice points arranged along the lines Δ (see figure 6). The following key lemma states that, for sufficiently small λ, all points $z \neq (0,0)$ where $\mathbf{v}(z) \neq \mathbf{w}(z)$ are transition points.

Lemma 4. *Let $A(r, \lambda)$ be as in equation (7). Then for all $r > 0$ there exists $\lambda^* > 0$ such that, for all $\lambda < \lambda^*$ and $z \in A(r, \lambda)$, we have*

$$\mathbf{v}(z) \neq \mathbf{w}(z) \implies z \in \Lambda \cup \{(0,0)\}.$$

We conclude this section by formulating a shadowing theorem, which states that for time scales corresponding to a first return to the domain X, every integrable orbit has a scaled return orbit that shadows it. Furthermore, this scaled orbit of the round-off map converges to the integrable orbit in the Hausdorff metric as $\lambda \to 0$.
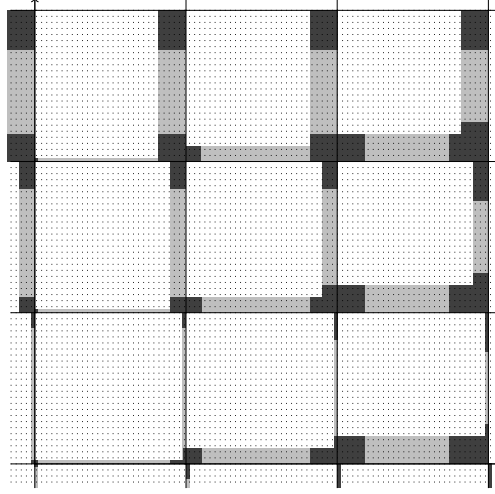
FIGURE 6. Structure of phase space. The boxes $B_{m,n}$, bounded by the set Δ, include regular domains (white), where the motion is integrable. The darker regions comprise the set Λ of transition points, given in (27). The darkest domains are the components $\Sigma_{m,n}$ of the set $\Sigma \subset \Lambda$, defined in (35).

Theorem 5. *For any $w \in \mathbb{R}^2$, let $\Pi(w)$ be the orbit of $\mathscr{P}$, and let $\mathcal{O}_\tau(R_\lambda(w))$ be the return orbit at the rounded lattice point. Then*

$$\lim_{\lambda \to 0} d_H\left(\Pi(w), \mathcal{O}_\tau(R_\lambda(w))\right) = 0,$$

where d_H is the Hausdorff distance on $\mathbb{R}^2$.

This result justifies the term 'integrable limit' assigned to the flow $\mathscr{P}$.

4. PROOFS FOR SECTION 3

PROOF OF LEMMA 4.   Let $r > 0$ be given, and let $A(r, \lambda)$ be as in equation (7). We show that if λ and $\|z\|_\infty$ are sufficiently small (and $z \neq 0$), then

$$\mathbf{v}(z) \neq \mathbf{w}(z) \quad \Rightarrow \quad R(F_\lambda^4(z)) \neq R(z).$$

Since $z \in A(r, \lambda)$, we have $z \in B_{m,n}$ for some $|m|, |n| \leq \lceil r \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, defined by the identity $\lceil x \rceil = -\lfloor -x \rfloor$. Through repeated applications of F_λ, we have

(28)
$$\begin{aligned}
F_\lambda(z) &= \lambda(-y + m, x) & R(F_\lambda(z)) &= (-(a+1), m), \\
F_\lambda^2(z) &= \lambda(-x - a - 1, -y + m) & R(F_\lambda^2(z)) &= (-(b+1), -(a+1)), \\
F_\lambda^3(z) &= \lambda(y - m - b - 1, -x - a - 1) & R(F_\lambda^3(z)) &= (c, -(b+1)), \\
F_\lambda^4(z) &= \lambda(x + a + c + 1, y - m - b - 1) & R(F_\lambda^4(z)) &= (d, c),
\end{aligned}$$

where the integers a, b, c, d are given by

(29)
$$\begin{aligned}
a + 1 &= \lceil \lambda(y - m) \rceil, \\
b + 1 &= \lceil \lambda(x + a + 1) \rceil, \\
c &= \lfloor \lambda(y - m - b - 1) \rfloor \\
d &= \lfloor \lambda(x + a + c + 1) \rfloor.
\end{aligned}$$

The integers a, b, c and d label the boxes in which each iterate occurs (figure 7), and also give an explicit expression for the round-off term $\lfloor \lambda x \rfloor$ at each step. Thus reading from the last of these equations, the discrete vector field $\mathbf{v}$ is given by

$$(30) \qquad \mathbf{v}(z) = F_\lambda^4(z) - z = \lambda(a + c + 1, -(m + b + 1)),$$

and z is a transition point whenever at least one of the equalities $d = m$ and $c = n$ on the box labels fails.
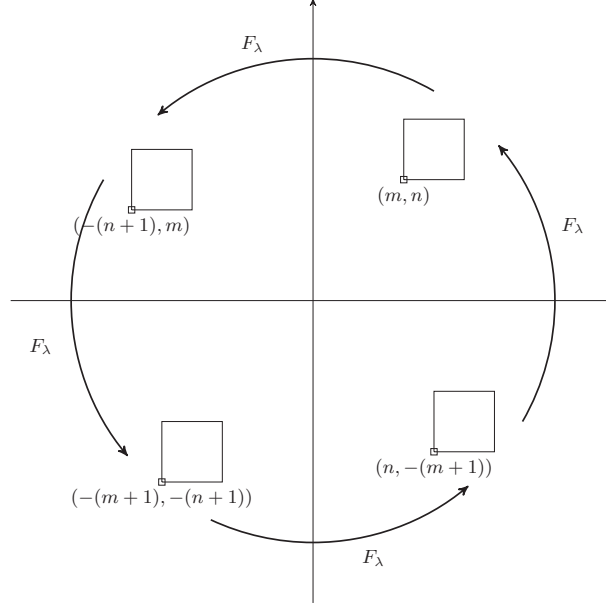

FIGURE 7. The boxes in which successive iterates of z under F_λ occur, for the typical case where $m = b = d$, $n = a = c$.

If the integers m, a, b, c are sufficiently small relative to the number of lattice points per unit length, i.e., if

$$(31) \qquad \max(|m|, |a + 1|, |m + b + 1|, |a + c + 1|) < 1/\lambda,$$

then the map F_λ^4 moves the point z at most one box in each of the x and y directions, so that the labels a, b, c and d satisfy

$$(32) \qquad b, d \in \{m - 1, m, m + 1\}, \qquad a, c \in \{n - 1, n, n + 1\}.$$

Similarly, (31) dictates that the discrepancy between each of the pairs (b, d), (a, c) cannot be too large:

$$(33) \qquad |b - d|, |a - c| \leq 1.$$

Letting $\lambda^* = 1/(2\lceil r \rceil + 3)$, we obtain

$$
\begin{aligned}
\max(|m|, |a + 1|, |m + b + 1|, |a + c + 1|) &\leq \max(|m| + |b + 1|, |a + 1| + |c|) \\
&\leq \max(2|m| + |b - m|, 2|n| + |a - n| + |c - n|) + 1 \\
&\leq 2\lceil r \rceil + 3 \leq 1/\lambda^*,
\end{aligned}
$$

so that (32) and (33) hold for all $\lambda < \lambda^*$. Then the expression (30) for $\mathbf{v}$, combined with the inequality (33), gives that $\mathbf{v}(z) = \mathbf{w}(z)$ if and only if

$$(34) \qquad\qquad m = b \qquad\qquad n = a = c.$$

Suppose now that z is not a transition point, so that $c = n$ and $d = m$, but that $\mathbf{v}(z) \neq \mathbf{w}(z)$, so that at least one of the equalities (34) fails. If $a \neq n$, straightforward manipulation of inequalities shows that the only combination of values which satisfies (32) is

$$a = n - 1,\ m = 0,\ b = -1,\ \lambda y = n.$$

In turn, using also the inequality (33), $b \neq m$ implies that

$$b = m - 1,\ n = 0,\ a = -1,\ \lambda x = m.$$

Combining these, we have $m = a + 1 = b + 1 = c = 0$, which corresponds to the unique point $z = (0,0)$. $\qquad\square$

We now use this result to prove propositions 1 and 3, given in sections 2 and 3, respectively.

PROOF OF PROPOSITION 1.   From equation (7), we have that the number of lattice points in the set $A(r, \lambda)$ is given by

$$\#A(r, \lambda) = \left(2 \left\lceil \frac{r}{\lambda} \right\rceil - 1 \right)^2.$$

By lemma 4, for sufficiently small λ, every non-zero point $z \in A(r, \lambda)$ satisfying $\mathbf{v}(z) \neq \mathbf{w}(z)$ is a transition point, so has $z \in \Lambda_{m,n}$ for some $m, n \in \mathbb{Z}$ with $|m|, |n| \leq \lceil r \rceil + 1$. Furthermore, every set $\Lambda_{m,n}$ is composed of two strips, each of unit length, and width approximately equal to $\lambda(2m + 1)$ and $\lambda(2n + 1)$, respectively. We can bound the number of lattice points in the set $\Lambda_{m,n}$ explicitly by

$$\#\Lambda_{m,n} \leqslant \frac{|2m + 1| + |2n + 1| + c}{\lambda},$$

for some positive constant c, independent of m and n. (Indeed $c = 3$ is sufficient – cf. the methods used in the proof of proposition 7.) It follows that for fixed $r > 0$, as $\lambda \to 0$ we have the estimate

$$\begin{aligned}
\mu_1(r, \lambda) &= 1 - \frac{\#\{z \in A(r, \lambda) : \mathbf{v}(z) \neq \mathbf{w}(z)\}}{\#A(r, \lambda)} \\
&\geq 1 - \frac{\#\left(A(r, \lambda) \cap \Lambda \right)}{\#A(r, \lambda)} \\
&\geq 1 - \frac{1}{\#A(r, \lambda)} \sum_{|m|,|n| \leq \lceil r \rceil + 1} \#\Lambda_{m,n} \\
&\geq 1 - \left(2 \left\lceil \frac{r}{\lambda} \right\rceil - 1 \right)^{-2} \sum_{|m|,|n| \leq \lceil r \rceil + 1} \frac{|2m + 1| + |2n + 1| + c}{\lambda} \\
&= 1 - O(\lambda).
\end{aligned}$$

Since $\mu_1(r, \lambda) \leq 1$, the proof is complete. $\qquad\square$

By construction, the piecewise-constant vector field $\mathbf{w}$ is parallel to the Hamiltonian vector field associated with $\mathscr{P}$ (cf. equation (13)). It follows that $\mathscr{P}$ is invariant under translation by $\mathbf{w}$ everywhere except across the discontinuities of $\mathbf{w}$, i.e., except at transition points, which occur near the vertices of integrable orbits.

In the above proof of proposition 1, we show that transition points $z \in \Lambda$ have zero density in the limit $\lambda \to 0$ in the following sense:

$$\forall r > 0 : \qquad \frac{\#\left(A(r, \lambda) \cap \Lambda\right)}{\#A(r, \lambda)} \to 0,$$

so that in any bounded region, and for sufficiently small λ, $\mathscr{P}$ is invariant under translation by $\mathbf{w}$ almost everywhere. In turn, lemma 4 implies that $\mathscr{P}$ is invariant under translation by $\mathbf{v}$, and hence under iterates of F_λ^4, almost everywhere. Again the exceptions to this rule occur near the vertices of integrable orbits.

To prove proposition 3 and theorem 5, we bound the variation in the Hamiltonian function $\mathscr{P}$ along perturbed orbits $\mathcal{O}_\tau(z)$. Since we know $\mathscr{P}$ is invariant under F_λ^4 at all points $z \notin \Lambda$, it remains to consider the change in $\mathscr{P}$ under F_λ, and under F_λ^4 at points $z \in \Lambda$.

PROOF OF PROPOSITION 3.

Let $r > 0$. Using lemma 4, we observe that for sufficiently small λ, $\mathscr{P}$ is invariant under F_λ^4 on $A(r, \lambda) \setminus \Lambda$. We begin by bounding the change in $\mathscr{P}$ under F_λ^4 on the set $A(r, \lambda) \cap \Lambda$.

For any $z, v \in \mathbb{R}^2$ we have:

$$|\mathscr{P}(z + v) - \mathscr{P}(z)| = \left| \int_{[z, z+v]} \nabla \mathscr{P}(\xi) \cdot d\xi \right|$$
$$\leq \max_{\xi \in [z, z+v]} \left(\|\nabla \mathscr{P}(\xi)\| \right) \|v\|,$$

where $[z, z + v]$ denotes the line segment joining the points z and $z + v$, $d\xi$ is the line element tangent to this segment, and $\nabla \mathscr{P}$ is the gradient of $\mathscr{P}$, given by:

$$\nabla \mathscr{P}(x, y) = (2\lfloor x \rfloor + 1, 2\lfloor y \rfloor + 1) \qquad (x, y) \in \mathbb{R}^2 \setminus \Delta.$$

If $z = \lambda(x, y)$ and $v = \mathbf{v}(z)$ is the discrete vector field, then for sufficiently small λ, equations (29) and (32) can be combined to give:

$$\|\mathbf{v}(z)\| \leq \lambda \sqrt{(|2\lfloor \lambda y \rfloor + 1| + 2)^2 + (|2\lfloor \lambda x \rfloor + 1| + 1)^2}$$
$$\leq \lambda \sqrt{(2|\lambda y| + 3)^2 + (2|\lambda x| + 2)^2}$$
$$\leq \lambda (2\|z\| + \sqrt{13}).$$

This inequality ensures that the length of the line segment $[z, z + v]$ goes to zero with λ, so that for sufficiently small λ, the piecewise-constant form of the gradient $\nabla \mathscr{P}$ gives:

$$\max_{\xi \in [z, z+v]} \left(\|\nabla \mathscr{P}(\xi)\| \right) \leq \sqrt{(|2\lfloor \lambda x \rfloor + 1| + 2)^2 + (|2\lfloor \lambda y \rfloor + 1| + 2)^2}$$
$$\leq \sqrt{(2|\lambda x| + 3)^2 + (2|\lambda y| + 3)^2}$$
$$\leq 2\|z\| + 3\sqrt{2}.$$

Substituting these into the first inequality, we have that for sufficiently small λ:

$$\left| \mathscr{P}(F_\lambda^4(z)) - \mathscr{P}(z) \right| = |\mathscr{P}(z + \mathbf{v}(z)) - \mathscr{P}(z)| \leq \lambda(2\|z\| + 3\sqrt{2})^2.$$

For the change in $\mathscr{P}$ under F_λ, if $z = \lambda(x, y)$, then by the same sort of analysis we have:

$$
\begin{aligned}
|\mathscr{P}(F_\lambda(z)) - \mathscr{P}(z)| &= |P(\lambda(\lfloor \lambda x \rfloor - y)) - P(\lambda y)| \\
&= |P(\lambda(y - \lfloor \lambda x \rfloor)) - P(\lambda y)| \\
&\leq \lambda|\lfloor \lambda x \rfloor| \left(|P'(\lambda y)| + 2\right) \\
&\leq \lambda|\lfloor \lambda x \rfloor| \left(|2\lfloor \lambda y \rfloor + 1| + 2\right) \\
&\leq \lambda(|\lambda x| + 1)\left(2|\lambda y| + 3\right) \\
&\leq 2\lambda(\|z\| + 2)^2,
\end{aligned}
$$

where P is the piecewise-affine function defined in equation (8). (See page 8 for the proof that P is even.) It follows that for any orbit contained in $A(r, \lambda)$, if $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq l < 4$:

$$
\left|\mathscr{P}(F_\lambda^{4k+l}(z)) - \mathscr{P}(z)\right| \leq m\lambda(2\|z\| + 3\sqrt{2})^2 + 2l\lambda(\|z\| + 2)^2
$$

where m is the number of transition points in the orbit of z under F_λ^4:

$$
m = \# \left(\{z, F_\lambda^4(z), \ldots, F_\lambda^{4k}(z)\} \cap \Lambda\right).
$$

Similar expressions hold in backwards time, for iterates of F_λ^{-4} and F_λ^{-1}. For fixed λ, this estimate bounds the perturbed orbit of a point $z \in (\lambda\mathbb{Z})^2$ to a polygonal annulus around the polygon $\Pi(z)$, which grows in thickness as the number of transition points in the orbit increases. Conversely, by taking a sufficiently small value of λ, we can ensure that the perturbed orbit of z stays arbitrarily close to the polygon $\Pi(z)$ for any finite number of transition points. Specifically, given any $\epsilon > 0$ and $m \in \mathbb{N}$, there is a critical value of λ below which any point $F_\lambda^{4k+l}(z)$ along the perturbed orbit satisfies

$$
\left|\mathscr{P}(F_\lambda^{4k+l}(z)) - \mathscr{P}(z)\right| < \epsilon,
$$

as long as the number of transition points encountered is not more than m:

$$
\# \left(\{z, F_\lambda^4(z), \ldots, F_\lambda^{4k}(z)\} \cap \Lambda\right) \leq m.
$$

Now let $w \in \mathbb{R}^2$ be given, and let $z = R_\lambda(w)$, so that $\mathcal{O}_\tau(z)$ is the return orbit which shadows the integrable orbit $\Pi(w)$. Furthermore let $r^\pm = P^{-1}(\mathscr{P}(w)) \pm 1$, where $P^{-1}(\mathscr{P}(w))$ is the abscissa of the intersection of $\Pi(w)$ with the positive x-axis. Since $\|z - w\| < \sqrt{2}\lambda$, using the same bounds as above, we have:

$$
\begin{aligned}
|\mathscr{P}(z) - \mathscr{P}(w)| &\leq \max_{\xi \in [w, z]} \left(\|\nabla \mathscr{P}(\xi)\|\right) \|z - w\| \\
&\leq \sqrt{2}\lambda(2\|w\| + 3\sqrt{2}),
\end{aligned}
$$

so that for small λ, the polygons $\Pi(z)$ and $\Pi(w)$ are close.

Pick $0 < \epsilon < 1$ and consider the polygons $\Pi(w)^\pm$ given by

$$
\Pi(w)^\pm = \{\xi : \mathscr{P}(\xi) = \mathscr{P}(w) \pm \epsilon\}.
$$

Without loss of generality, we may assume that neither of these polygons is critical. Thus each of these integrable orbits intersects as many boxes as it has sides. For the larger polygon $\Pi(w)^+$, the number of sides (see theorem 2) is given by

$$
4 \left(2 \left\lfloor \sqrt{\mathscr{P}(w) + \epsilon} \right\rfloor + 1\right),
$$

and we let this number be m. Furthermore, we note that all lattice points in the interior of $\Pi(w)^+$ are elements of the set $A(r^+, \lambda)$, since:

$$\max\{\, \|\xi\|_\infty : \ \xi \in \Pi(w)^+ \,\} = P^{-1}(\mathscr{P}(w) + \epsilon) < r^+.$$

Then, as we showed above, for sufficiently small λ, the perturbed orbit $\mathcal{O}_\tau(z)$ is bounded between $\Pi(w)^\pm$ as long as the number of transition points encountered does not exceed m. By construction (cf. equation (27)), the return orbit $\mathcal{O}_\tau(z)$ contains at most one transition point for every box it intersects. However, as long as the perturbed orbit is bounded above by the integrable orbit $\Pi(w)^+$, it cannot intersect more boxes than this integrable orbit in any one revolution around the origin. Hence the perturbed orbit cannot intersect more than m boxes in any one revolution, and its number of transition points is bounded above as follows:

$$\#\left(\mathcal{O}_\tau(z) \cap \Lambda\right) \leq m.$$

Now we consider the total number of points in the return orbit $\mathcal{O}_\tau(z)$. Since the perturbed orbit is bounded below by the integrable orbit $\Pi(w)^-$, it must contain a point ξ, close to the positive x-axis, with x-coordinate not less than:

$$P^{-1}(\mathscr{P}(w) - \epsilon) > r^-.$$

Similarly it must contain a point ξ, close to the negative x-axis, with x-coordinate not greater than $-r^-$. The return orbit moves between these points via the action of F_λ^4, i.e., by translations of the vector field $\mathbf{v}$. Hence, for sufficiently small λ, the number of steps required for the orbit to move from one point to the other in, say, the upper half-plane, is bounded below by the distance $2r^-$ divided by the maximal length of $\mathbf{v}$ along the orbit:

$$\{\lambda(x, y) \in \mathcal{O}_\tau(z) : y > 0\} \geq \frac{2r^-}{\lambda(2\|z\| + \sqrt{13})} \geq \frac{2r^-}{\lambda(2\|w\| + 4)}.$$

Thus, as $\lambda \to 0$, we have the estimate

$$\begin{aligned}
\mu_2(w, \lambda) &= 1 - \frac{\#\{\xi \in \mathcal{O}_\tau(z) : \mathbf{v}(\xi) \neq \mathbf{w}(\xi)\}}{\#\mathcal{O}_\tau(z)}, \\
&\geq 1 - \frac{\#\left(\mathcal{O}_\tau(z) \cap \Lambda\right)}{\#\mathcal{O}_\tau(z)}, \\
&\geq 1 - \lambda \frac{m(2\|w\| + 4)}{4r^-}, \\
&= 1 - O(\lambda).
\end{aligned}$$

Since $\mu_2(w, \lambda) \leq 1$, the proof is complete. $\qquad\square$

PROOF OF THEOREM 5.

Let $w \in \mathbb{R}^2$ be given, and let $z = R_\lambda(w)$, so that $\mathcal{O}_\tau(z)$ is the return orbit which shadows the integrable orbit $\Pi(w)$. We have already seen, in the proof of proposition 3, that by taking a sufficiently small value of λ, we can ensure that the perturbed orbit of z stays arbitrarily close to the polygon $\Pi(z)$. It follows that:

$$\forall \xi \in \mathcal{O}_\tau(z) : \qquad d_H(\xi, \Pi(w)) = O(\lambda)$$

as $\lambda \to 0$.

Neighbouring points $\xi, \xi + \mathbf{v}(\xi)$ in the return orbit $\mathcal{O}_\tau(z)$ are also $O(\lambda)$-close as $\lambda \to 0$, so the result follows. $\qquad\square$

5. Regular domains and statement of the main theorems

In this section we state theorems A and B, after the necessary preparation. Then we briefly discuss the conditions of theorem B.

In section 2, the positive real line was partitioned into critical intervals $I^e, e \in \mathscr{E}$, each corresponding to a distinct polygon class of the integrable system; accordingly, the set Γ of critical polygons partitioned the plane. In section 3 we reduced the near-integrable round-off map F_λ to a return map Φ of a thin domain X, placed along the symmetry axis. In this section we partition the set X into sub-domains which play the same role for the perturbed orbits as the intervals I^e for the integrable orbits.

By theorem 5, the return orbit $\mathcal{O}_\tau(z)$ of a point $z \in X$ shadows the orbit $\Pi(z)$ of the integrable Hamiltonian $\mathscr{P}$. However, the quantity $\mathscr{P}$ is not constant along perturbed orbits. To deal with this problem, we introduce the following sequence of sets

$$X^e = \{z \in X \: : \: \forall w \in \mathcal{O}_\tau(z), \, \mathscr{P}(w) \in I^e\} \qquad e \in \mathscr{E}.$$

It remains to match the vertex list associated with I^e. To this end, it is necessary to replace the sets X^e by smaller **regular domains**, and then prove that, in the limit, these domains have full density in X.

We start by defining the **edges** of $\mathcal{O}_\tau(z)$, as the non-empty sets of the form

$$B_{m,n} \cap \mathcal{O}_\tau(z), \qquad m, n \in \mathbb{Z}.$$

For sufficiently small λ, consecutive edges of $\mathcal{O}_\tau(z)$ must lie in adjacent boxes, and transitions between edges occur when the orbit meets the set Λ given in equation (27). Thus we call the set $\mathcal{O}_\tau(z) \cap \Lambda$ the set of **vertices** of $\mathcal{O}_\tau(z)$. By analogy with the vertices of the polygons, we say that the return orbit $\mathcal{O}_\tau(z)$ has a vertex on $x = m$ of **type** v if there exists a point $w \in \mathcal{O}_\tau(z)$ such that

$$w \in B_{m,v} \cap F_\lambda^{-4}(B_{m-1,v}) \qquad \text{or} \qquad w \in B_{m-1,v} \cap F_\lambda^{-4}(B_{m,v}).$$

Similarly for a vertex on $y = n$ of type v. A perturbed orbit is **critical** if it has a vertex whose type is undefined, i.e., if there exists $w \in \mathcal{O}_\tau(z)$ such that

$$w \in B_{m,n} \cap F_\lambda^{-4}(B_{m\pm1,n\pm1})$$

for some $m, n \in \mathbb{Z}$ (see figure 8).

By excluding points whose perturbed orbit is critical, we'll construct a subset $\tilde{X}^e$ of X^e with the following properties: $i)$ for all $z \in \tilde{X}^e$ the orbit $\mathcal{O}_\tau(z)$ has the same sequence of vertex types as $\Pi(z)$; $ii)$ the union of all sets $\tilde{X}^e$ still has full density in X, as $\lambda \to 0$.

We now give the construction of $\tilde{X}^e$. Let the set $\Sigma \subset \Lambda$ be given by

$$(35) \qquad \Sigma = \bigcup_{m,n \in \mathbb{Z}} \Sigma_{m,n},$$

where

$$\Sigma_{m,n} = \{z \in \Lambda : \|z - (m,n)\|_\infty \le \lambda(\, \|(2m+1, 2n+1)\|_\infty + 2)\,\}$$

and $\|(u,v)\|_\infty = \max(|u|, |v|)$. The set $\Sigma_{m,n}$ is a small domain, adjacent to the integer point (m,n) (see figure 6).

If $z \in X^e$ for some $e \in \mathscr{E}$, we say that z is **regular** if two properties hold: firstly, z itself is not a vertex of $\mathcal{O}_\tau(z)$, i.e., $z \in X \setminus \Lambda$, and secondly the orbit $\mathcal{O}_\tau(z)$ does not intersect the set Σ. Points which are not regular are called **irregular**. Then the set $\tilde{X}^e$ is defined as

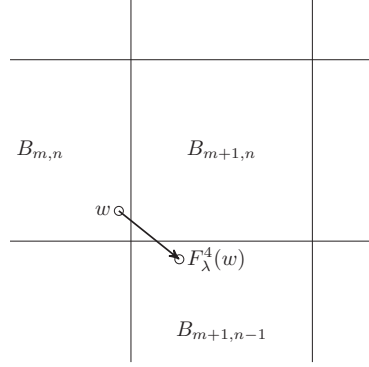$$(36) \qquad \tilde{X}^e = \{z \in X^e : \mathscr{P}(z) \in \tilde{I}^e(\lambda)\},$$

FIGURE 8. A critical vertex $w \in B_{m,n} \cap F_\lambda^{-4}(B_{m\pm 1,n\pm 1})$.

where $\tilde{I}^e(\lambda) \subset I^e$ is the largest interval such that all points in $\tilde{X}^e$ are regular.

In principle, the interval $\tilde{I}^e(\lambda)$ need not be uniquely defined, and may be empty. However, the following proposition ensures that $\tilde{I}^e(\lambda)$ is well-defined for all sufficiently small λ, and indeed that the irregular points have zero density in X as $\lambda \to 0$.

Proposition 6. *Let $\tilde{X}^e$ and $\tilde{I}^e$ be as above. Then, for all $e \in \mathcal{E}$, we have*

$$\lim_{\lambda \to 0} \frac{|\tilde{I}^e(\lambda)|}{|I^e|} = 1.$$

PROOF. Consider $z \in X$ such that $\mathscr{P}(z) \in I^e$ for some $e \in \mathcal{E}$. If the orbit of z strays between polygon classes, i.e., if $z \notin X^e$, then we have

$$\exists w \in \mathcal{O}_\tau(z): \qquad \mathscr{P}(w) \notin I^e.$$

However, in the proof of theorem 5 in section 4, we showed that the maximum variation in $\mathscr{P}$ along an orbit $\mathcal{O}_\tau(z)$ is of order λ, as $\lambda \to 0$:

$$\forall z \in X, \forall w \in \mathcal{O}_\tau(z): \qquad \mathscr{P}(w) - \mathscr{P}(z) = O(\lambda).$$

Hence we have

(37) $$\mathscr{P}(z) = \mathscr{P}(w) + O(\lambda) = \begin{cases} e + O(\lambda) & \mathscr{P}(w) \leq e \\ e' + O(\lambda) & \mathscr{P}(w) \geq e' \end{cases}$$

where e' is the successor of e in the sequence $\mathcal{E}$. In both cases, $\mathscr{P}(z)$ is near the boundary of I^e.

If $z \in X^e$ but z is irregular, then either $z \in \Lambda$ or the orbit of z intersects the set Σ. If $z \in \Lambda$, then one of its coordinates must be nearly integer:

$$d_H(z, \Delta) = O(\lambda),$$

where the set Δ was defined in (12). However, as the domain X^e lies in an $O(\lambda)$-neighbourhood of the symmetry line $\mathrm{Fix}\, G$, it follows that both coordinates must be nearly integer, giving

$$d_H(z, \mathbb{Z}^2) = O(\lambda).$$

Again, $\mathscr{P}(z)$ lies in a $O(\lambda)$-neighbourhood of the boundary of I^e.

Similarly, if there is a point $w \in \mathcal{O}_\tau(z) \cap \Sigma$, then, by construction,

$$d_H(w, \mathbb{Z}^2) = O(\lambda).$$

Hence we have again the estimate (37).

Combining these observations, we have

$$\frac{|\tilde{I}^e(\lambda)|}{|I^e|} = 1 - \frac{|I^e \setminus \tilde{I}^e(\lambda)|}{|I^e|} = 1 - \frac{O(\lambda)}{|I^e|}$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We now give an explicit representation of $\tilde{X}^e$. Let v_1 be the first entry in the vertex list of the critical number e. By the construction (36) of $\tilde{X}^e$, we have $\tilde{X}^e \subset B_{v_1,v_1} \setminus \Lambda$. Hence, by lemma 4, the discrete vector field in $\tilde{X}^e$ satisfies

$$\mathbf{v}(z) = \mathbf{w}(z) = \mathbf{w}_{v_1,v_1} \qquad\qquad z \in \tilde{X}^e.$$

Consequently by the definition (24) of the Poincare section X, if $z = \lambda(x, y) \in \tilde{X}^e$ then

$$-(2v_1 + 1) \leq x - y < 2v_1 + 1.$$

Hence the set $\tilde{X}^e$ is given by:

(38) $\qquad \tilde{X}^e = \{z = \lambda(x, y) \in (\lambda\mathbb{Z})^2 : -(2v_1 + 1) \leq x - y < 2v_1 + 1, \ \mathscr{P}(z) \in \tilde{I}_e\}.$

Now we show that the sequence of sets $\tilde{X}^e$ fulfil their objective, which was to exclude all points $z \in X$ whose perturbed orbit is critical in the sense defined above.

Proposition 7. *If $z \in \tilde{X}^e$ for some $e \in \mathcal{E}$, then the perturbed orbit of z is not critical.*

PROOF. Suppose that $z \in X$ with $\mathscr{P}(z) \in I^e$ is critical, i.e., there exists $w \in \mathcal{O}_\tau(z)$ such that

$$w \in B_{m,n} \qquad \text{and} \qquad F_\lambda^4(w) = w + \mathbf{v}(w) \in B_{m\pm1,n\pm1}$$

for some $m, n \in \mathbb{Z}$. We will show that $z \notin \tilde{X}^e$. For simplicity, we assume that m and n are both non-negative, so that by the orientation of the vector field in the first quadrant:

$$F_\lambda^4(w) = w + \mathbf{v}(w) \in B_{m+1,n-1}.$$

Recalling the proof of lemma 4, the expression (30) for the perturbed vector field $\mathbf{v}$ at the point $w = \lambda(x, y)$ implies that

(39) $\qquad\qquad\qquad\qquad \mathbf{v}(w) = \lambda(a + c + 1, -(m + b + 1)),$

where the integers a, b, c are given by (29). By assumption, $w + \mathbf{v}(w) \in B_{m+1,n-1}$, which implies that

$$c = n - 1 \qquad \text{and} \qquad d = m + 1.$$

It follows that the difference between the values λx, λy and the integers $m+1$, n, respectively, is bounded according to

$$-\lambda(a + n) = -\lambda(a + c + 1) \leq \lambda x - (m + 1) < 0$$
$$0 \leq \lambda y - n < \lambda(m + b + 1).$$

Combining this observation with the bounds (32) on a and b gives

$$
\begin{aligned}
\|w - (m+1, n)\|_\infty &\leq \lambda \max(|a+n|, |m+b+1|) \\
&\leq \lambda \max(|2n+1| + |n-a| + 1, |2m+1| + |m-b|) \\
&\leq \lambda(\|2m+1, 2n+1\|_\infty + 2).
\end{aligned}
$$

Hence $w \in \Sigma$ and z is irregular, so $z \notin \tilde{X}^e$. The cases where m or n are negative proceed similarly.   □

Below, in equation (56) of section 7, we shall define a sequence of lattices $\mathbb{L}^e \subset (\lambda\mathbb{Z})^2$, $e \in \mathscr{E}$, independent of λ up to scaling, such that within the domain $\tilde{X}^e$, the return map Φ is equivariant under the group of translations generated by $\mathbb{L}^e$. Formally,

Theorem A. *For every $e \in \mathscr{E}$, and all sufficiently small λ, the map Φ commutes with translations by the elements of $\mathbb{L}^e$ on the domain $\tilde{X}^e$*

$$
(40) \qquad \forall l \in \mathbb{L}^e, \ \forall z, z+l \in \tilde{X}^e, \quad \Phi(z+l) \equiv \Phi(z) + l \quad (\mathrm{mod} \ \mathbf{w}_{v_1,v_1}).
$$

There is a critical value of λ, depending on e, above which the statement of the theorem is empty, as $\tilde{X}^e$ is insufficiently populated for a pair of points $z, z+l \in \tilde{X}^e$ to exist. We write (mod $\mathbf{a}$) for some vector $\mathbf{a}$ to denote congruence modulo the one-dimensional module $\langle \mathbf{a} \rangle$ generated by $\mathbf{a}$. The congruence under the local integrable vector field $\mathbf{w}_{v_1,v_1}$ in equation (40) is necessary for the case that $\Phi(z) + l \notin X$.

Furthermore, we define the fraction of symmetric, minimal orbits in $\tilde{X}^e$

$$
\delta(e, \lambda) = \frac{\#\{z \in \tilde{X}^e : G(\mathcal{O}(z)) = \mathcal{O}(z), \Phi(z) = z\}}{\#\tilde{X}^e},
$$

and prove the following result on the persistence of such orbits in the limit $\lambda \to 0$.

Theorem B. *Let $e \in \mathscr{E}$, and let $(v_1, \ldots, v_k)$ be the vertex list of the corresponding polygon class. If $2v_1 + 1$ or $2v_k + 1$ is coprime to $2v_j + 1$ for all other vertex types v_j, i.e., if*

$$
(41) \qquad \exists\, i \in \{1, k\}, \quad \forall j \in \{1, \ldots k\}, \quad v_j \neq v_i \Rightarrow \gcd(2v_i + 1, 2v_j + 1) = 1,
$$

then, for sufficiently small λ, the number of symmetric fixed points of Φ in $\tilde{X}^e$ modulo $\mathbb{L}^e$ is independent of λ. Thus the asymptotic density of symmetric fixed points in $\tilde{X}^e$ converges, and its value is given by

$$
(42) \qquad \lim_{\lambda \to 0} \delta(e, \lambda) = \frac{1}{(2\lfloor \sqrt{e} \rfloor + 1)(2\lfloor \sqrt{e/2} \rfloor + 1)}.
$$

As in theorem A, the smallness of λ serves only to ensure that $\tilde{X}^e$ is sufficiently populated for all congruence classes modulo $\mathbb{L}^e$ to be represented.

The condition (41) on the orbit code is clearly satisfied for infinitely many critical numbers e, e.g., those for which either $2\lfloor \sqrt{e} \rfloor + 1$ or $2\lfloor \sqrt{e/2} \rfloor + 1$ is a prime number. The first violation occurs at $e = 49$ (see Figure 4), where $2v_1 + 1 = 9$ and $2v_k + 1 = 15$ have a common factor. We contrast this to the case of $e = 52$, where $2v_k + 1 = 15$ and $2v_2 + 1 = 9$ have a common factor, but $2v_1 + 1 = 11$ is prime, so the condition (41) holds. Numerical experiments show that the density of values of e for which (41) holds decays very slowly, reaching $1/2$ for $e \approx 500,000$.

The stated condition on the orbit code is actually stronger than that we require in the proof. This was done to simplify the formulation of the theorem. We remark that the weaker

condition is still not necessary for the validity of the density expression (42). At the same time, there are values of e for which the density of symmetric minimal orbits deviates from the given formula, and convergence is not guaranteed. Our numerical experiments show that these deviations are small, and don't seem connected to new dynamical phenomena. More significant are the fluctuations in the density of non-symmetric orbits. Its dependence on e is considerably less predictable than for symmetric orbits, see figure 9.
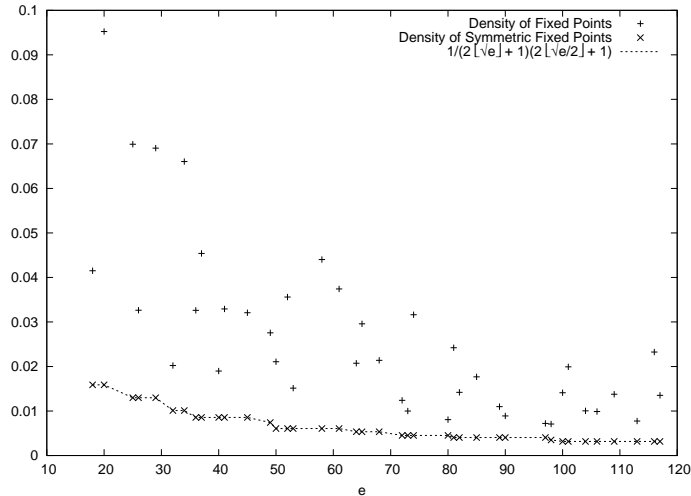

FIGURE 9. The density of symmetric minimal orbits, as a function of the critical number e (calculated for suitably small values of the parameter λ). The solid line represents the estimate (42). The scattered points correspond to the density of all minimal orbits, symmetric and non-symmetric.

The asymptotic density of symmetric fixed points in $\tilde{X}^e$ provides an obvious lower bound for the overall density of fixed points, which we denote $\eta(e, \lambda)$:

$$\eta(e, \lambda) = \frac{\#\{z \in \tilde{X}^e : \Phi(z) = z\}}{\#\tilde{X}^e}.$$

Corollary 8. *Let $e \in \mathscr{E}$ satisfy the condition (41) of theorem B. Then the asymptotic density of fixed points in $\tilde{X}^e$ is bounded below as follows:*

$$(43) \qquad \liminf_{\lambda \to 0} \eta(e, \lambda) = \frac{1}{(2\lfloor \sqrt{e} \rfloor + 1)(2\lfloor \sqrt{e/2} \rfloor + 1)}.$$

Note that we do not suggest that the density $\eta(e, \lambda)$ converges as $\lambda \to 0$, regardless of whether the condition (41) is satisfied or not.

6. THE STRIP MAP Ψ

In section 3, we saw that all non-zero points $z \in (\lambda \mathbb{Z})^2$ where the discrete vector field $\mathbf{v}(z)$ deviates from the Hamiltonian vector field $\mathbf{w}(z)$ lie in the set of transition points Λ, defined in (27). In order to study the dynamics at these points, where the perturbations from the integrable limit occur, we define a transit map Ψ to Λ which we call the **strip map**:

$$\Psi : (\lambda \mathbb{Z})^2 \to \Lambda \qquad \Psi(z) = F_\lambda^{4t(z)}(z),$$

where the transit time t to Λ is well-defined for all points excluding the origin:

$$t(z) = \min\{i \in \mathbb{N} : F_\lambda^{4i}(z) \in \Lambda\} \qquad z \neq (0,0).$$

(Since the origin plays no role in the present construction, to simplify notation we shall write $(\lambda\mathbb{Z})^2$ for $(\lambda\mathbb{Z})^2 \setminus \{(0,0)\}$ in the rest of the paper, where appropriate.) By abuse of notation, we define Ψ^{-1} to be the transit map to Λ under F_λ^{-1}. Note that Ψ^{-1} is the inverse of Ψ only on Λ.

If $z \in B_{m,n} \setminus \Lambda$ for some $m, n \in \mathbb{Z}$, then lemma 4 (page 14) implies that $\Psi(z)$ satisfies

$$(44) \qquad\qquad\qquad\qquad \Psi(z) = z + t(z)\mathbf{w}_{m,n},$$

where $\mathbf{w}_{m,n}$ is the value of the Hamiltonian vector field $\mathbf{w}$ in the box $B_{m,n}$. If $z \in \Lambda_{m,n}$, then we may have $\mathbf{v}(z) \neq \mathbf{w}(z)$, so the expression becomes

$$(45) \qquad\qquad\qquad \Psi(z) = z + \mathbf{v}(z) + (t(z) - 1)\mathbf{w}_{m,n}.$$

In the previous section, we identified the set $\mathcal{O}_\tau(z) \cap \Lambda$ as the set of vertices of the perturbed orbit $\mathcal{O}_\tau(z)$. Thus, within each quarter-turn, the strip map Ψ represents transit to the next vertex. For $1 \leq j \leq k$, where k is the length of the vertex list at z, we say that the orbit $\mathcal{O}_\tau(z)$ **meets the jth vertex** at the point $\Psi^j(z) \in \Lambda$. For $z \in X$ regular, the polygon $\Pi(z)$ and the return orbit $\mathcal{O}_\tau(z)$ are non-critical, by construction, and the number of sides of each is given by equation (18). Thus the full set of vertices of $\mathcal{O}_\tau(z)$ is given by

$$\mathcal{O}_\tau(z) \cap \Lambda = \bigcup_{i=0}^{3} \bigcup_{j=1}^{2k-1} \{(\Psi^j \circ F_\lambda^i)(z)\}.$$

Recall that the vertices of a polygon (or orbit) are numbered in the clockwise direction —the orientation of the integrable vector field $\mathbf{w}$. Hence the first $2k - 1$ vertices (those lying in the first quarter-turn) are given by $(\Psi^j(z))$, $1 \leq j \leq 2k - 1$. The action of F_λ moves points from one quadrant to the next in the opposing (anti-clockwise) direction, so that the vertices $((\Psi^j \circ F)(z))$, $1 \leq j \leq 2k - 1$ are the last $2k - 1$ vertices. Thus the following proposition is a simple consequence of the number of vertices of a given polygon class.

Proposition 9. *Let e be a critical number, and let k be the length of the vertex list of the corresponding polygon class. Then the return map Φ on $\tilde{X}^e$ is related to Ψ via*

$$(46) \qquad\qquad \Phi(z) \equiv (\Psi^{2k} \circ F_\lambda)(z) \pmod{\mathbf{w}_{v_1,v_1}}, \qquad z \in \tilde{X}^e$$

where v_1 is the type of the first vertex and $\mathbf{w}_{v_1,v_1}$ is the value of the integrable vector field $\mathbf{w}$ at z.

PROOF. Let $z \in \tilde{X}^e$. By the preceding discussion, the last vertex in the orbit $\mathcal{O}_\tau(z)$ is given by

$$(\Psi^{2k-1} \circ F_\lambda)(z) \in \Lambda_{v_1,v_1}.$$

The point $\Phi(z)$ satisfies $\Phi(z) \in B_{v_1,v_1} \setminus \Lambda$. Using the expression (44) for Ψ applied to $\Phi(z)$, we have

$$(\Psi^{2k} \circ F_\lambda)(z) \equiv \Phi(z) \pmod{\mathbf{w}_{v_1,v_1}}$$

as required. $\square$

For $z \in X$ regular, we use the vertices $(\Psi^j(z))_{1 \leq j \leq 2k-1}$ in the first quarter-turn to define a sequence of natural numbers $\sigma(z)$ called the **orbit code** of z, which encapsulates how the asymptotic orbit $\mathcal{O}_\tau(z)$ deviates from $\Pi(z)$.

Suppose the jth vertex of $\Pi(z)$ is a vertex of type v_j lying on $y = n$, and the orbit $\mathcal{O}_\tau(z)$ meets its corresponding vertex at $\Psi^j(z)$. We define the pair (x_j, y_j) via

$$(47) \qquad \Psi^j(z) = \lambda \left(\left\lceil \frac{v_j}{\lambda} \right\rceil + x_j, \left\lceil \frac{n}{\lambda} \right\rceil + y_j \right),$$

where $x_j \geq 0$, and $|y_j|$, which is (essentially) the number of lattice points between $\Psi^j(z)$ and the line $y = n$, is small relative to $1/\lambda$. Using similar arguments to those in the proof of proposition 7 one can show that y_j satisfies

$$-(2v_j + 1) \leq y_j < 0 \qquad \text{or} \qquad 0 \leq y_j < 2v_j + 1,$$

depending whether the integrable vector field is oriented in the positive or negative y-direction. In both cases, the possible values of y_j form a complete set of residues modulo $2v_j + 1$. Hence the jth element σ_j of the orbit code $\sigma(z)$ is defined to be the unique residue $\sigma_j \in \{0, 1, \ldots, 2v_j\}$ which is congruent to y_j:

$$(48) \qquad \sigma_j \equiv y_j \pmod{2v_j + 1}.$$

We call y the **integer coordinate** of the vertex and x the **non-integer coordinate**. Similarly, if the jth vertex lies on $x = m$, then the jth element σ_j of the orbit code is defined to be the residue congruent to x_j modulo $2v_j + 1$. In this case x is the integer coordinate and y is the non-integer coordinate.

For all vertices in the first quadrant, the fact that orbits progress clockwise under the action of F_λ^4 means that y_j will be non-negative wherever y is the integer coordinate, and x_j will be negative wherever x is the integer coordinate:

$$(49) \qquad -(2v_j + 1) \leq x_j < 0 \qquad \text{or} \qquad 0 \leq y_j < 2v_j + 1.$$

Thus the value of σ_j is given explicitly by

$$(50) \qquad \sigma_j = x_j + 2v_j + 1 \qquad \text{or} \qquad \sigma_j = y_j,$$

respectively.

In addition to the values σ_j for $1 \leq j \leq 2k - 1$ we consider σ_{-1}, which corresponds to the last vertex *before* the symmetry line, i.e., to the point $\Psi^{-1}(z)$. Thus the orbit code of z is a sequence $\sigma(z) = (\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$, such that

$$0 \leq \sigma_{-1} < 2v_1 + 1,$$
$$0 \leq \sigma_j < 2v_j + 1, \qquad 1 \leq j \leq 2k - 1,$$

where the v_j are the vertex types.

In the next proposition we consider how a perturbed orbit behaves at its vertices. We find that the regularity of z ensures that the discrete vector field $\mathbf{v}$ matches the Hamiltonian vector field $\mathbf{w}$ in the integer coordinate at $\Psi^j(z)$. The possible discrepancy in the non-integer coordinate is determined by the value of σ_j.

Proposition 10. *Let e be a critical number and let k be the length of the vertex list of the corresponding polygon class. For any $z \in \tilde{X}^e$ and any $j \in \{-1, 1, 2, \ldots, 2k - 1\}$, let m, n be such that $\Psi^j(z) \in \Lambda_{m,n}$. Then the discrete vector field at the jth vertex of $\mathcal{O}_\tau(z)$ is given by*

$$\mathbf{v}(\Psi^j(z)) = \mathbf{w}_{m,n} + \lambda \epsilon_j(\sigma_j) \mathbf{e},$$

where ϵ_j is a function of the jth entry σ_j of the orbit code $\sigma(z)$ and $\mathbf{e}$ is the unit vector in the direction of the non-integer coordinate of the vertex.

PROOF. If $z \in X$ is regular, then by proposition 7 the perturbed orbit $\mathcal{O}(z)$ is not critical. Thus for any vertex w, which, by construction, satisfies

$$w \in \Lambda_{m,n} \qquad F_\lambda^4(w) = w + \mathbf{v}(w) \in B_{m,n}$$

for some $m, n \in \mathbb{Z}$, we must have either $w \in B_{m,n\pm1}$ or $w \in B_{m\pm1,n}$. For definiteness we suppose that $w \in B_{m,n+1}$, so that the vertex w lies on $y = n + 1$. The cases where $w \in B_{m,n-1}$ or $w \in B_{m\pm1,n}$ are similar.

Now the proof proceeds very much as that of proposition 7. The perturbed vector field $\mathbf{v}(w)$, with $R(w) = (m, n+1)$, is given by equation (39), with a, b, c, d as in (29). In this case, $R(F_\lambda^4(w)) = (m, n)$ implies that

$$c = n \qquad \text{and} \qquad d = m,$$

and according to (32), the remaining integers a and b satisfy

$$a \in \{n, n + 1\}, \qquad\qquad b = m.$$

Thus we have

(51)
$$\mathbf{v}(w) = \lambda(n + a + 1, -(2m + 1))$$
$$= \mathbf{w}_{m,n} + \lambda(a - n)\mathbf{e},$$

where $\mathbf{e} = (1, 0)$ is the unit vector in the x-direction, the non-integer coordinate direction of the vertex.

If $w = \Psi^j(z)$, then the coefficient of the difference between $\mathbf{v}(w)$ and $\mathbf{w}_{m,n}$ in the x-direction is given by

$$\epsilon_j = a - n$$
$$= \lceil \lambda(y - m) \rceil - (n + 1)$$
$$= \begin{cases} 1 & \lambda y - (n + 1) > \lambda m \\ 0 & \text{otherwise.} \end{cases}$$

As in equation (47), we write

$$y = \left\lceil \frac{n + 1}{\lambda} \right\rceil + y_j,$$

where y_j satisfies the second inequality in (49). Hence, by (50), we have $y_j = \sigma_j$, and the function $\epsilon_j = \epsilon_j(\sigma_j)$ is given by

$$\epsilon_j = \begin{cases} 1 & \sigma_j > m \\ 0 & \text{otherwise} \end{cases}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that the function ϵ_j depends on j via m. In what follows we shall write ϵ_j, omitting the argument.

Applying proposition 10 to equation (45), we have that if $\Psi^j(z) \in \Lambda_{m,n}$, then the transit between vertices satisfies

(52)
$$\Psi^{j+1}(z) = \Psi^j(z) + \lambda\epsilon_j\mathbf{e} + t\mathbf{w}_{m,n},$$

where $t = t(\Psi^j(z))$ is the transit time. Hence we think of an orbit as moving according to the integrable vector field at all points except the vertices, where there is a mismatch between integrable and non-integrable dynamics, and points are given a small 'kick' in the non-integer coordinate direction.

7. Lattice structure and proof of main theorems

In this section we prove theorems A and B, stated in section 5.

For $e \in \mathscr{E}$, suppose the vertex list $V(e) = (v_1, \ldots, v_k)$ contains l distinct entries. We define the sequence $(\iota(j))_{1 \leq j \leq l}$ such that the $\iota(j)$th entry in the vertex list is the jth distinct entry. Since all repeated entries are consecutive, it follows that the vertex list has the form

$$(53) \qquad V(e) = (v_{\iota(1)}, \ldots, v_{\iota(1)}, v_{\iota(2)}, \ldots, v_{\iota(2)}, \ldots, v_{\iota(l)}, \ldots, v_{\iota(l)}),$$

with $v_{\iota(1)} = v_1$ and $v_{\iota(l)} = v_k$. We define the vector $\mathbf{L} = \mathbf{L}(e, \lambda)$ as:

$$(54) \qquad \mathbf{L} = \frac{\lambda q}{2v_1 + 1} \, (1, 1),$$

where the natural number $q = q(e)$ is defined as follows

$$(55) \qquad q = \mathrm{lcm}((2v_{\iota(1)} + 1)^2, (2v_{\iota(1)} + 1)(2v_{\iota(2)} + 1), \ldots, (2v_{\iota(l-1)} + 1)(2v_{\iota(l)} + 1)).$$

Here the least common multiple runs over $(2v_1 + 1)^2$ and all products of the form $(2v_j + 1)(2v_{j+1} + 1)$, where v_j and v_{j+1} are consecutive, distinct vertex types. Finally, the lattice $\mathbb{L}^e = \mathbb{L}^e(\lambda) \subset (\lambda \mathbb{Z})^2$ of theorem A is given by

$$(56) \qquad \mathbb{L}^e = \left\langle \mathbf{L}, \frac{1}{2} \left(\mathbf{L} - \mathbf{w}_{v_1, v_1} \right) \right\rangle,$$

where $\langle \cdots \rangle$ denotes the $\mathbb{Z}$-module generated by a set of vectors, and the vector $\mathbf{w}_{v_1, v_1}$ given by (6) is the Hamiltonian vector field $\mathbf{w}$ in the domain $\tilde{X}^e$ (figure 10).
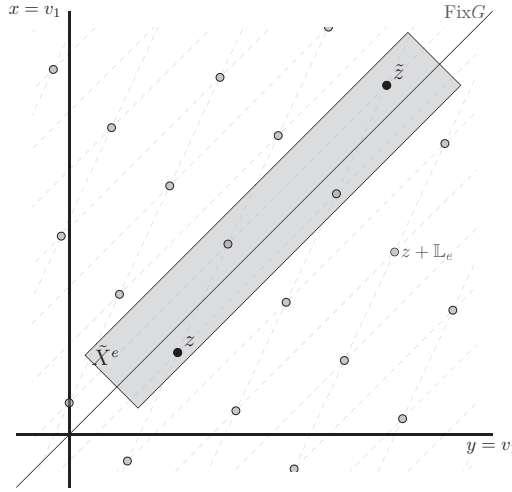

FIGURE 10. The lattice $\mathbb{L}^e$.

Any $z, \tilde{z} \in \tilde{X}^e$ which are congruent modulo $\mathbb{L}^e$ are related by:

$$\tilde{z} = z + \frac{1}{2} \left((2a + b)\mathbf{L} - b\mathbf{w}_{v_1, v_1} \right),$$

where $a, b \in \mathbb{Z}$ are the coordinates of $\tilde{z} - z$ relative to the module basis. We note that the vector $\mathbf{L}$ is parallel to the symmetry line $\mathrm{Fix}\,G$, and hence parallel to the strip $\tilde{X}^e$, whereas

the vector $\mathbf{w}_{v_1,v_1}$ is perpendicular to it. It follows that both a and b are determined uniquely by the value of the coefficient $2a + b$, because if $z = \lambda(x, y)$, then

$$x \geq y \quad \Rightarrow \quad b \in \{0, 1\},$$
$$x < y \quad \Rightarrow \quad b \in \{-1, 0\}.$$

The point z itself corresponds to $a = b = 0$.

We prove theorems A & B via several lemmata. The first and most significant step is to show that the orbit codes $\sigma(z)$ of points $z \in \tilde{X}^e$ are in one-to-one correspondence with the equivalence classes of $\tilde{X}^e$ modulo $\mathbb{L}^e$. We do this by constructing a sequence of nested lattices whose congruence classes are the cylinder sets of the orbit code.

We define recursively a finite integer sequence (q_j), $j = 1, \ldots, 2k - 1$, as follows:

$$q_1 = (2v_1 + 1)^2$$

$$(57) \qquad q_j = \begin{cases} q_{j-1} & \text{if } v_j = v_{j-1} \\ \text{lcm}((2v_j + 1)(2v_{j-1} + 1), q_{j-1}) & \text{if } v_j \neq v_{j-1} \end{cases} \qquad j > 1.$$

Then we let

$$(58) \qquad\qquad p_j = q_j/(2v_j + 1) \qquad j = 1, \ldots, 2k - 1.$$

By construction, p_j is also an integer. After defining the associated sequence of vectors

$$(59) \qquad\qquad \mathbf{L}_j = \frac{\lambda q_j}{2v_1 + 1} \, (1, 1),$$

we let the lattices $\mathbb{L}_j^e$ be the $\mathbb{Z}$-modules with basis

$$(60) \qquad\qquad \mathbb{L}_j^e = \left\langle \mathbf{L}_j, \frac{1}{2} \left(\mathbf{L}_j - \mathbf{w}_{v_1,v_1} \right) \right\rangle.$$

By construction

$$\mathbb{L}_{2k-1}^e \subseteq \mathbb{L}_{2k-2}^e \subseteq \cdots \subseteq \mathbb{L}_1^e \subset (\lambda\mathbb{Z})^2.$$

We claim that for all $1 \leq j \leq 2k - 1$, the closed form expression for q_j is given by

$$(61) \qquad q_j = \text{lcm}((2v_{\iota(1)} + 1)^2, (2v_{\iota(1)} + 1)(2v_{\iota(2)} + 1), \ldots, (2v_{\iota(i-1)} + 1)(2v_{\iota(i)} + 1)),$$

where i is the number of distinct entries in the list $(v_1, v_2, \ldots, v_j)$. That the lowest common multiple (61) runs over all products $(2v_j + 1)(2v_{j+1} + 1)$ of consecutive, distinct vertex types follows from the form (53) of the vertex list and the symmetry (20) of the vertex types. Furthermore, since all distinct vertex types occur within the first k vertex types, the expression (61) implies that the sequence (q_j) is eventually stationary:

$$(62) \qquad\qquad q_j = q, \qquad \mathbb{L}_j^e = \mathbb{L}^e, \qquad k \leq j \leq 2k - 1,$$

where q and $\mathbb{L}^e$ are given by equations (55) and (56).

For given e, the following result details the role of the $\mathbb{L}_j^e$ as cylinder sets of the orbit code. Applying the result for $j = 2k - 1$, along with the observation (62), implies that two points share the same orbit code if and only if they are congruent modulo $\mathbb{L}^e$.

Lemma 11. *Let e be a critical number, let k be the length of the vertex list of the corresponding polygon class, and let p_j and $\mathbb{L}_j^e$ be as above. For any $1 \leq j \leq 2k - 1$ and all $z, \tilde{z} \in \tilde{X}^e$, the following three statements are equivalent:*

  (i) the orbit codes of z and $\tilde{z}$ match up to the jth entry,

(ii) z and $\tilde{z}$ are congruent modulo $\mathbb{L}_j^e$,

(iii) the points $\Psi^j(z)$ and $\Psi^j(\tilde{z})$ are congruent modulo $\lambda p_j \mathbf{e}$, where $\mathbf{e}$ is the unit vector in the direction of the non-integer coordinate of the jth vertex.

PROOF. Let $e \in \mathcal{E}$, let $\mathbb{L}_j^e$ be as above, and let $z, \tilde{z} \in \tilde{X}^e$ with orbit codes $(\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$, and $(\tilde{\sigma}_{-1}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{2k-1})$, respectively. We proceed by induction on j, with two induction hypotheses. Firstly we suppose that (i) is equivalent to (ii), so that for any $1 \leq j \leq 2k-1$:

(H1) $$(\sigma_{-1}, \sigma_1, \ldots, \sigma_j) = (\tilde{\sigma}_{-1}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_j) \quad \Leftrightarrow \quad \tilde{z} \equiv z \quad (\mathrm{mod}\ \mathbb{L}_j^e).$$

Thus any such $\tilde{z}$ is related to z via

(63) $$\tilde{z} = z + \frac{1}{2}\left((2a+b)\mathbf{L}_j - b\mathbf{w}_{v_1,v_1}\right),$$

for $a \in \mathbb{Z}$ and $b \in \{0,1\}$ or $b \in \{0,-1\}$, as appropriate. Secondly, we suppose that (ii) is equivalent to (iii). In particular,

(H2) $$\tilde{z} = z + \frac{1}{2}\left((2a+b)\mathbf{L}_j - b\mathbf{w}_{v_1,v_1}\right) \quad \Leftrightarrow \quad \Psi^j(\tilde{z}) = \Psi^j(z) + \lambda(2a+b)p_j\mathbf{e},$$

where $\mathbf{e}$ is the unit vector in the direction of the non-integer coordinate of that vertex.

We begin with the base case $j = 1$. Suppose that the first vertex of a polygon in class e lies on $y = v_1$, so that y is its integer coordinate (if x is the integer coordinate, then the analysis is identical). By symmetry, the previous vertex lies on $x = v_1$ and its integer coordinate is x. Using the properties of Ψ given in equations (44) and (45), applied to $z \in B_{v_1,v_1} \setminus \Lambda$ and $\Psi^{-1}(z) \in \Lambda_{v_1,v_1}$ respectively, we have

(64) $$\Psi(z) \equiv z \quad (\mathrm{mod}\ \mathbf{w}_{v_1,v_1}),$$

(65) $$\Psi^{-1}(z) + \mathbf{v}(\Psi^{-1}(z)) \equiv z \quad (\mathrm{mod}\ \mathbf{w}_{v_1,v_1}).$$

Furthermore by proposition 10:

$$\mathbf{v}(\Psi^{-1}(z)) = \mathbf{w}_{v_1,v_1} + \lambda\epsilon_{-1}\mathbf{e},$$

where $\mathbf{e} = (0,1)$ is the non-integer coordinate vector for the (-1)th vertex. Thus if $z = \lambda(\lceil v_1/\lambda \rceil + x, \lceil v_1/\lambda \rceil + y)$, by the definition (48) of the orbit code, the x- and y-components of equations (65) and (64), respectively, give us that the first two entries in the orbit code $\sigma(z)$ satisfy

$$x \equiv \sigma_{-1} \quad (\mathrm{mod}\ 2v_1 + 1),$$
$$y \equiv \sigma_1 \quad (\mathrm{mod}\ 2v_1 + 1).$$

It follows that $z, \tilde{z} \in \tilde{X}^e$ share the partial code (σ_{-1}, σ_1) if and only if

$$\tilde{z} \equiv z \quad \left(\mathrm{mod}\ (\lambda(2v_1+1)\mathbb{Z})^2\right).$$

The lattice $\mathbb{L}_1^e$ is given by (cf. (60))

$$\mathbb{L}_1^e = \left\langle \mathbf{L}_1, \frac{1}{2}\left(\mathbf{L}_1 - \mathbf{w}_{v_1,v_1}\right)\right\rangle,$$

where $\mathbf{L}_1 = \lambda p_1(1,1)$, $p_1 = 2v_1 + 1$ and

$$\frac{1}{2}\left(\mathbf{L}_1 - \mathbf{w}_{v_1,v_1}\right) = \frac{\lambda}{2}(p_1 - p_1, p_1 + p_1) = \lambda p_1 \mathbf{e}.$$

Thus $\mathbb{L}_1^e = (\lambda(2v_1+1)\mathbb{Z})^2$ and the first hypothesis holds.

Now let $z, \tilde{z} \in \tilde{X}^e$ satisfy (63) with $j = 1$. If $\Psi(z) = F_\lambda^{4t}(z) = z + t\mathbf{w}_{v_1,v_1}$, where $t \in \mathbb{N}$ is the transit time to Λ, then the identities

$$\tilde{z} + (t + a + b)\mathbf{w}_{v_1,v_1} = \Psi(z) + \frac{1}{2}(2a + b)\left(\mathbf{L}_1 + \mathbf{w}_{v_1,v_1}\right)$$
$$= \Psi(z) + \lambda(2a + b)p_1\mathbf{e}$$

with $\mathbf{e} = (1, 0)$ show that $\tilde{z}$ has transit time $t + a + b$, and therefore $\Psi(\tilde{z}) = \Psi(z) + \lambda(2a+b)p_1\mathbf{e}$, as required (see figure 11). This completes the basis for induction.
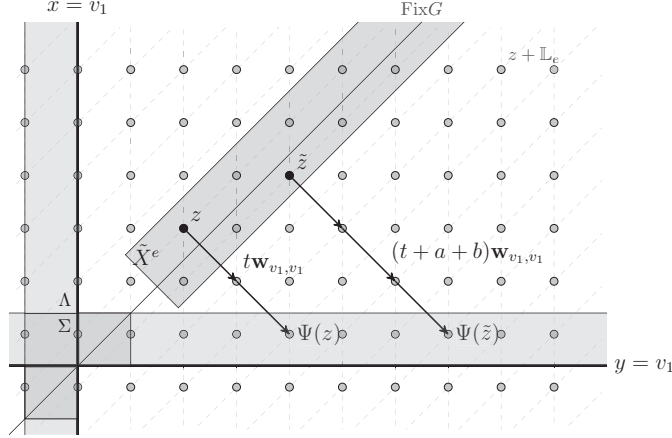

FIGURE 11. The points z, $\tilde{z}$, $\Psi(z)$ and $\Psi(\tilde{z})$.

Now we proceed with the inductive step. Let (H1) and (H2) hold for some $j \geq 1$. Then z and $\tilde{z}$ are related as in equation (63), for some a, b. We think of $\tilde{\sigma}_{j+1}$, the $(j + 1)$th entry of the orbit code of $\tilde{z}$, as a function of (a, b). We suppose that the jth vertex lies on $y = n$ for some $n \in \mathbb{Z}$ (again the case in which the vertex lies on $x = m$ is identical). Let the pair (x_j, y_j) be defined from $\Psi^j(z)$ via equation (47). Similarly, $\Psi^j(\tilde{z})$ defines the pair $(\tilde{x}_j, \tilde{y}_j)$.

By (H2), $\Psi^j(\tilde{z})$ satisfies

$$\Psi^j(\tilde{z}) = \Psi^j(z) + \lambda(2a + b)p_j\mathbf{x},$$

where $\mathbf{x} = \mathbf{e} = (1, 0)$. Combining this expression with equation (52), applied to $\Psi^j(\tilde{z}) \in \Lambda_{v_j,n-1}$, we obtain

$$\Psi^{j+1}(\tilde{z}) = \Psi^j(\tilde{z}) + \lambda\epsilon_j\mathbf{x} + \tilde{t}\mathbf{w}_{v_j,n-1}$$
(66)
$$= \Psi^j(z) + \lambda(2a + b)p_j\mathbf{x} + \lambda\epsilon_j\mathbf{x} + \tilde{t}\mathbf{w}_{v_j,n-1}.$$

where $\tilde{t}$ is the transit time of $\tilde{z}$ to Λ.

There are now two cases to consider.

Case 1: $v_j = v_{j+1}$.

In this case the jth and $(j + 1)$th vertices lie on parallel lines, which we take to be $y = n$ and $y = n - 1$, so $\Psi^{j+1}(z)$ is given by

$$\Psi^{j+1}(z) = \lambda\left(\left\lceil\frac{v_j}{\lambda}\right\rceil + x_{j+1}, \left\lceil\frac{n - 1}{\lambda}\right\rceil + y_{j+1}\right),$$

and similarly for $\Psi^{j+1}(\tilde{z})$. According to the definitions (58) and (60), we have $p_j = p_{j+1}$ and $\mathbb{L}_j^e = \mathbb{L}_{j+1}^e$. Thus, to show that (H1) continues to hold, we need to show that $\tilde{\sigma}_{j+1} = \sigma_{j+1}$ for all (a, b). Similarly we need to show that the vector $\Psi^{j+1}(\tilde{z}) - \Psi^{j+1}(z)$ is equal to the vector $\Psi^j(\tilde{z}) - \Psi^j(z)$ of hypothesis (H2).

Because y is the integer coordinate of both jth and $(j + 1)$th vertices, the transit time is the same for the orbits of z and $\tilde{z}$. Therefore equations (52) and (66) with $\tilde{t} = t$ give us that

$$\Psi^{j+1}(\tilde{z}) = \Psi^j(z) + \lambda(2a + b)p_j\mathbf{x} + \lambda\epsilon_j\mathbf{x} + t\mathbf{w}_{v_j, n-1},$$
$$= \Psi^{j+1}(z) + \lambda(2a + b)p_j\mathbf{x},$$

and the second hypothesis (H2) remains satisfied. Furthermore, $\Psi^{j+1}(\tilde{z})$ and $\Psi^{j+1}(z)$ have the same integer (y) coordinate. It follows that, by the definition (48) of the orbit code, $\tilde{\sigma}_{j+1} = \sigma_{j+1}$ and (H1) is also satisfied.

By the y-component of (66), the value of σ_{j+1} is determined explicitly by the congruences

$$(67) \qquad \left\lceil \frac{n-1}{\lambda} \right\rceil + \sigma_{j+1} \equiv \left\lceil \frac{n}{\lambda} \right\rceil + \sigma_j \pmod{2v_j + 1}.$$

Equation (67) shows that, if $v_j = v_{j+1}$, then there is a map $\sigma_j \mapsto \sigma_{j+1}$, which is a bijection of a set of congruence classes.

Case 2: $v_j \neq v_{j+1}$.

In this case the jth and $(j + 1)$th vertices lie on perpendicular lines. We take these to be the lines $y = n$ and $x = v_j + 1$, respectively, so that $v_{j+1} = n - 1$ and $\Psi^{j+1}(z)$ is given by

$$\Psi^{j+1}(z) = \lambda \left(\left\lceil \frac{v_j + 1}{\lambda} \right\rceil + x_{j+1}, \left\lceil \frac{n-1}{\lambda} \right\rceil + y_{j+1} \right).$$

(If x is the integer co-ordinate, then the analysis is identical.) We shall demonstrate the form of $\mathbb{L}_{j+1}^e$ by identifying those pairs (a, b) for which $\tilde{\sigma}_{j+1} = \sigma_{j+1}$.

Taking the x-coordinate of equation (66), and recalling the explicit form (50) of the orbit code, we see that $\tilde{\sigma}_{j+1}$ is determined by

$$(68) \qquad \left\lceil \frac{v_j}{\lambda} \right\rceil + x_j + (2a + b)p_j + \epsilon_j + \tilde{t}(2v_{j+1} + 1) = \left\lceil \frac{v_j + 1}{\lambda} \right\rceil + \tilde{\sigma}_{j+1} - (2v_{j+1} + 1).$$

We think of this as an integer equation of the form $A(2a + b) + B\tilde{t} = C$, which has solutions $2a + b \in \mathbb{Z}$ and $\tilde{t} \in \mathbb{N}$ for some given value of $\tilde{\sigma}_{j+1}$ if and only if

$$C = \left\lceil \frac{v_j + 1}{\lambda} \right\rceil + \tilde{\sigma}_{j+1} - (2v_{j+1} + 1) - \left\lceil \frac{v_j}{\lambda} \right\rceil - x_j - \epsilon_j$$

is sufficiently large and $C \equiv 0 \pmod{\gcd(A, B)}$, i.e., if λ is sufficiently small and $\tilde{\sigma}_{j+1}$ satisfies the congruence

$$(69) \qquad \tilde{\sigma}_{j+1} \equiv \left\lceil \frac{v_j}{\lambda} \right\rceil + x_j + \epsilon_j - \left\lceil \frac{v_j + 1}{\lambda} \right\rceil \pmod{\gcd(p_j, 2v_{j+1} + 1)}.$$

To find the lattice $\mathbb{L}_{j+1}^e$, we need to solve this equation in the case $\tilde{\sigma}_{j+1} = \sigma_{j+1}$.

By assumption, the point z, given by the module coordinates $a = b = 0$, corresponds to the solution $2a + b = 0$, $\tilde{t} = t$, for some transit time $t \in \mathbb{N}$. Hence the general solution of (68)

is given by

$$\tilde{t} = t - s \frac{p_j}{\gcd(p_j, 2v_{j+1} + 1)}, \tag{70}$$

$$2a + b = s \frac{2v_{j+1} + 1}{\gcd(p_j, 2v_{j+1} + 1)}, \tag{71}$$

for $s \in \mathbb{Z}$. The second of these equations implies that s must have the same parity as $2a + b$, so we can write $s = 2\tilde{a} + b$, where $\tilde{a} \in \mathbb{Z}$ and $b \in \{0, \pm 1\}$ for an appropriate choice of sign. Substituting this expression into equation (63), the points $\tilde{z}$ for which $\tilde{\sigma}_{j+1} = \sigma_{j+1}$ are given by

$$\tilde{z} = z + \frac{1}{2} \left(s \frac{2v_{j+1} + 1}{\gcd(p_j, 2v_{j+1} + 1)} \mathbf{L}_j - b\mathbf{w}_{v_1, v_1} \right)$$

$$= z + \frac{1}{2} \left((2\tilde{a} + b)\mathbf{L}_{j+1} - b\mathbf{w}_{v_1, v_1} \right).$$

The last equality is justified by the identities

$$\frac{2v_{j+1} + 1}{\gcd(p_j, 2v_{j+1} + 1)} \mathbf{L}_j = \frac{(2v_{j+1} + 1)(2v_j + 1)}{\gcd((2v_j + 1)p_j, (2v_{j+1} + 1)(2v_j + 1))} \frac{\lambda q_j}{2v_1 + 1} (1, 1)$$

$$= \frac{\lambda q_{j+1}}{2v_1 + 1} (1, 1) = \mathbf{L}_{j+1}$$

where we have used the relationship $\mathrm{lcm}(a, b) = ab/\gcd(a, b)$. Therefore the first hypothesis (H1) remains satisfied.

Substituting the general solution (70) and (71) into equation (66) with $\mathbf{x} = \mathbf{e} = (1, 0)$, and using equation (52), we find

$$\Psi^{j+1}(\tilde{z}) = \Psi^j(z) + \lambda(2a + b)p_j\mathbf{x} + \lambda\epsilon_j\mathbf{x} + \tilde{t}\mathbf{w}_{v_j, v_{j+1}}$$

$$= \Psi^{j+1}(z) + \frac{sp_j}{\gcd(p_j, 2v_{j+1} + 1)} \left(\lambda(2v_{j+1} + 1)\mathbf{x} - \mathbf{w}_{v_j, v_{j+1}} \right)$$

$$= \Psi^{j+1}(z) + \frac{\lambda(2\tilde{a} + b)p_j}{\gcd(p_j, 2v_{j+1} + 1)} (2v_j + 1)\mathbf{y}$$

$$= \Psi^{j+1}(z) + \lambda(2\tilde{a} + b)p_{j+1}\mathbf{y}$$

where $\mathbf{y} = (0, 1)$. Thus the points where these $\tilde{z}$ meet the $(j + 1)$th vertex share the same integer coordinate. So hypothesis (H2) also remains satisfied, completing the induction.

Thus hypotheses (H1) and (H2) hold for all $1 \le j \le 2k - 1$, and the equivalence of the three statements follows. $\qquad\square$

Corollary 12. *Let e be a critical number, let k be the length of the vertex list of the corresponding polygon class, and let j be in the range $1 \le j \le 2k - 1$. Then two points z and $\tilde{z}$ in $\tilde{X}^e$ have the same orbit code if and only if the points $\Psi^j(z)$ and $\Psi^j(\tilde{z})$ are congruent modulo $\lambda q/(2v_j + 1)\mathbf{e}$, where $\mathbf{e}$ is the unit vector in the direction of the non-integer coordinate of the jth vertex.*

PROOF. Recall that for all $1 \le j \le 2k - 1$:

$$\mathbb{L}^e \subseteq \mathbb{L}_j^e.$$

Thus any two points which are congruent modulo $\mathbb{L}^e$ are also congruent modulo $\mathbb{L}^e_j$. In particular, if

$$\tilde{z} = z + \frac{1}{2}\left((2a+b)\mathbf{L} - b\mathbf{w}_{v_1,v_1}\right),$$

$$= z + \frac{1}{2}\left((2a+b)\frac{q}{q_j}\mathbf{L}_j - b\mathbf{w}_{v_1,v_1}\right),$$

then by the hypothesis (H2) of lemma 11 we have:

$$\Psi^j(\tilde{z}) = \Psi^j(z) + \lambda\left((2a+b)\frac{q}{q_j}\right)p_j\,\mathbf{e}$$

$$= \Psi^j(z) + \lambda(2a+b)\frac{q}{2v_j+1}\,\mathbf{e}$$

as required. $\qquad\square$

Lemma 11 shows the equivalence between orbit codes and congruence classes of $\mathbb{L}^e$. To complete the proof of theorem A, we show that the orbit code $\sigma(z)$ determines uniquely the behaviour of z under the return map Φ.

PROOF OF THEOREM A

Consider two points $z, z+l \in \tilde{X}^e$ for some $l \in \mathbb{L}^e$ given by

$$l = \frac{1}{2}\left((2a+b)\mathbf{L} - b\mathbf{w}_{v_1,v_1}\right).$$

These two points have the same orbit code and reach the $(2k-1)$th vertex at the points $\Psi^{2k-1}(z), \Psi^{2k-1}(z+l) \in \Lambda_{v_1,-(v_1+1)}$, which, by corollary 12, are congruent modulo $\lambda q/(2v_{2k-1}+1)\mathbf{e}$, where $\mathbf{e}$ is the unit vector in the non-integer direction. In particular, $\Psi^{2k-1}(z)$ and $\Psi^{2k-1}(z+l)$ are related via

$$\Psi^{2k-1}(z+l) = \Psi^{2k-1}(z) + \lambda(2a+b)p_{2k-1}\mathbf{e},$$

(72)
$$= \Psi^{2k-1}(z) + \lambda(2a+b)\frac{q}{(2v_1+1)}\,\mathbf{e},$$

where we have replaced v_{2k-1} by v_1 using the symmetry (20) of the vertex types. We will show that the points where they reach the last vertex are related by a similar equation:

(73)
$$(\Psi^{2k-1}\circ F_\lambda)(z+l) = (\Psi^{2k-1}\circ F_\lambda)(z) + \lambda(2a+b)\frac{q}{(2v_1+1)}\,\mathbf{e}^\perp,$$

where the unit vector $\mathbf{e}^\perp$, the non-integer direction of the last vertex, is perpendicular to $\mathbf{e}$.

The last vertex of the return orbit $\mathcal{O}_\tau(z)$ lies in the set Λ_{v_1,v_1}, so must be close to the image under F_λ of the $(2k-1)$th vertex (see figure 12). If the $(2k-1)$th vertex lies on the line $x = v_1 + 1$, it is a simple exercise to show that these two points are in fact equal, i.e., that $(\Psi^{2k-1}\circ F_\lambda)(z) = (F_\lambda\circ\Psi^{2k-1})(z)$ for any $z \in \tilde{X}^e$. We consider the less obvious case in which the $(2k-1)$th vertex lies on the line $y = -v_1$ and the non-integer direction is $\mathbf{e} = (1,0)$. By the orientation of the vector field in the fourth quadrant, the orbit of the point z reaches this vertex at the point $\Psi^{2k-1}(z)$ given by:

(74)
$$\Psi^{2k-1}(z) = \lambda\left(\left\lceil\frac{v_1}{\lambda}\right\rceil + x_{2k-1}, \left\lceil\frac{-v_1}{\lambda}\right\rceil + y_{2k-1}\right),$$

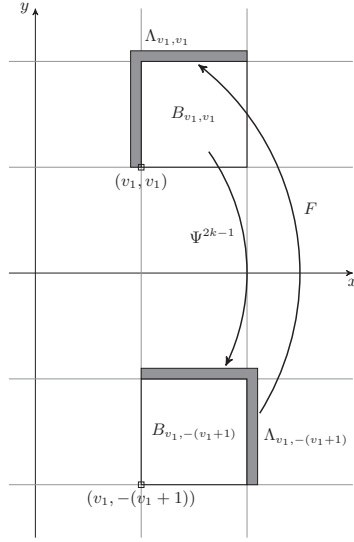where $x_{2k-1} \geq 0$ and $0 \leq y_{2k-1} < 2v_1 + 1$.

FIGURE 12. The $(2k-1)$th and last vertices, joined by the action of F_λ.

Applying F_λ to (72) and substituting the expression (74), we get:

$$(F_\lambda \circ \Psi^{2k-1})(z+l) = (F_\lambda \circ \Psi^{2k-1})(z) + \lambda(2a+b)\frac{(2v_k+1)\,p}{(2v_1+1)}\,\mathbf{e}^\perp$$

$$= \lambda\left(v_1 - \left\lceil\frac{-v_1}{\lambda}\right\rceil - y_{2k-1}, \left\lceil\frac{v_1}{\lambda}\right\rceil + x_{2k-1}\right) + \lambda(2a+b)\frac{(2v_k+1)\,p}{(2v_1+1)}\,\mathbf{e}^\perp,$$

where the non-integer direction of the last vertex is $\mathbf{e}^\perp = (0,1)$. By equation (49), if the first component of this point satisfies

$$-(2v_1+1) \le \left(v_1 - \left\lceil\frac{-v_1}{\lambda}\right\rceil - y_{2k-1}\right) - \left\lceil\frac{v_1}{\lambda}\right\rceil < 0,$$

then the point is the last vertex on $x = v_1$ that we seek for all (a,b):

(75) $(\Psi^{2k-1} \circ F_\lambda)(z+l) = (F_\lambda \circ \Psi^{2k-1})(z+l).$

If the above inequality is not satisfied, then it must be the upper bound that fails; since v_1 is non-negative, y_{2k-1} satisfies $0 \le y_{2k-1} < 2v_1 + 1$, and the absolute value of the two ceiling functions differ by at most one, the lower bound must hold. In this case $(F_\lambda \circ \Psi^{2k-1})(z+l) \in B_{v_1,v_1}$, and we apply F_λ^{-4} to find:

$$(\Psi^{2k-1} \circ F_\lambda)(z+l) = (F_\lambda^{-3} \circ \Psi^{2k-1})(z+l)$$

$$= (F_\lambda \circ \Psi^{2k-1})(z+l) - \mathbf{v}((F_\lambda^{-3} \circ \Psi^{2k-1})(z+l))$$

(76) $$= (F_\lambda \circ \Psi^{2k-1})(z+l) - \mathbf{w}_{v_1,v_1} - \lambda\epsilon\,\mathbf{e}^\perp,$$

where the error term ϵ is independent of (a,b) by proposition 10. In both cases (75) and (76) the relationship (73) follows.

Using (73), the property (45) of Ψ, and the expression (46) for Φ, we obtain

$$\Phi(z+l) \equiv (\Psi^{2k-1} \circ F_\lambda)(z+l) + \mathbf{v}((\Psi^{2k-1} \circ F_\lambda)(z+l)) \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv (\Psi^{2k-1} \circ F_\lambda)(z) + \lambda(2a+b)\frac{q}{(2v_1+1)}\mathbf{e}^\perp + \mathbf{v}((\Psi^{2k-1} \circ F_\lambda)(z)) \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv \Phi(z) + \lambda(2a+b)\frac{q}{(2v_1+1)}\mathbf{e}^\perp \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv \Phi(z) + \frac{\lambda}{2}(2a+b)\frac{q}{(2v_1+1)}\left((\mathbf{e}^\perp + \mathbf{e}) + (\mathbf{e}^\perp - \mathbf{e})\right) \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv \Phi(z) + \frac{1}{2}(2a+b)\left(\mathbf{L} \pm \frac{q}{(2v_1+1)^2}\mathbf{w}_{v_1,v_1}\right) \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv \Phi(z) + \frac{1}{2}\left((2a+b)\mathbf{L} - b\mathbf{w}_{v_1,v_1}\right) \quad (\text{mod } \mathbf{w}_{v_1,v_1})$$

$$\equiv \Phi(z) + l \quad (\text{mod } \mathbf{w}_{v_1,v_1}),$$

where we have also used the fact that $(\mathbf{e}^\perp + \mathbf{e}) = (1,1)$, $(\mathbf{e}^\perp - \mathbf{e}) = \pm(1,-1)$, and that $q/(2v_1+1)^2$ is odd. This completes the proof of theorem A. $\qquad\square$

The set Θ^e of possible orbit codes is a subset of the product space

$$\Theta^e \subseteq \{0,1,\ldots,2v_1\} \times \prod_{j=1}^{2k-1} \{0,1,\ldots,2v_j\}.$$

Denoting by $\#\mathbb{L}^e$ the scaled co-volume of $\mathbb{L}^e$, namely

$$(77) \qquad\qquad\qquad \#\mathbb{L}^e = |(\lambda\mathbb{Z})^2/\mathbb{L}^e|,$$

the total number of possible orbit codes is given by

$$(78) \qquad\qquad |\Theta^e| = \#\mathbb{L}^e = -\frac{1}{2\lambda^2}\det(\mathbf{L}, \mathbf{w}_{v_1,v_1}) = q.$$

We note that although the lattice $\mathbb{L}^e$ is independent of λ (up to scaling), Θ^e is not.

In the next lemma, we identify the orbit codes which correspond to symmetric fixed points of Φ. Subsequently, in lemmas 14 and 15, we identify values of e for which the number of codes which satisfy the conditions of lemma 13 is independent of λ. The proof of theorem B will then follow.

Lemma 13. *For any $e \in \mathcal{E}$ with vertex list $(v_1,\ldots,v_k)$, $z \in \tilde{X}^e$ and sufficiently small λ, the point z is a symmetric fixed point of Φ if and only if its orbit code $\sigma(z) = (\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$ satisfies:*

$$(i) \quad \sigma_{-1} = \sigma_1, \qquad\qquad (ii) \quad 2\sigma_k \equiv \lfloor\sqrt{e}\rfloor \quad (\text{mod } 2\lfloor\sqrt{e}\rfloor + 1).$$

PROOF. It is a standard property of any reversible map F, with $F = G \circ F^{-1} \circ G$, that it can be written as the composition of two involutions:

$$F = H \circ G \qquad\quad H = F \circ G \qquad\quad G^2 = H^2 = \text{Id}.$$

It follows that every symmetric periodic orbit of F intersects the union of the fixed spaces of these involutions, $\text{Fix } G \cup \text{Fix } H$, at exactly two points (which coincide for period 1) [9]. Furthermore these two points are maximally separated in time, in the sense that if the minimal

period of the orbit is s, then the transit time from one to the other is approximately $s/2$. More precisely, if s is even, the orbit must intersect one of the following sets:

$$\text{(79)} \qquad \text{Fix}\,G \cap F^{s/2}(\text{Fix}\,G) \qquad \text{or} \qquad \text{Fix}\,H \cap F^{s/2}(\text{Fix}\,H),$$

whereas if s is odd, the orbit must intersect the set

$$\text{(80)} \qquad \text{Fix}\,G \cap F^{(s-1)/2}(\text{Fix}\,H).$$

For our map F_λ, we have already introduced the involution G and its fixed space, considered now as a subset of the rescaled lattice $(\lambda\mathbb{Z})^2$:

$$G(x,y) = (y,x) \qquad \text{Fix}\,G = \{\lambda(x,y) \in (\lambda\mathbb{Z})^2 : x = y\}.$$

A simple calculation shows that the involution H and its fixed space are given by:

$$\text{(81)} \qquad H(x,y) = (\lfloor \lambda y \rfloor - x, y) \qquad \text{Fix}\,H = \{\lambda(x,y) \in (\lambda\mathbb{Z})^2 : 2x = \lfloor \lambda y \rfloor\}.$$

Take $e \in \mathscr{E}$ and a point $z \in \tilde{X}^e$. Suppose that z is a symmetric fixed point of Φ. If z is non-zero, the orbit of z intersects the set $\text{Fix}\,G \cup \text{Fix}\,H$ at exactly two points, and as z is a fixed point of Φ, these two points must occur within a single revolution. Hence we have:

$$|\mathcal{O}_\tau(z) \cap (\text{Fix}\,G \cup \text{Fix}\,H)| = 2.$$

We begin by considering which points in the return orbit $\mathcal{O}_\tau(z)$ can lie in $\text{Fix}\,G$. Since the domain $\tilde{X}^e$ lies in an $O(\lambda)$-neighbourhood of the positive half of the symmetry line $\text{Fix}\,G$, we may have:

$$z = \Phi(z) \in \text{Fix}\,G.$$

Equally, the orbit may intersect the negative half of the symmetry line, which occurs if:

$$F^4(z) = G(z) \qquad \Leftrightarrow \qquad F^2(z) \in \text{Fix}\,G.$$

Points in $\text{Fix}\,H$ lie on disjoint vertical line segments of length one, in an $O(\lambda)$-neighbourhood of the y-axis. Recall that the polygon $\Pi(z)$ intersects the axes at vertices of type $v_k = \lfloor \sqrt{e} \rfloor$, and hence intersects the y-axis in the boxes:

$$B_{0,\lfloor \sqrt{e} \rfloor} \qquad \text{and} \qquad B_{0,-(\lfloor \sqrt{e} \rfloor + 1)}.$$

If $\lfloor \sqrt{e} \rfloor$ is even, it follows that the relevant segment H_+^e of $\text{Fix}\,H$ is given by:

$$H_+^e = \{\lambda(x,y) \in (\lambda\mathbb{Z})^2 : 2x = \lfloor \lambda y \rfloor = \lfloor \sqrt{e} \rfloor\},$$

which lies in the positive half-plane. Similarly if $\lfloor \sqrt{e} \rfloor$ is odd, the relevant segment H_-^e of $\text{Fix}\,H$ is given by:

$$H_-^e = \{\lambda(x,y) \in (\lambda\mathbb{Z})^2 : 2x = \lfloor \lambda y \rfloor = -(\lfloor \sqrt{e} \rfloor + 1)\},$$

which lies in the negative half-plane.

Hence we see that the return orbit of z cannot intersect $\text{Fix}\,H$ twice in a single revolution, and intersects $\text{Fix}\,G$ twice if and only if:

$$z \in \text{Fix}\,G \cap F^{-2}(\text{Fix}\,G).$$

By (79), the latter implies that z is periodic with period 4, and we have already observed that there are no points with minimal period 4. Thus the only non-trivial possibility for a symmetric fixed point occurs when the return orbit of z intersects both $\text{Fix}\,G$ and $\text{Fix}\,H$. Conversely, equation (80) ensures that this is also a sufficient condition.

The proof now proceeds in two parts.

(i) $\mathcal{O}_\tau(z)$ intersects $\mathrm{Fix}\,G$ if and only if $\sigma_{-1} = \sigma_1$.

If $z = \lambda(x, y)$, then the property $\sigma_{-1} = \sigma_1$ is satisfied if and only if:

$$x \equiv y \pmod{2v_1 + 1}.$$

If $y = x$ then clearly $z \in \mathrm{Fix}\,G$. The width of the strip $\tilde{X}^e$, given by (38), ensures that the only other possibility is $y = x + (2v_1 + 1)$, in which case:

$$
\begin{aligned}
F^4(z) &= z + \mathbf{w}_{v_1, v_1} \\
&= \lambda(x + (2v_1 + 1), y - (2v_1 + 1)) \\
&= \lambda(y, x) = G(z).
\end{aligned}
$$

This corresponds to $F^2(z) \in \mathrm{Fix}\,G$.

(ii) $\mathcal{O}_\tau(z)$ intersects $\mathrm{Fix}\,H$ if and only if $2\sigma_k \equiv \lfloor\sqrt{e}\rfloor \pmod{2\lfloor\sqrt{e}\rfloor + 1}$.

Instead of considering the sets H^e_+ and H^e_- directly, we consider their images under G and F_λ, respectively, which lie in a neighbourhood of the x-axis:

$$
\tag{82} G(H^e_+) = \{\lambda(x, y) \in (\lambda\mathbb{Z})^2 : \ 2y = \lfloor\lambda x\rfloor = \lfloor\sqrt{e}\rfloor\},
$$

$$
\tag{83} F_\lambda(H^e_-) = \{\lambda(x, y) \in (\lambda\mathbb{Z})^2 : \ 2y = \lfloor-\lambda(x+1)\rfloor = -(\lfloor\sqrt{e}\rfloor + 1)\}.
$$

In (83), we assume that $\lambda(\lfloor\sqrt{e}\rfloor + 1)/2 < 1$, so that $F_\lambda(w) = \lambda(-1 - y, x)$ for all $w = \lambda(x, y) \in H^e_-$. The orbit $\mathcal{O}_\tau(z)$ intersects $\mathrm{Fix}\,H$ if and only if it intersects the relevant one of these sets, according to the parity of $\lfloor\sqrt{e}\rfloor$.

The polygon $\Pi(z)$ intersects the x-axis at the kth vertex, where k is the length of the vertex list $V(e)$. The return orbit $\mathcal{O}_\tau(z)$ reaches the kth vertex at the point $\Psi^k(z)$, given in the notation of (47) by

$$
\Psi^k(z) = \lambda\left(\left\lceil\frac{\lfloor\sqrt{e}\rfloor}{\lambda}\right\rceil + x_k,\, y_k\right),
$$

where, by (50), $y_k = \sigma_k$ is non-negative. Hence if $\lfloor\sqrt{e}\rfloor$ is even, $\mathcal{O}_\tau(z)$ intersects $\mathrm{Fix}\,H$ if and only if:

$$
\Psi^k(z) \in G(H^e_+) \qquad \Leftrightarrow \qquad \sigma_k = \lfloor\sqrt{e}\rfloor/2.
$$

If $\lfloor\sqrt{e}\rfloor$ is odd, then $\mathcal{O}_\tau(z)$ intersects $\mathrm{Fix}\,H$ if and only if:

$$
\begin{aligned}
F^4_\lambda(\Psi^k(z)) \in F_\lambda(H^e_-) \qquad \Leftrightarrow \qquad \sigma_k &= -(\lfloor\sqrt{e}\rfloor + 1)/2 + (2v_k + 1) \\
&= (3\lfloor\sqrt{e}\rfloor + 1)/2.
\end{aligned}
$$

The congruence $2\sigma_k \equiv \lfloor\sqrt{e}\rfloor \pmod{2\lfloor\sqrt{e}\rfloor + 1}$ covers both of these cases, which completes the proof. $\qquad\square$

For all $e \in \mathcal{E}$ and sufficiently small λ, the set $\tilde{X}^e$ —see equation (38)— is non-empty and contains at least one element from every congruence class modulo $\mathbb{L}^e$. We now seek to identify the number of congruence classes whose orbit code satisfies the conditions of lemma 13.

As discussed in the proof of lemma 13, the points $z = \lambda(x, y) \in \tilde{X}^e$ whose orbit code $\sigma(z) = (\sigma_{-1}, \sigma_1, \dots, \sigma_{2k-1})$ satisfies $\sigma_{-1} = \sigma_1$ are precisely those satisfying

$$y = x \qquad \text{or} \qquad y = x + (2v_1 + 1).$$

All such points lie on one of two lines, parallel to the first generator $\mathbf{L}$ of the lattice $\mathbb{L}^e$. Furthermore all points on one line are congruent to those on the other, as they are connected

by the second generator $(\mathbf{L} - \mathbf{w}_{v_1,v_1})/2$. Hence the number of points satisfying this condition modulo $\mathbb{L}^e$ is

$$(84) \qquad |\mathbf{L}| = \frac{\#\mathbb{L}^e}{2v_1 + 1} = \frac{q}{2v_1 + 1},$$

where we have used the expression (78) for $\#\mathbb{L}^e$.

It remains to determine what fraction of the orbit codes with $\sigma_{-1} = \sigma_1$ satisfy the second condition of lemma 13. We do this by identifying values of e for which all possible values of σ_k occur with equal frequency, independently of λ.

Clearly if $k = 1$, i.e., if a polygon class has just one vertex in the first octant ($e = 0$ – a square), then the points $z = \lambda(x, y) \in \tilde{X}^e$ with $\sigma_k = \sigma_1 = \sigma^*$ satisfy

$$x \equiv y \equiv \sigma^* \pmod{2v_1 + 1},$$

for any given $\sigma^* \in \{0, 1, \ldots, 2v_1\}$. Such points form a fraction

$$\frac{q}{(2v_1 + 1)^2}$$

of all points modulo $\mathbb{L}^e$. Hence all possible values of σ_k occur with equal frequency modulo $\mathbb{L}^e$. More generally if $v_k = v_1$, i.e., if all vertices of the polygon class have the same type ($e = 0, 2, 8$), then the same applies. This follows from the fact that, for any congruence class of $\mathbb{L}^e$, the map $\sigma_j \mapsto \sigma_{j+1}$ is a permutation of the set $\{0, 1, \ldots, 2v_j\}$ whenever $v_j = v_{j+1}$, as we saw in case 1 of the proof of lemma 11.

The following lemma deals with the case that a polygon class has two or more distinct vertex types.

Lemma 14. *Let $e \in \mathscr{E}$. Suppose that the vertex list $(v_1, v_2, \ldots, v_k)$ of the associated polygon class has at least two distinct entries and satisfies*

$$(85) \qquad \gcd(2v_{\iota(l)} + 1, p_{\iota(l-1)}) = 1,$$

where $(\iota(i))_{i=1}^l$ is the sequence of distinct vertex types defined in (53). Then for every $z \in \tilde{X}^e$, all $1 \leq j < \iota(l)$, all $\sigma^* \in \{0, 1, \ldots, 2v_k\}$, and all sufficiently small λ, the number of points in the set $(z + \mathbb{L}_j^e)/\mathbb{L}^e$ whose orbit code has kth entry σ^* is

$$(86) \qquad \frac{1}{2v_k + 1} |\mathbb{L}_j^e/\mathbb{L}^e|.$$

PROOF. Pick $e \in \mathscr{E}$ such that the coprimality condition (85) holds. Let $z \in \tilde{X}^e$ have orbit code $\sigma(z) = (\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$ and let the pair (x_j, y_j) be defined as in equation (47), where $1 \leq j < \iota(l)$.

We have to show that all possible values of σ_k occur with equal frequency among points in $z + \mathbb{L}_j^e$ modulo $\mathbb{L}^e$. It suffices to prove that the expression (86) holds for $j = \iota(l) - 1$, since all cylinder sets of $\mathbb{L}_j^e$ with index $j < \iota(l) - 1$ can be written as a union of cylinder sets of $\mathbb{L}_{\iota(l)-1}^e$.

We let $j = \iota(l) - 1$, so that

$$v_{j+1} = v_{\iota(l)} = v_k,$$

and consider the congruence class of z modulo $\mathbb{L}_j^e$. Since $z + \mathbb{L}_j^e$ is a cylinder set in the sense of lemma 11, the orbit codes of all points $\tilde{z} \in z + \mathbb{L}_j^e$ match up to the jth entry σ_j. Let the $(j+1)$th entry of the orbit code of such a $\tilde{z}$ be $\tilde{\sigma}_{j+1}$. We wish to show that all possible values of $\tilde{\sigma}_{j+1}$ occur with equal frequency.

By construction $v_j \neq v_{j+1}$, so the possible values of $\tilde{\sigma}_{j+1}$ are determined by case 2 of the proof of lemma 11. In the course of the proof, we saw that the occurrence of points $\tilde{z}$ with some fixed value of $\tilde{\sigma}_{j+1}$ correspond to solutions of an integer equation, given in the case where y is the non-integer coordinate of the jth vertex by equation (68). (A similar equation holds when x is the non-integer coordinate.) Each solution $(2a+b, \tilde{t}) \in \mathbb{Z} \times \mathbb{N}$ determines the module coordinates (a, b) of $\tilde{z} - z$ in $\mathbb{L}_j^e$ and the transit time $\tilde{t}$ of $\tilde{z}$ from the jth vertex to the $(j+1)$th.

Solutions of (68) occur for all values of $\tilde{\sigma}_{j+1}$ satisfying the congruence (69), and the condition that λ be sufficiently small ensures that at least one such solution is realised by a point $\tilde{z} \in \tilde{X}^e$. By construction, each distinct value of $\tilde{\sigma}_{j+1}$ which has a solution defines a unique point in $z + \mathbb{L}_j^e$ modulo $\mathbb{L}_{j+1}^e$, which is isomorphic to the module $\mathbb{L}_j^e / \mathbb{L}_{j+1}^e$. However due to the coprimality condition (85), the modulus of the congruence (69) is unity. Hence solutions occur for all possible values of $\tilde{\sigma}_{j+1}$, and each corresponds to a unique congruence class of $z + \mathbb{L}_j^e$ modulo $\mathbb{L}_{j+1}^e$. Furthermore, by (62), the lattices $\mathbb{L}_{j+1}^e$ and $\mathbb{L}^e$ are equal, hence all possible values of $\tilde{\sigma}_{j+1}$ occur with equal frequency in $z + \mathbb{L}_j^e$ modulo $\mathbb{L}^e$.

If $j+1 = \iota(l) = k$ then this completes the proof. If $\iota(l) < k$, take i in the range $\iota(l) \leq i < k$. By the definition of $\iota(l)$ as the index of the last distinct vertex type, we have $v_i = v_{i+1} = v_k$. As discussed above, the map $\sigma_i \mapsto \sigma_{i+1}$ is a permutation of the set $\{0, 1, \ldots, 2v_i\}$ whenever $v_i = v_{i+1}$. Hence the equal frequency of the possible values of $\tilde{\sigma}_i$ implies that of $\tilde{\sigma}_{i+1}$ and the result follows. $\qquad \square$

In the previous section (equation (48)), we defined the jth entry σ_j of the orbit code via the congruence $\sigma_j \equiv y_j \pmod{2v_j + 1}$, where y is the integer coordinate of the relevant vertex, and the pair (x_j, y_j) is defined by equation (47). Similarly, we define the sequence $\gamma(z)$ such that its jth entry γ_j satisfies:

$$(87) \qquad \qquad \gamma_j \equiv x_j \quad \left(\mathrm{mod} \; \frac{q}{2v_j + 1} \right),$$

where x is the non-integer coordinate of the vertex. It follows from corollary 12 that, for any $1 \leq j \leq 2k-1$, and any two points $z, \tilde{z} \in \tilde{X}^e$:

$$\tilde{z} \equiv z \pmod{\mathbb{L}^e} \qquad \Leftrightarrow \qquad (\sigma_j, \gamma_j) = (\tilde{\sigma}_j, \tilde{\gamma}_j).$$

In the following lemma, we use $\gamma(z)$ to identify polygon classes where, among points with $\sigma_{-1} = \sigma_1$ and for each j in $1 \leq j \leq 2k-1$, all possible values of $\sigma_j \in \{0, 1, \ldots, 2v_j\}$ occur with equal frequency modulo $\mathbb{L}^e$, independently of λ.

Lemma 15. *Let $e \in \mathscr{E}$ and suppose that the vertex list $(v_1, v_2, \ldots, v_k)$ of the associated polygon class is such that $2v_1 + 1$ is coprime to $2v_j + 1$ for all other vertex types v_j:*

$$(88) \qquad \qquad \gcd(2v_1 + 1, 2v_j + 1) = 1 \qquad 2 \leq j \leq k, \; v_j \neq v_1.$$

Then for sufficiently small λ, for all j in $1 \leq j \leq 2k-1$, and all $\sigma^* \in \{0, 1, \ldots, 2v_j\}$, the number n_j of points $z \in \tilde{X}^e$ modulo $\mathbb{L}^e$ whose orbit code $(\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$ has $\sigma_{-1} = \sigma_1$ and $\sigma_j = \sigma^*$ is given by:

$$(89) \qquad \qquad n_j = \frac{\#\mathbb{L}^e}{(2v_1 + 1)(2v_j + 1)}.$$

PROOF. We use induction on j. Consider points z whose orbit code satisfies $\sigma_{-1} = \sigma_1$ and has jth value σ_j, for some arbitrary $\sigma_j \in \{0, 1, \ldots, 2v_j\}$ and $j \in \{1, \ldots, 2k-1\}$. Let the sequence $\gamma(z)$ be denoted $(\gamma_{-1}, \gamma_1, \ldots, \gamma_{2k-1})$. Our induction hypotheses are that:

(i) equation (89) holds, where the coprimality condition (88) ensures that n_j is a natural number;

(ii) for each residue $r \in \{0, 1, \ldots, n_j - 1\}$ modulo n_j, there is a unique z modulo $\mathbb{L}^e$ satisfying

$$\gamma_j \equiv r \pmod{n_j}.$$

The base case is $j = 1$. The points $z = \lambda(x, y) \in \tilde{X}^e$ with $\sigma_{-1} = \sigma_1$ for some fixed value of σ_1 satisfy:

$$x \equiv y \equiv \sigma_1 \pmod{2v_1 + 1}.$$

Such points are congruent modulo $(\lambda(2v_1 + 1)\mathbb{Z})^2$, hence the number of such points modulo $\mathbb{L}^e$ is

$$\frac{\#\mathbb{L}^e}{(2v_1 + 1)^2} = \frac{q}{(2v_1 + 1)^2} = n_1.$$

By lemma 11, if z is one such point, then any other point $\tilde{z}$ reaches the first vertex at

$$\Psi(\tilde{z}) = \Psi(z) + \lambda s p_1 \mathbf{e}$$

for some $s \in \mathbb{Z}$, where $p_1 = 2v_1 + 1$ and $\mathbf{e}$ is the unit vector in the non-integer coordinate direction. Then by the construction of $\gamma(z)$, if $\gamma(\tilde{z}) = (\tilde{\gamma}_{-1}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{2k-1})$, the value of $\tilde{\gamma}_1$ is related to γ_1 by

$$\tilde{\gamma}_1 \equiv \gamma_1 + s(2v_1 + 1) \pmod{(2v_1 + 1)\, n_1}.$$

By corollary 12, z and $\tilde{z}$ are congruent modulo $\mathbb{L}^e$ if and only if $\tilde{\gamma}_1 = \gamma_1$. Thus the n_1 distinct points modulo $\mathbb{L}^e$ correspond to distinct values of s modulo n_1. Furthermore, if we consider the value of $\tilde{\gamma}_1$ modulo n_1, we have:

$$\tilde{\gamma}_1 \equiv \gamma_1 + s(2v_1 + 1) \pmod{n_1}.$$

Now $2v_1 + 1$ is coprime to the modulus, as by the coprimality condition (88) and the construction (55) of q, two is the highest power of $(2v_1 + 1)$ that divides q. It follows that each distinct value of $\tilde{\gamma}_1$ is distinct modulo n_1. This completes the base case.

To proceed with the inductive step, we suppose that the above hypotheses hold for some $j \in \{1, \ldots, k-1\}$. In the proof of lemma 11 we used equation (52) to describe the behaviour of points as they move from one vertex to the next in two cases. The first case occurs when $v_j = v_{j+1}$, so that the jth and $(j+1)$th vertices lie on parallel lines, and $n_j = n_{j+1}$. In this case, the value of σ_{j+1} is determined uniquely by the value of σ_j. In particular, we saw that if the jth vertex lies on $y = n$ and the $(j+1)$th vertex lies on $y = n - 1$, then σ_{j+1} and σ_j are related by equation (67).

We can use the same methods, considering this time the non-integer component of equation (52), to show that γ_{j+1} is determined by the pair (σ_j, γ_j) via:

$$\gamma_{j+1} \equiv \gamma_j + \epsilon_j + (2n - 1)t \pmod{(2v_1 + 1)\, n_j},$$

where $\epsilon_j = \epsilon_j(\sigma_j)$, and $t = t(\sigma_j)$ is the transit time between vertices. The one-to-one relationship between σ_j and σ_{j+1}, ensures that there are $n_{j+1} = n_j$ points with $\sigma_{-1} = \sigma_1$ that achieve any given value of σ_{j+1} at the $(j+1)$th vertex. Then for any given value of σ_j, which uniquely determines σ_{j+1}, the above congruence establishes a one-to-one relationship between γ_j and γ_{j+1} modulo $(2v_1 + 1)n_j$. Because this bijection is a translation, it also

holds modulo n_j. In other words, there is a skew-product map of residue classes modulo n_j: $(\sigma_j, \gamma_j) \mapsto (\sigma_{j+1}, \gamma_{j+1})$. This completes the inductive step for the first case.

In the second case, where $v_j \neq v_{j+1}$, the jth and $(j+1)$th vertices lie on perpendicular lines. Again referring to the proof of lemma 11, taking equation (68) modulo $2v_{j+1} + 1$ gives the following expression for σ_{j+1} in terms of the pair (σ_j, γ_j):

$$\left\lceil \frac{v_j + 1}{\lambda} \right\rceil + \sigma_{j+1} \equiv \left\lceil \frac{v_j}{\lambda} \right\rceil + \gamma_j + \epsilon_j \quad (\mathrm{mod}\ 2v_{j+1} + 1).$$

Here we were able to replace x_j with γ_j as, by the construction (55) of q, $2v_{j+1}+1$ is a divisor of the modulus $q/(2v_j + 1) = (2v_1 + 1)n_j$ which defines γ_j. If the coprimality condition (88) holds, then $2v_{j+1} + 1$ also divides n_j. Hence for any given pair (σ_j, σ_{j+1}), there are $n_j/(2v_{j+1} + 1)$ values of γ_j modulo n_j for which the following congruence is satisfied:

$$(90) \qquad \gamma_j \equiv \left\lceil \frac{v_j + 1}{\lambda} \right\rceil + \sigma_{j+1} - \left\lceil \frac{v_j}{\lambda} \right\rceil - \epsilon_j + s(2v_{j+1} + 1) \quad (\mathrm{mod}\ n_j)$$

where $s \in \mathbb{Z}$. The total number of points with any given value of σ_{j+1} is thus:

$$(2v_j + 1) \times \frac{n_j}{2v_{j+1} + 1} = n_{j+1},$$

which completes the inductive step for hypothesis (i).

Taking the second component of equation (52) modulo n_{j+1} gives an expression for γ_{j+1} in terms of the pair (σ_j, γ_j):

$$(91) \qquad \gamma_{j+1} \equiv \left\lceil \frac{n}{\lambda} \right\rceil + \sigma_j + (2v_j + 1)t - \left\lceil \frac{n-1}{\lambda} \right\rceil \quad (\mathrm{mod}\ n_{j+1}),$$

where $t = t(\sigma_j, \gamma_j)$. For a given pair (σ_j, σ_{j+1}), t is given by equation (68). Hence taking equation (68) modulo n_j, a multiple of $2v_{j+1} + 1$, and using the expression (90) for γ_j, it follows that the values of t satisfy

$$t + 1 \equiv \frac{\lceil (v_j + 1)/\lambda \rceil + \sigma_{j+1} - \lceil v_j/\lambda \rceil - \gamma_j - \epsilon_j}{2v_{j+1} + 1} \quad \left(\mathrm{mod}\ \frac{n_j}{2v_{j+1} + 1} \right)$$

$$\equiv -s \quad (\mathrm{mod}\ n_j/(2v_{j+1} + 1)),$$

where $s \in \mathbb{Z}$. Thus t takes all values modulo $n_j/(2v_{j+1} + 1) = n_{j+1}/(2v_j + 1)$. In turn, equation (91) implies that γ_{j+1} takes all values satisfying:

$$\gamma_{j+1} \equiv \left\lceil \frac{n}{\lambda} \right\rceil + \sigma_j - \left\lceil \frac{n-1}{\lambda} \right\rceil \quad (\mathrm{mod}\ 2v_j + 1).$$

Applying this argument for all values $\sigma_j \in \{0, 1, \ldots, 2v_j\}$, we get a complete residue class modulo n_{j+1} as required. This completes the inductive step for hypothesis (ii). $\qquad\square$

PROOF OF THEOREM B

Let $e \in \mathcal{E}$ be given, let $(v_1, \ldots, v_k)$ be the corresponding vertex list, and let σ^* be the unique element of the set $\{0, 1, \ldots, 2v_k\}$ that satisfies

$$2\sigma^* \equiv \lfloor \sqrt{e} \rfloor \quad (\mathrm{mod}\ 2\lfloor \sqrt{e} \rfloor + 1).$$

For $e = 0, 2, 8$ all elements of the vertex list are the same. This case is dealt with by the discussion preceding lemma 14. Thus we assume that the vertex list contains at least two distinct elements.

Suppose first that $2v_1+1$ is coprime to $2v_j+1$ for all $v_j \neq v_1$. Then for sufficiently small λ, lemma 15 states that the number n_k of points in $\tilde{X}^e \cap \mathrm{Fix}\, G$ modulo $\mathbb{L}^e$ whose orbit code has kth entry σ^* is given by equation (89), with $j = k$. Furthermore, since all points $z \in \tilde{X}^e$ whose orbit code $(\sigma_{-1}, \sigma_1, \ldots, \sigma_{2k-1})$ satisfies $\sigma_{-1} = \sigma_1$ are congruent to some point in $\tilde{X}^e \cap \mathrm{Fix}\, G$, it follows that n_k is the number of points in $\tilde{X}^e$ modulo $\mathbb{L}^e$ satisfying the conditions (i) and (ii) of lemma 13. Therefore the number of symmetric fixed points of Φ in $\tilde{X}^e$ modulo $\mathbb{L}^e$ is independent of λ and given by n_k.

Similarly if $2v_k + 1$ is coprime to $2v_j + 1$ for all $v_j \neq v_k$, then $2v_k + 1$ is coprime to $q_{\iota(l-2)}$, given in closed form by (61). If follows that the condition (85) of lemma 14 holds, since the recursive expression (57) for $q_{\iota(l-1)}$ gives us that:

$$
\begin{aligned}
p_{\iota(l-1)} &= \frac{q_{\iota(l-1)}}{2v_{\iota(l-1)} + 1} \\
&= \frac{\mathrm{lcm}((2v_{\iota(l-1)} + 1)(2v_{\iota(l-2)} + 1), q_{\iota(l-2)})}{2v_{\iota(l-1)} + 1} \\
&= \frac{(2v_{\iota(l-2)} + 1)\, q_{\iota(l-2)}}{\gcd((2v_{\iota(l-1)} + 1)(2v_{\iota(l-2)} + 1), q_{\iota(l-2)})}.
\end{aligned}
$$

Applying lemma 14 for $j = 1$, we have that for every cylinder set of $\mathbb{L}_1^e$, the number of points modulo $\mathbb{L}^e$ in the cylinder set whose orbit code has kth entry σ^* is given by

$$
\frac{1}{2v_k + 1} |\mathbb{L}_1^e / \mathbb{L}^e| = \frac{1}{2v_k + 1} \frac{\# \mathbb{L}^e}{(2v_1 + 1)^2} = \frac{n_k}{2v_1 + 1}.
$$

There are $2v_1 + 1$ cylinder sets of $\mathbb{L}_1^e$ whose associated orbit code satisfies $\sigma_{-1} = \sigma_1$. Hence, as before, the number of points in $\tilde{X}^e$ modulo $\mathbb{L}^e$ satisfying the conditions (i) and (ii) of lemma 13 is

$$
(2v_1 + 1) \times \frac{n_k}{2v_1 + 1} = n_k,
$$

and the number of symmetric fixed points of Φ in $\tilde{X}^e$ modulo $\mathbb{L}^e$ follows. This completes the proof of the first statement.

We have shown that for sufficiently small λ, and if (41) holds, then the fraction of symmetric fixed points of Φ in each fundamental domain of $\mathbb{L}^e$ is

$$
\frac{1}{(2v_1 + 1)(2v_k + 1)} = \frac{1}{(2\lfloor \sqrt{e/2} \rfloor + 1)(2\lfloor \sqrt{e} \rfloor + 1)},
$$

where we have used equations (21) and (22), respectively. It remains to show that the density $\delta(e, \lambda)$ of symmetric fixed points in $\tilde{X}^e$ converges to this fraction as $\lambda \to 0$.

By equation (38), the domain $\tilde{X}^e$ is a subset of the lattice $(\lambda \mathbb{Z})^2$, bounded by a rectangle lying parallel to the symmetry line $\mathrm{Fix}\, G$. Similarly, a fundamental domain of the lattice $\mathbb{L}^e$ is constructed by taking the points bounded by the parallelogram Ω, given by

$$
\Omega = \{\alpha \mathbf{L} + \frac{\beta}{2}(\mathbf{L} - \mathbf{w}_{v_1, v_1}) : \ \alpha, \beta \in [0, 1)\},
$$

where the generator $\mathbf{L}$ is also parallel to the symmetry line. These parallelograms tile the plane by translation: $\Omega + \mathbb{L}^e = \mathbb{R}^2$.

The width of $\tilde{X}^e$ (taken in the direction perpendicular to $\mathrm{Fix}\, G$) is exactly twice that of Ω, independently of λ, as shown in Figure 10. The number of parallelograms which fit lengthwise into $\tilde{X}^e$, however, goes to infinity as λ goes to zero. In particular, the number of

parallelograms which can be contained in the interior of the rectangle bounding $\tilde{X}^e$ is given by

$$2 \left\lfloor \frac{|\tilde{I}^e|}{\lambda |\mathbf{L}|} \right\rfloor - 8,$$

where $\lfloor |\tilde{I}^e|/\lambda|\mathbf{L}| \rfloor$ is the number of times that the vector $\mathbf{L}$ fits lengthways into the rectangle, and we subtract 8 for the parallelograms which intersect the boundary. Each of these parallelograms contains a complete fundamental domain of $\mathbb{L}^e$, and their contribution to $\delta(e, \lambda)$ dominates in the limit $\lambda \to 0$.

Explicitly, we have:

$$\delta(e, \lambda) = \frac{\#\mathbb{L}^e}{\#\tilde{X}^e} \left(\frac{2 \left\lfloor |\tilde{I}^e|/\lambda|\mathbf{L}| \right\rfloor - 8}{(2v_1 + 1)(2v_k + 1)} + O(1) \right)$$

$$= \frac{q}{2(2v_1 + 1) \left\lfloor |\tilde{I}^e|/\lambda \right\rfloor} \left(\frac{2 \left\lfloor (2v_1 + 1)|\tilde{I}^e|/\lambda q \right\rfloor - 8}{(2v_1 + 1)(2v_k + 1)} + O(1) \right)$$

$$= \frac{1}{(2v_1 + 1)(2v_k + 1)} + O(\lambda),$$

as $\lambda \to 0$. $\qquad \qquad \square$

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