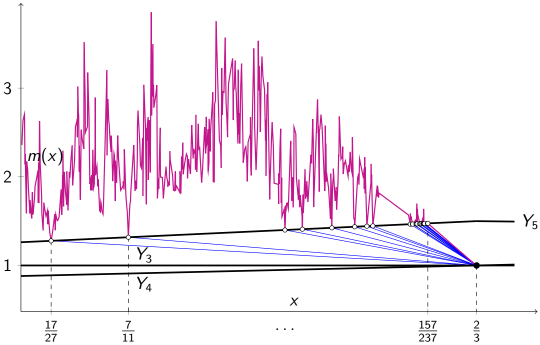

The Mean-Median Map

Jonathan Hoseana and Franco Vivaldi

Queen Mary
University of London

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Strong terminating conjecture [Schultz & Shiflett, 2005]

The mean-median sequence of any initial real sequence is eventually constant.

Several conjectures have been formulated on mean-median sequences: they are all open.

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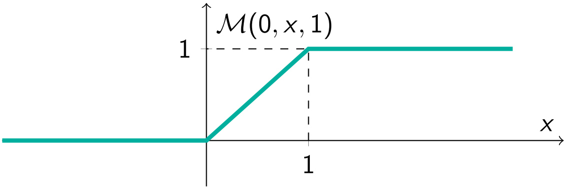
- Each new term is the difference of two diverging quantities.
- The median is a non-smooth function of the data.



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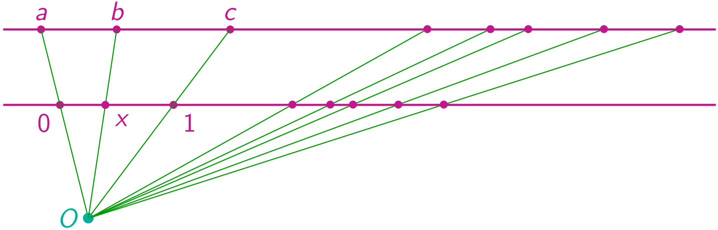

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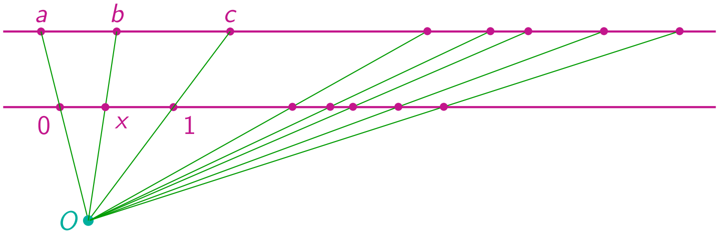

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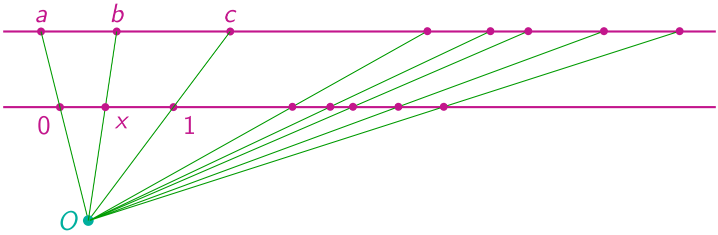

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- It suffices to study $(0, x, 1)$, $0 \leqslant x \leqslant 1$.
- [Chamberland & Martelli, 2007]: $\frac{1}{2} \leqslant x \leqslant \frac{2}{3}$ suffices.

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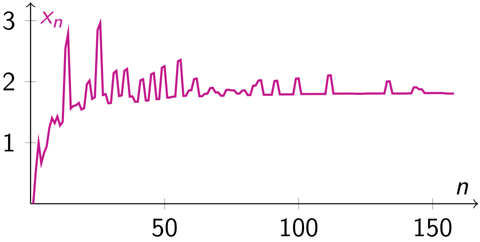

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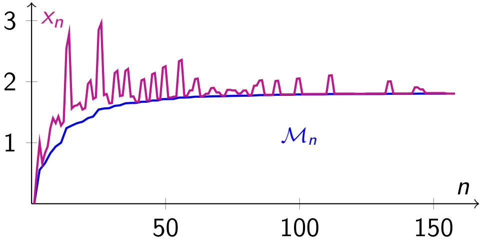

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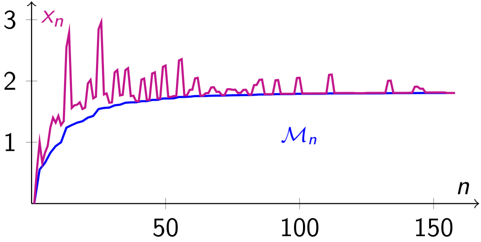

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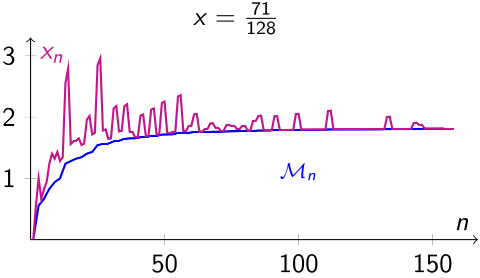

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- Convergence without stabilisation has never been observed.

Invariant modules

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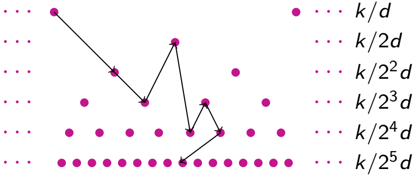

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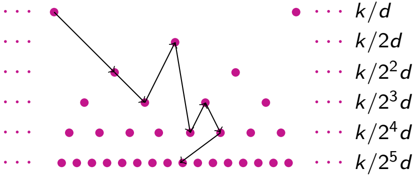

How deep do orbits sink?

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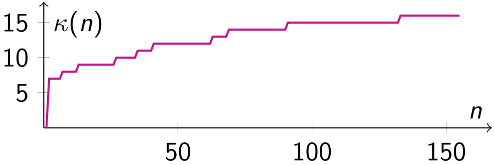

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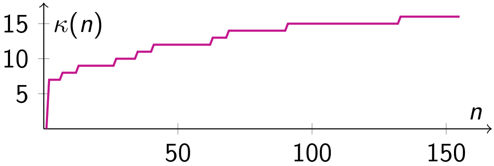

- **Substantial cancellations slow down the growth of $\kappa(n)$.**

From numbers to piecewise-affine functions

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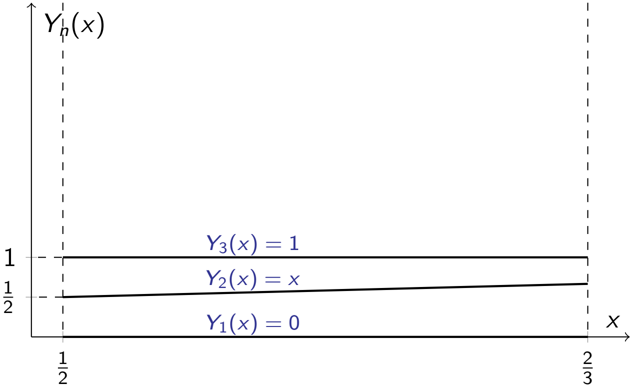
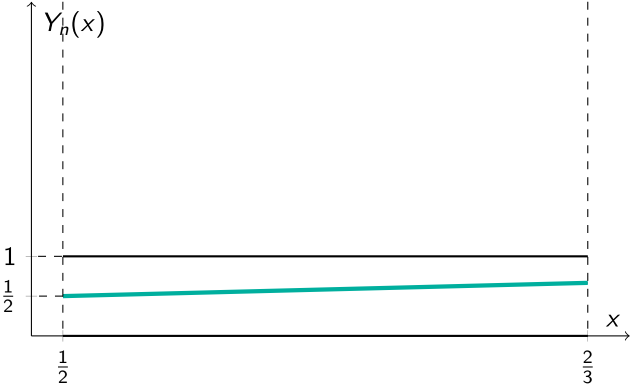


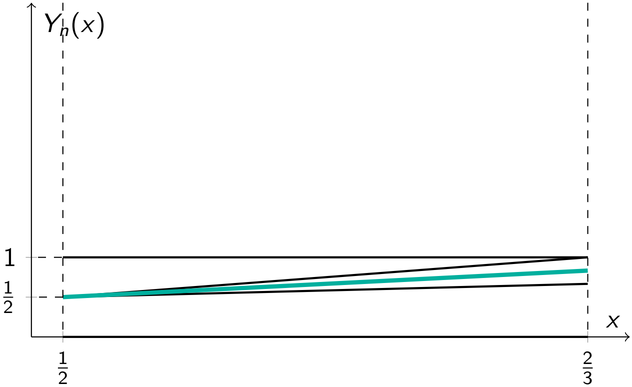
We view the initial sequence $(0, x, 1)$ as a sequence of affine functions.

From numbers to piecewise-affine functions



$\mathcal{M}_3$

From numbers to piecewise-affine functions



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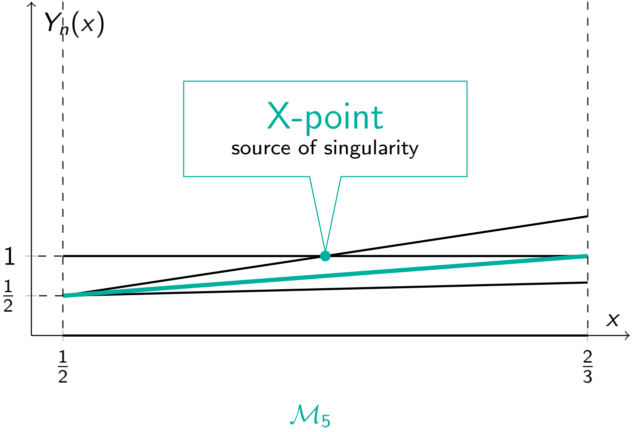
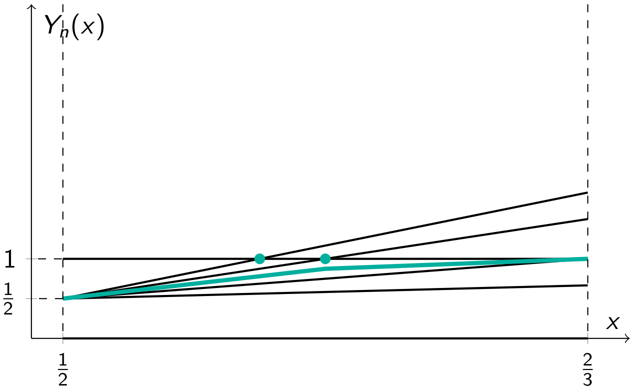


X-point

source of singularity

From numbers to piecewise-affine functions



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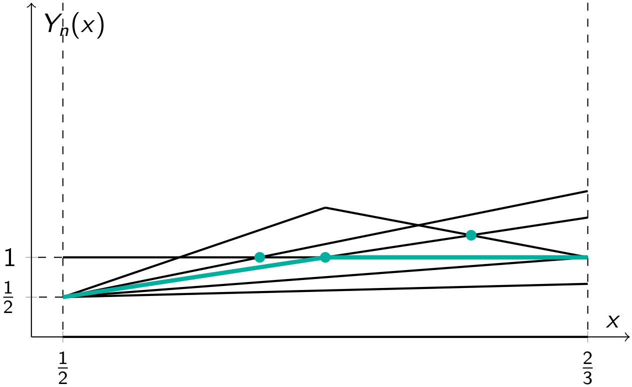
From numbers to piecewise-affine functions



$\mathcal{M}_7$

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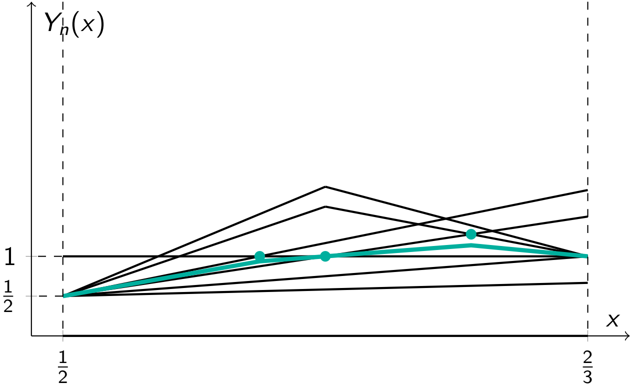

$\mathcal{M}_8$

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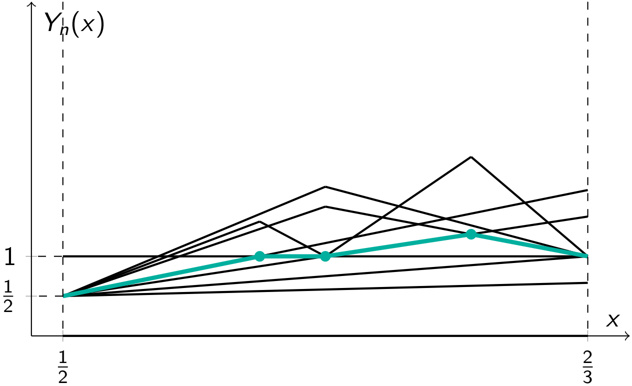

$\mathcal{M}_9$

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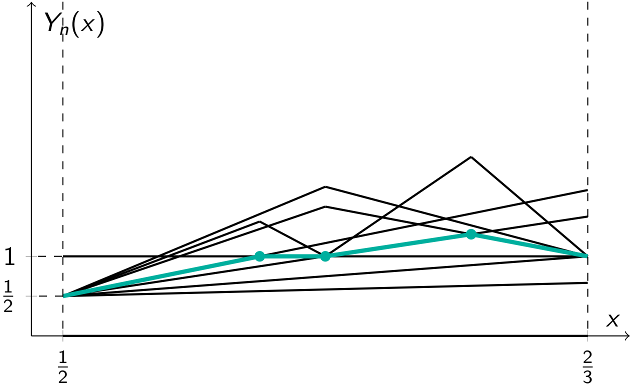

$\mathcal{M}_9$

How ubiquitous are the singularities?

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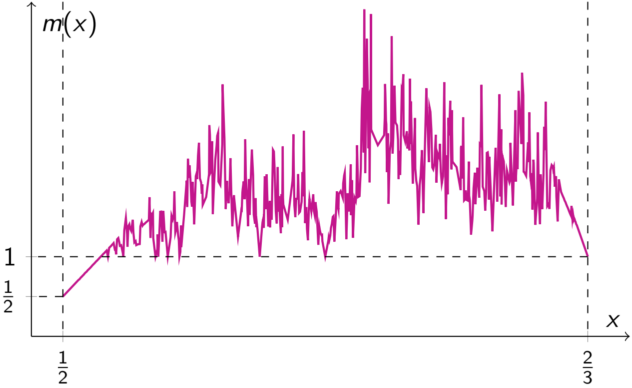

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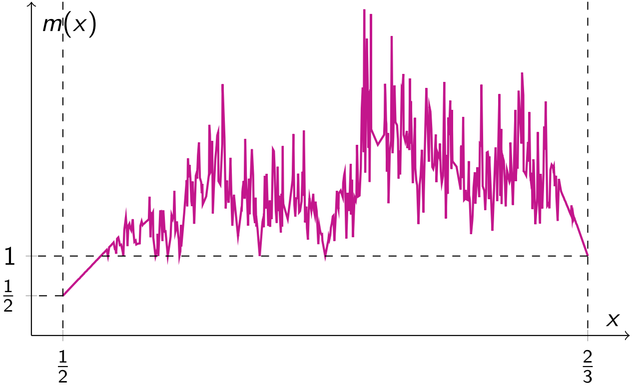
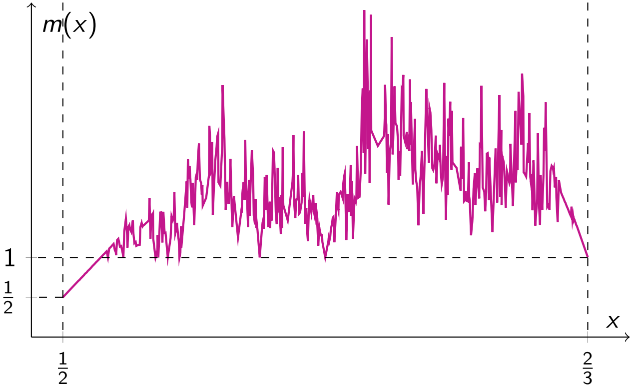


Conjecture [Chamberland & Martelli, 2007]

The limit function of $(0, x, 1)$ is continuous.

The limit function: what do we know?



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- Near 1/2 [Schultz & Shiflett, 2005]

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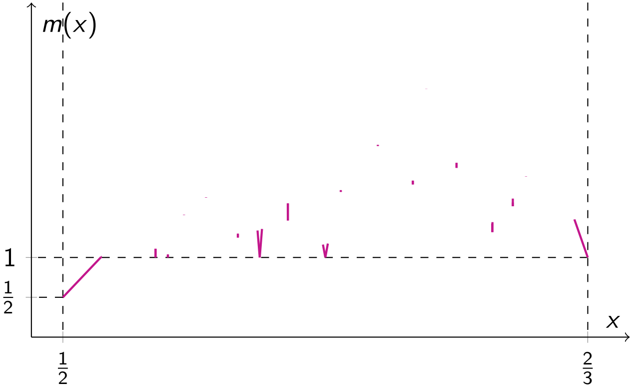

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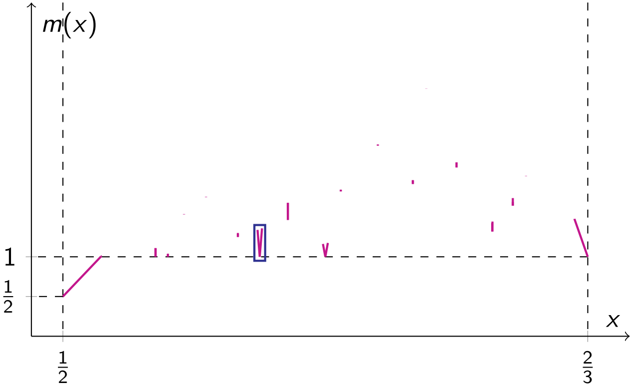

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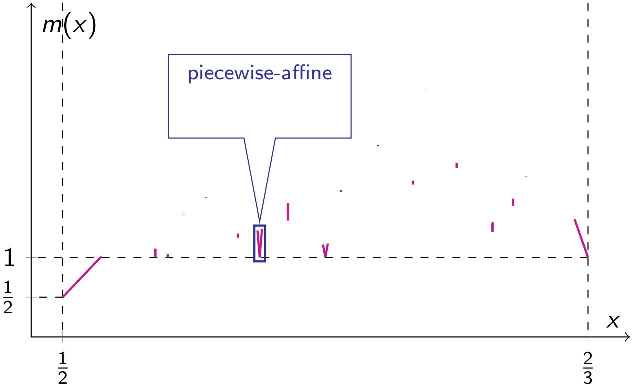

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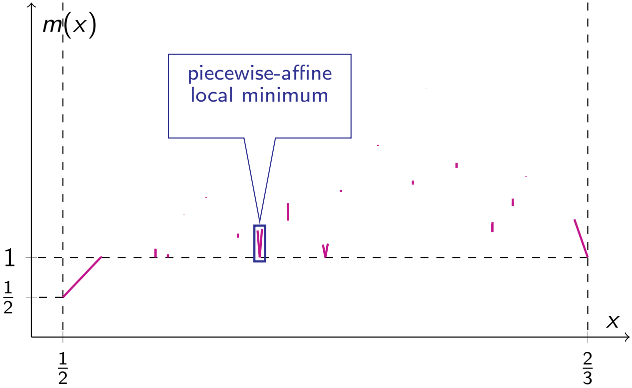

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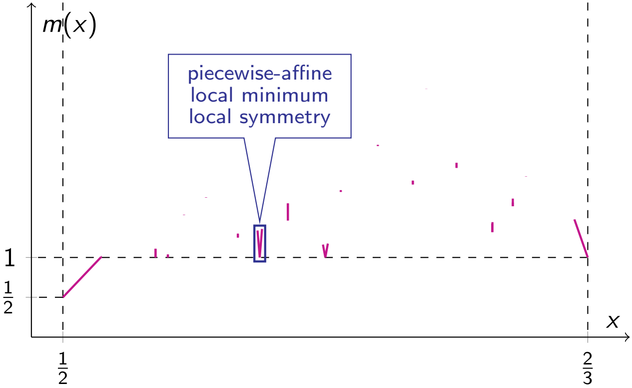

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Limit function vs. Takagi function

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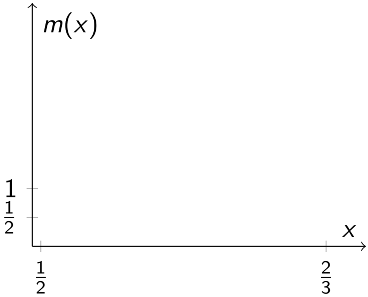

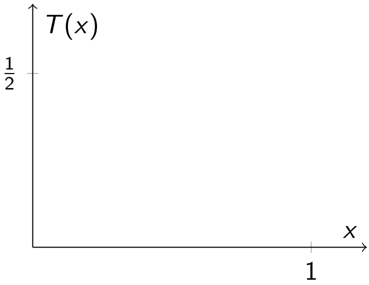

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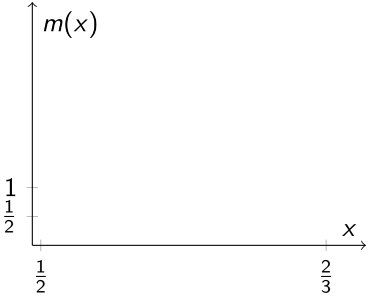

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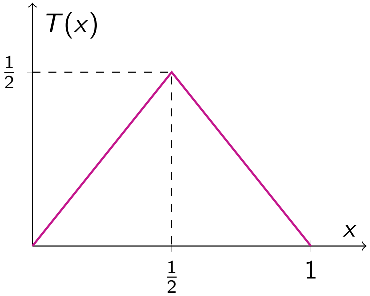

1st partial sum

a

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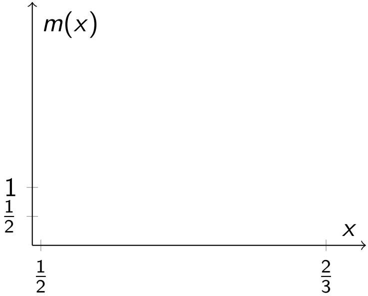
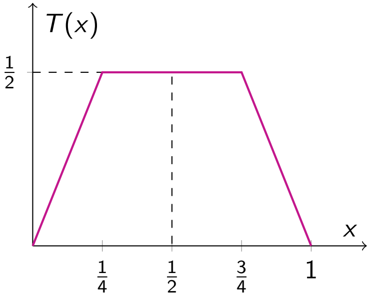
$$[x] = \min\{|x-n| : n \in \mathbb{Z}\}$$



a

a



2nd partial sum

a

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

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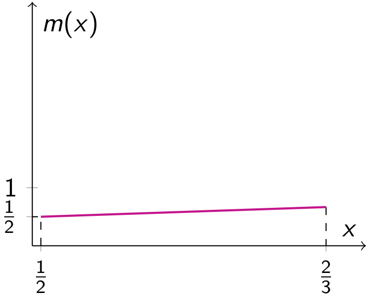
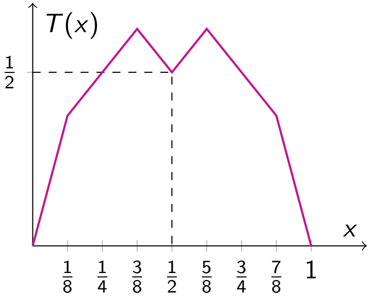


3rd partial sum

a



3rd partial sum

a

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

$$\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$$

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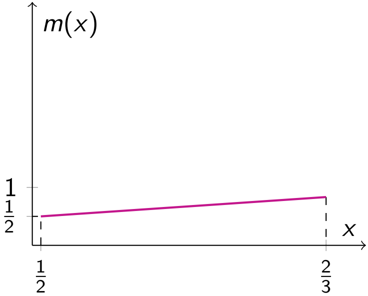
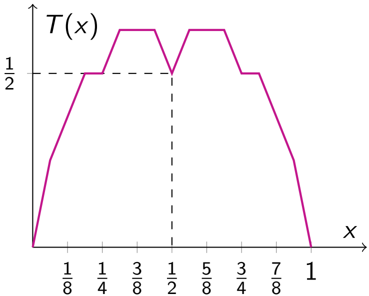


4th partial sum



4th partial sum

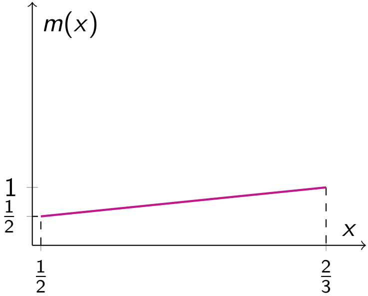
Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

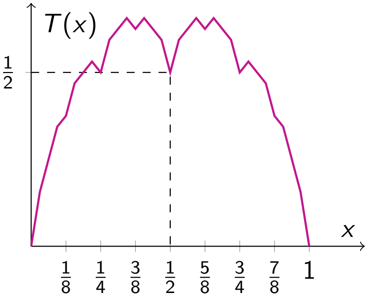
$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

$$\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$$

$$[x] = \min\{|x-n| : n \in \mathbb{Z}\}$$



5th partial sum



5th partial sum

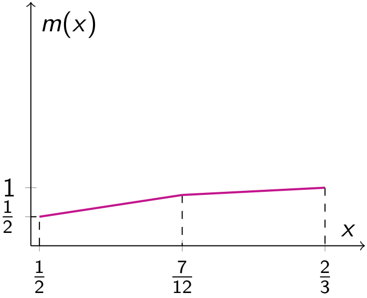
Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

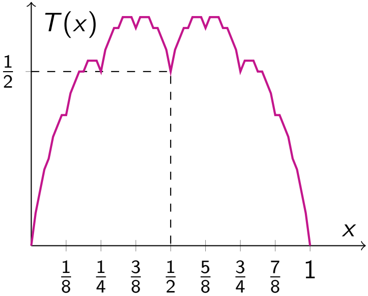
$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

$$\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$$

$$[x] = \min\{|x-n| : n \in \mathbb{Z}\}$$



6th partial sum

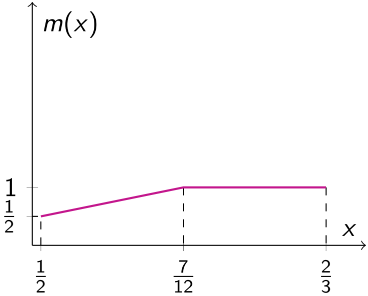


6th partial sum

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$$

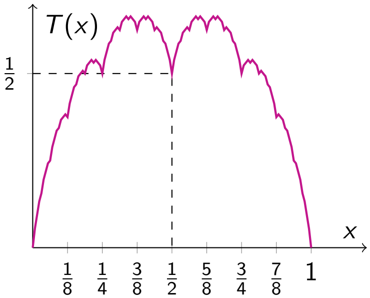


7th partial sum

a

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

$$[x] = \min\{|x-n| : n \in \mathbb{Z}\}$$



7th partial sum

a

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

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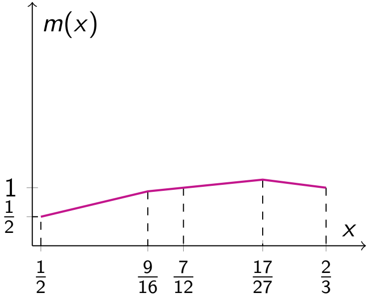
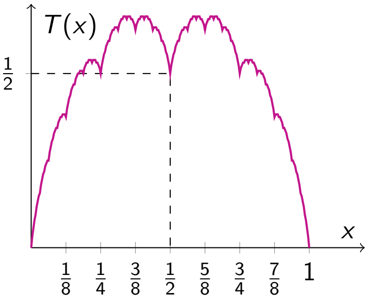


8th partial sum

a



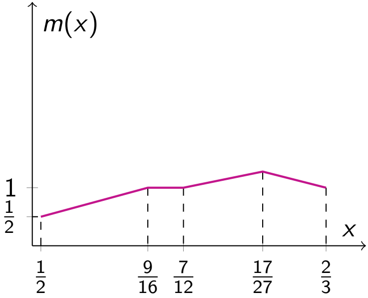
8th partial sum

a

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$$

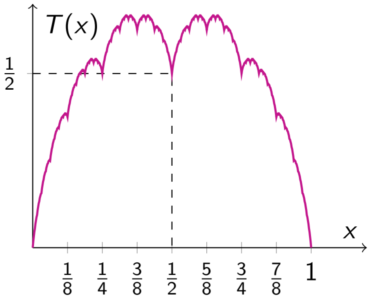


9th partial sum

a

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

$$[x] = \min\{|x-n| : n \in \mathbb{Z}\}$$



9th partial sum

a

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

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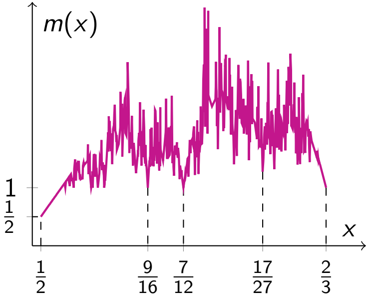
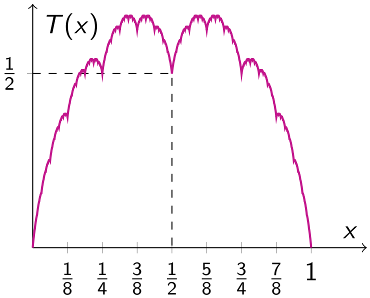


The limit function



The Takagi function

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

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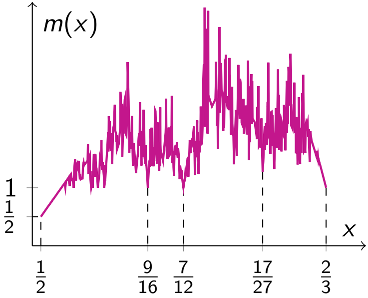
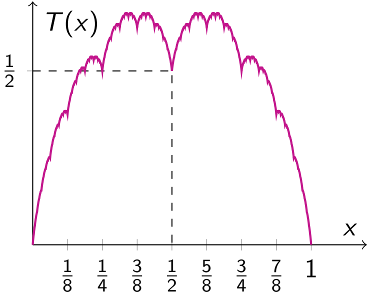


a

a



Continuous, unbounded variation everywhere, 1-D [Takagi, 1903]

Limit function vs. Takagi function

$$\boxed{m(x) = \mathcal{M}_3 + \sum_{n=4}^{\infty} \Delta\mathcal{M}_n}$$

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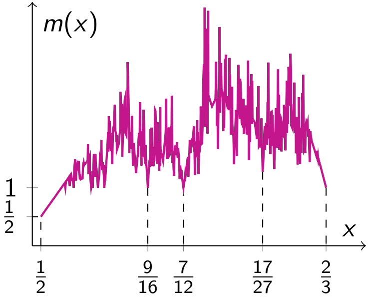

??

$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{[2^n x]}{2^n}}$$

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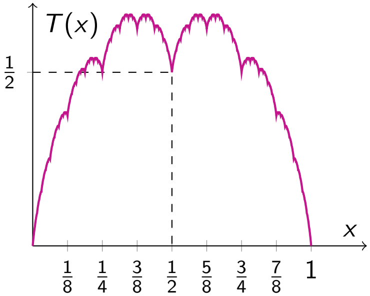

Continuous, unbounded variation everywhere, 1-D [Takagi, 1903]

The limit function near an X-point

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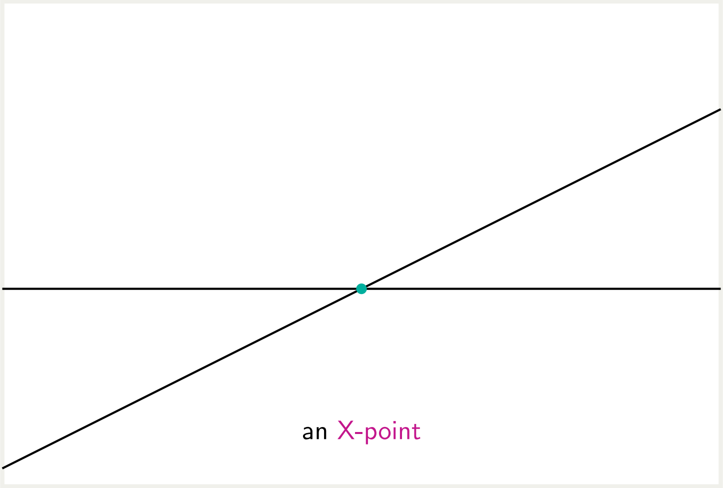
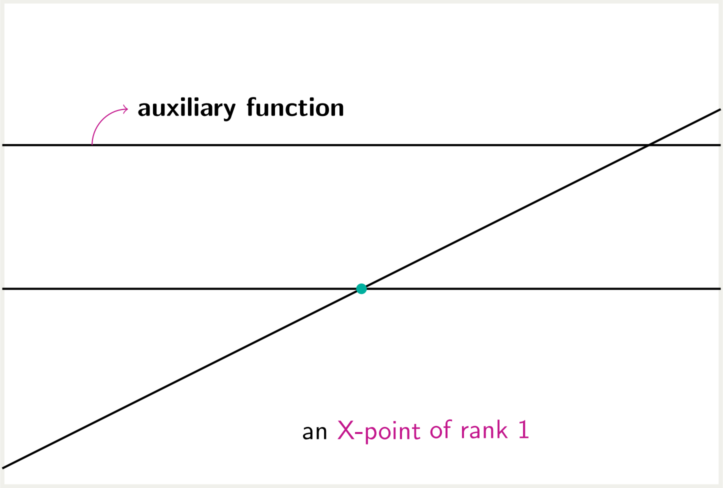


The limit function near an X-point



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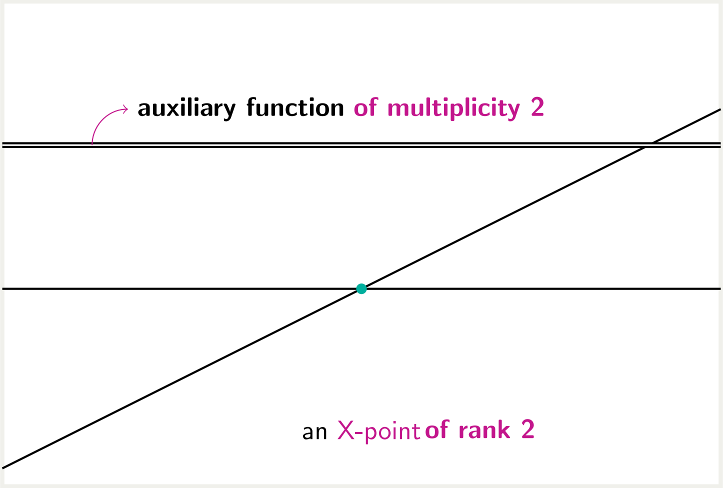

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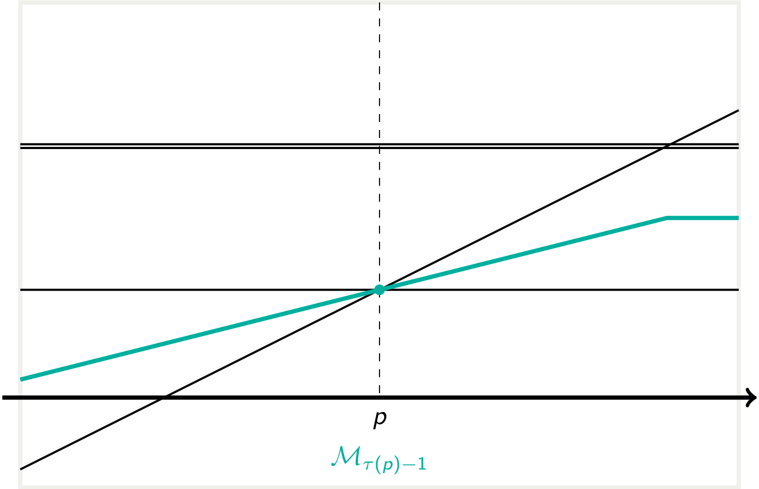

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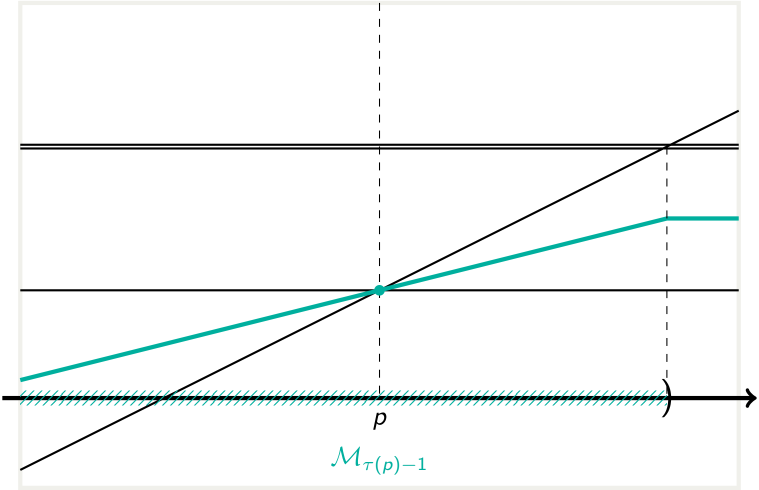

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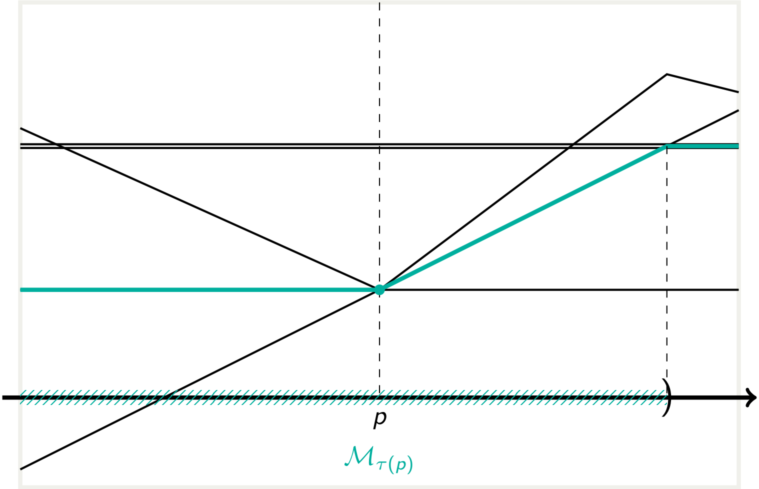
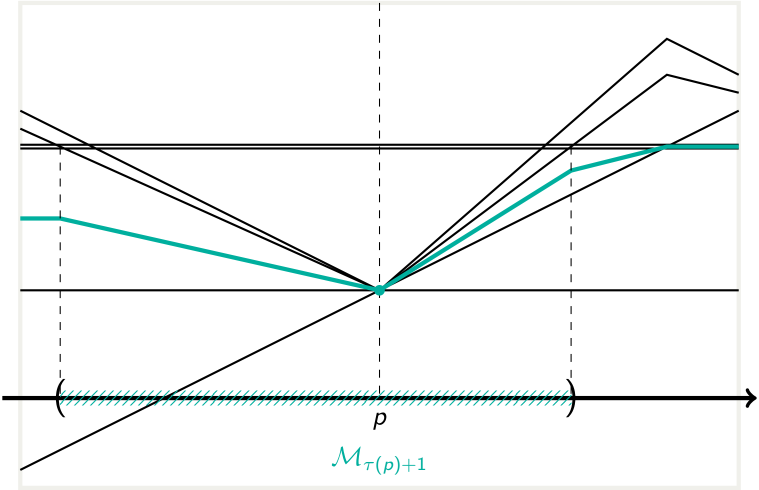


The limit function near an X-point



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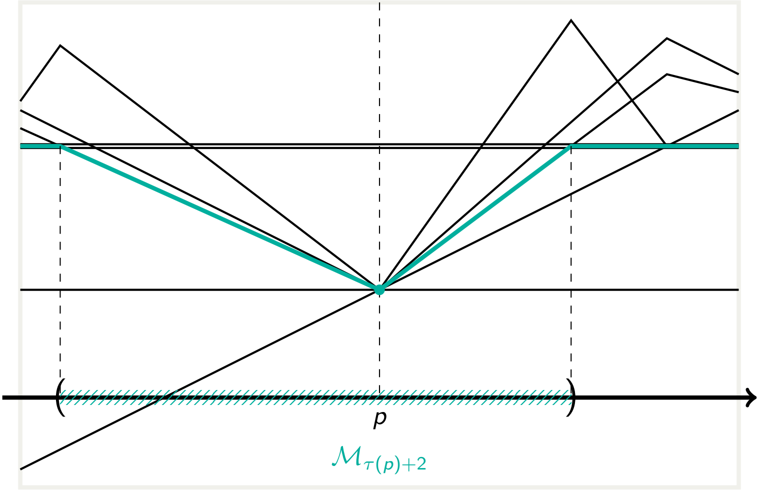

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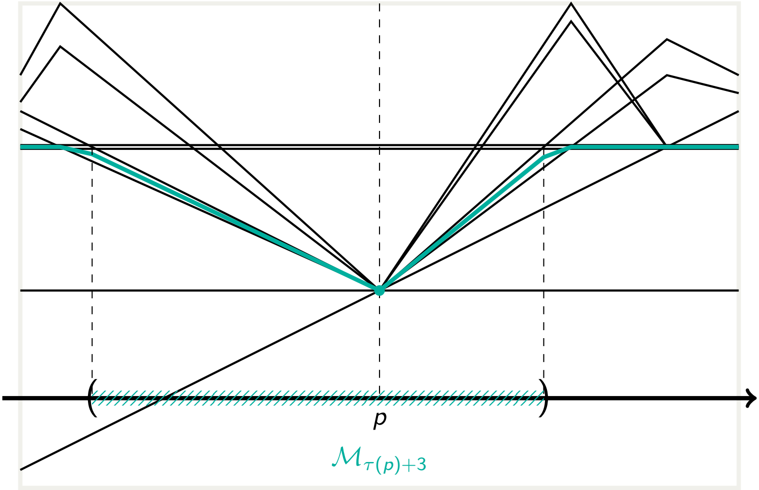

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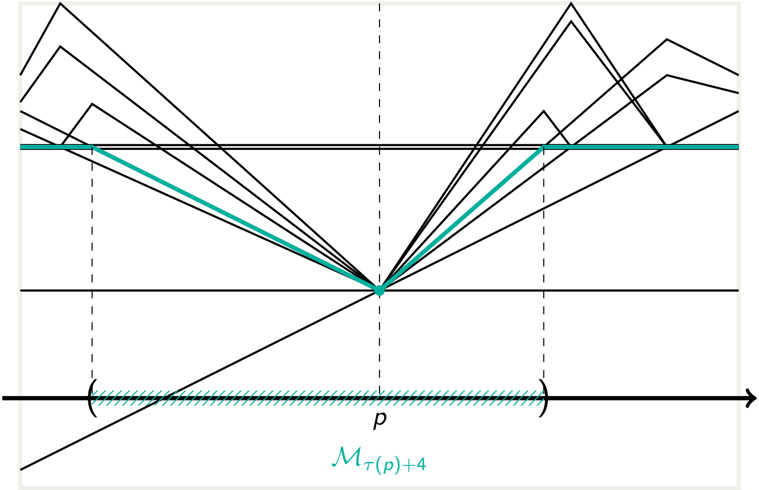

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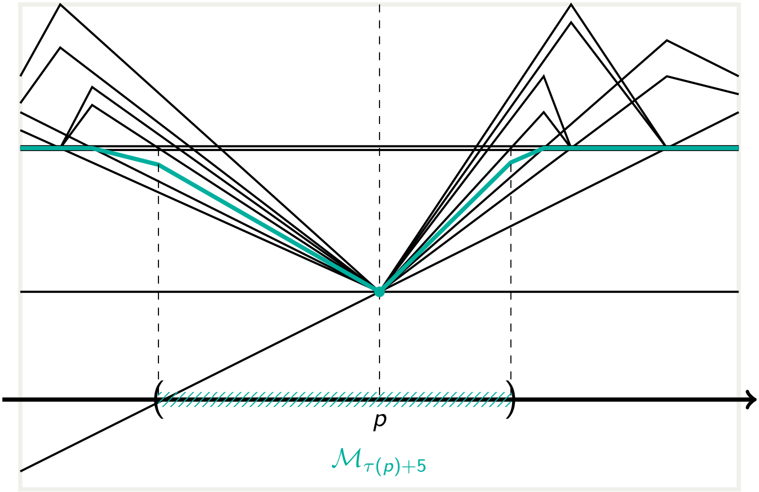

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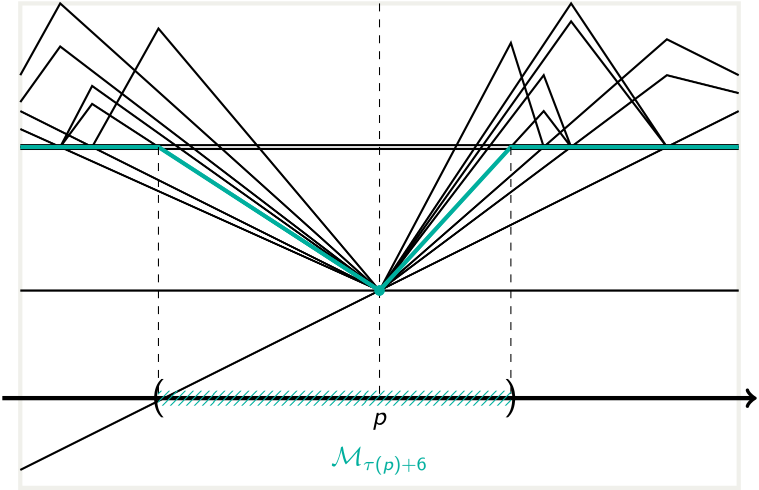

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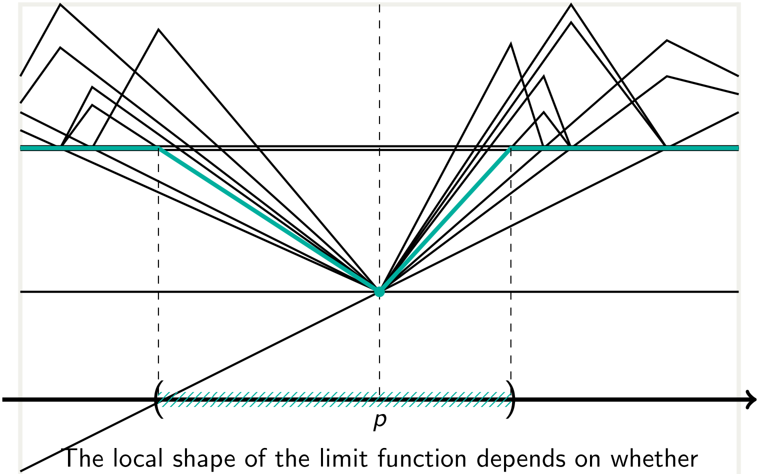
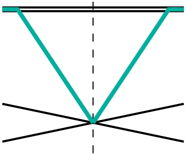
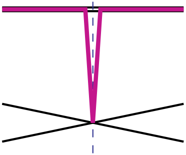


The local shape of the limit function depends on whether the interval of regularity of the median remains finite or shrinks to p.

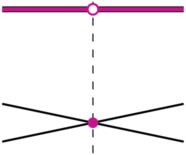
Theorem. *The limit function near an X-point of rank at least 2:*



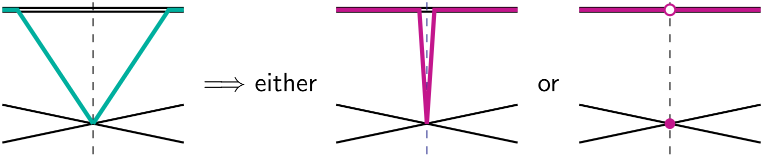
$\Longrightarrow$ either



or



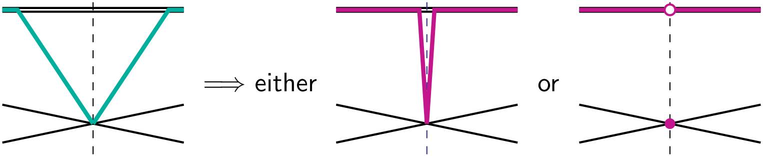
Theorem. *The limit function near an X-point of rank at least 2:*



$\Longrightarrow$ either … or

In our computations, we never encountered the second scenario.

Theorem. *The limit function near an X-point of rank at least 2:*

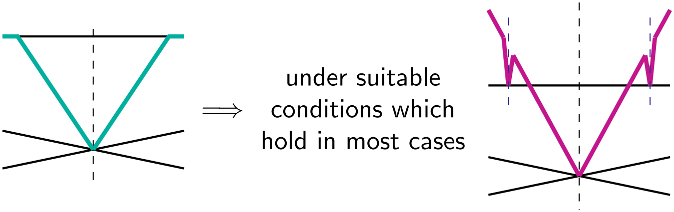


$\Longrightarrow$ either

or

In our computations, we never encountered the second scenario.

Theorem. *The limit function near an X-point of rank 1:*

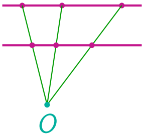


$\Longrightarrow$ under suitable conditions which hold in most cases

Symmetry near an X-point

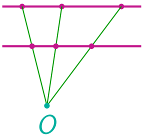
Symmetry near an X-point

Recall:



Symmetry near an X-point

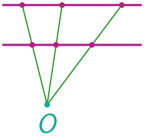
Recall:



Aim: Establish this near an X-point.

Symmetry near an X-point

Recall:



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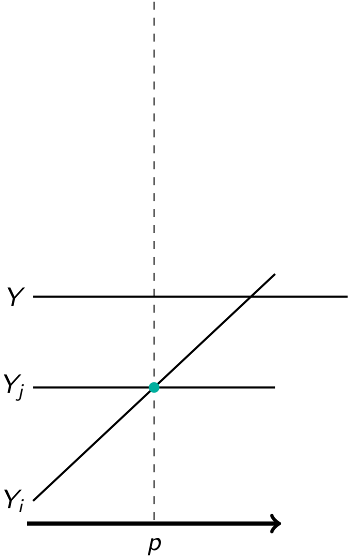

Symmetry near an X-point

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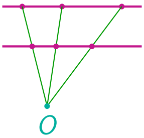

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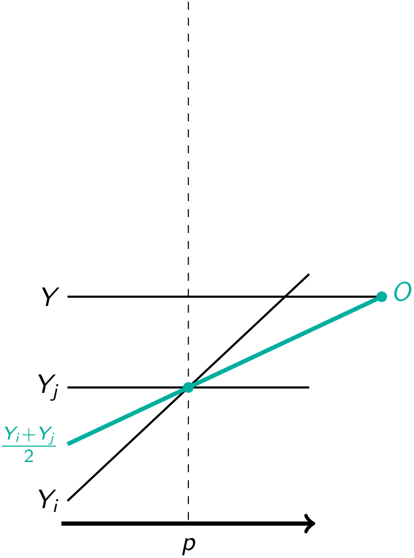

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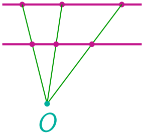

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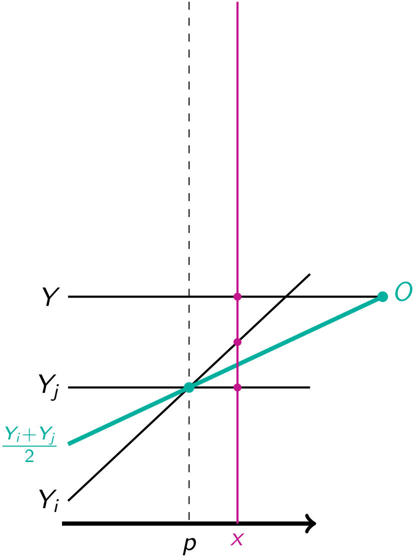
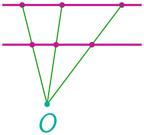
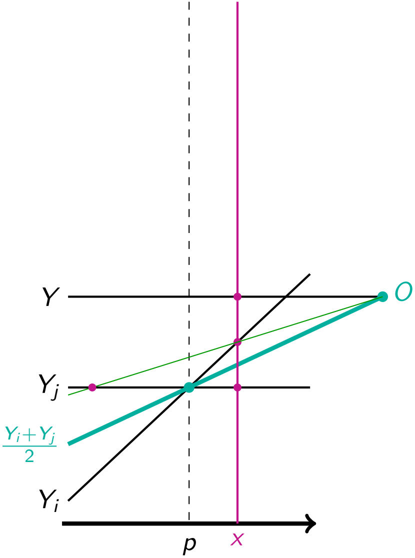


Symmetry near an X-point

Recall:

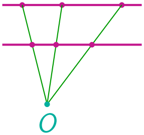




Aim: Establish this near an X-point.

Symmetry near an X-point

Recall:



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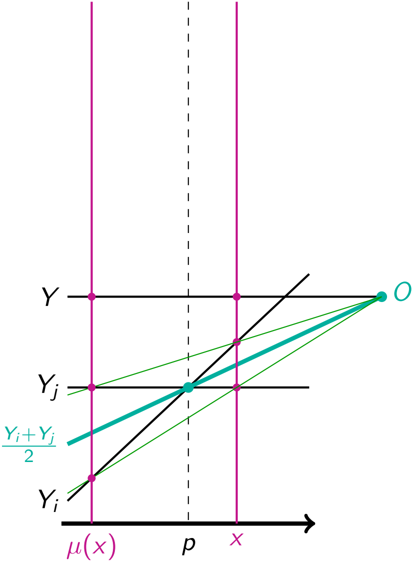
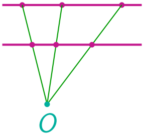
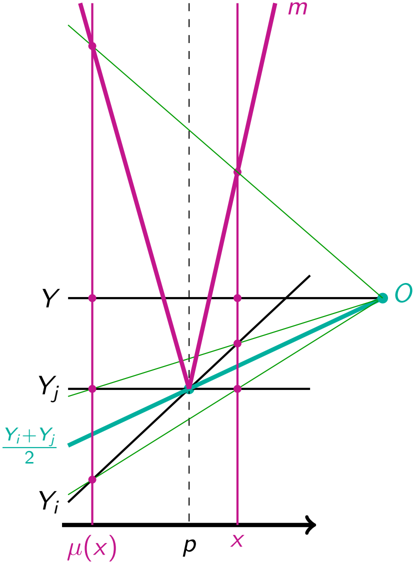


Symmetry near an X-point

Recall:



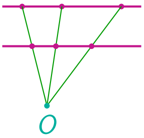


Aim: Establish this near an X-point.

Q: Does the limit function also obey this symmetry?

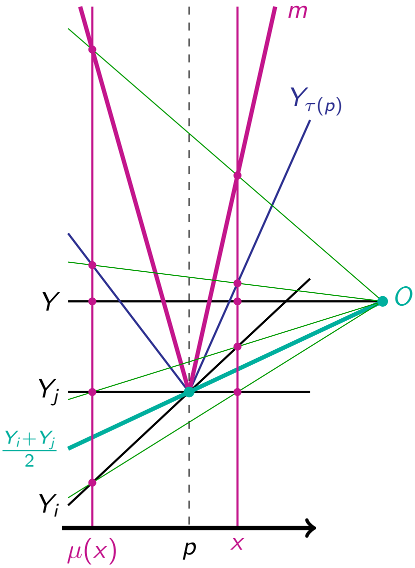
Symmetry near an X-point

Recall:



Aim: Establish this near an X-point.

Q: Does the limit function also obey this symmetry?



Theorem

If $Y_{\tau(p)}$ is an affine combination of Y, $\min\{Y_i, Y_j\}$, and $\max\{Y_i, Y_j\}$, then all functions Y_n, $n \geqslant \tau(p)$, and hence the limit, obey the symmetry.

Normal form near an X-point

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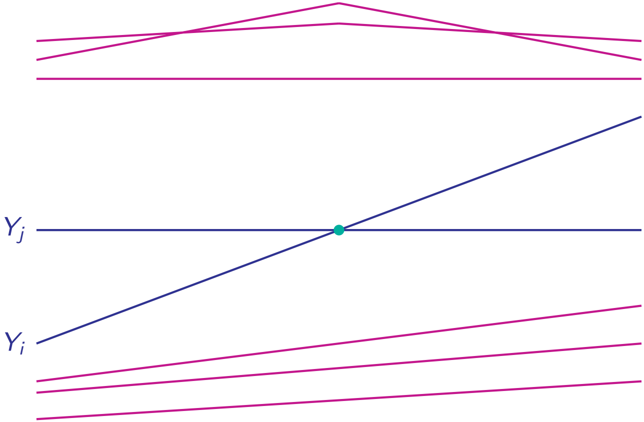

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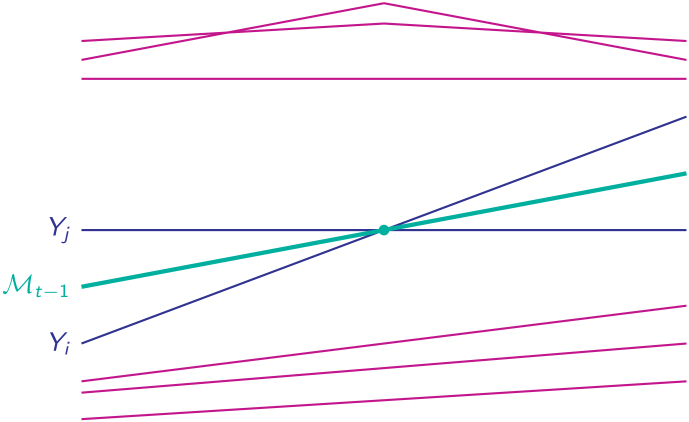

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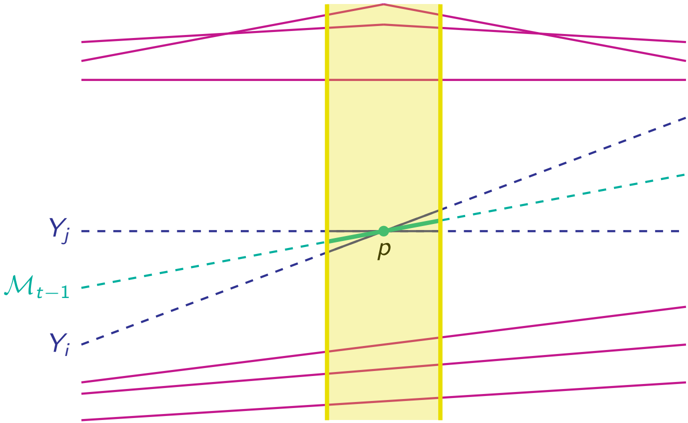

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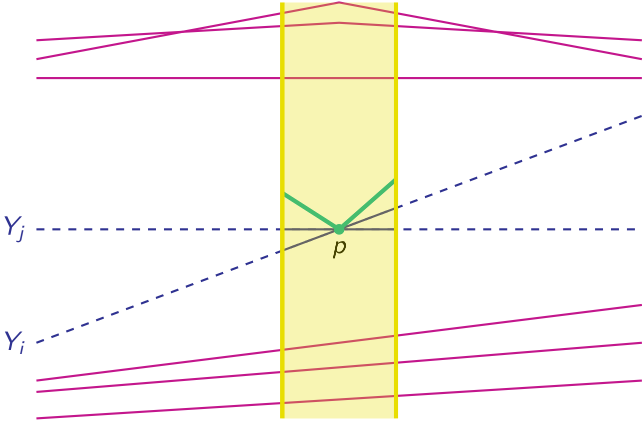

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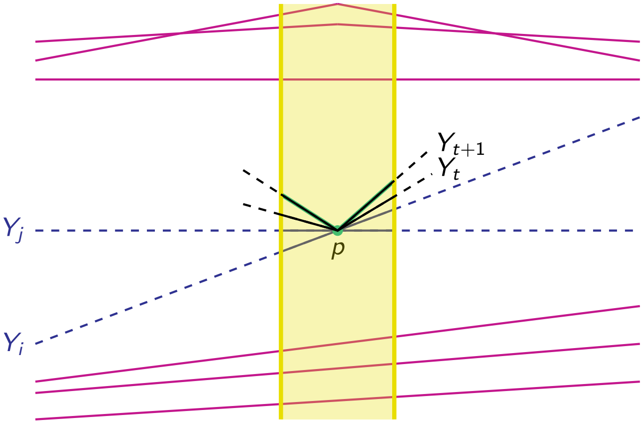

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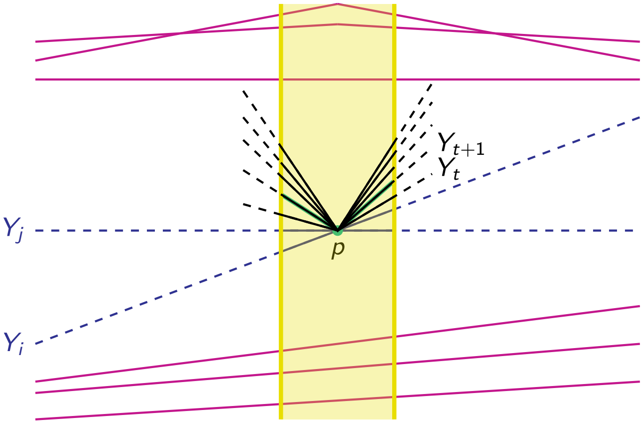

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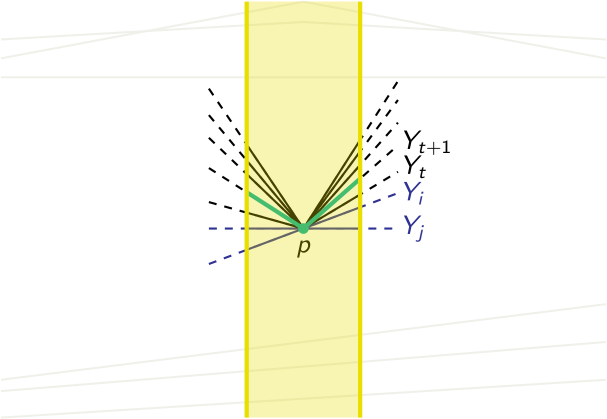

- Sufficiently close to a stabilised X-point, the dynamics is largely independent from pre-stabilisation data.

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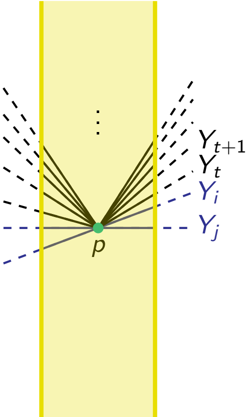

- A one-parameter family of dynamical systems involving only the functions passing through the X-point.

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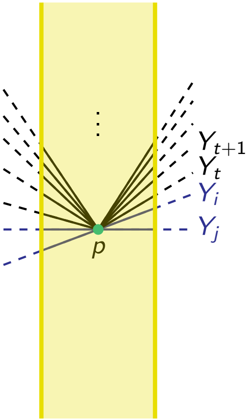

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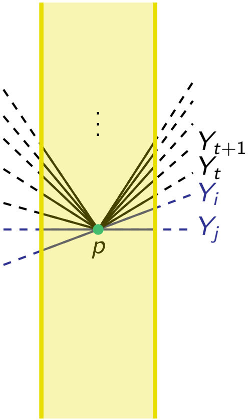

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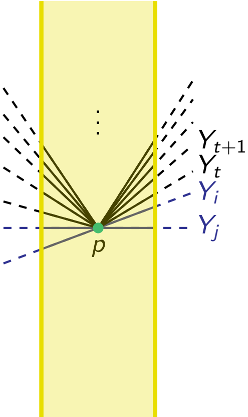

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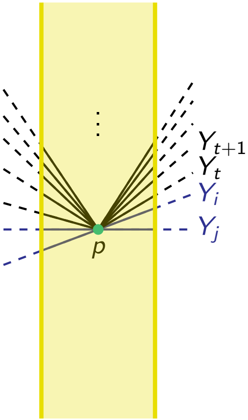

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- If this sequence stabilises, then the mean-median map stabilises in a small neighbourhood of **all X-points with the given stabilisation time**.

Normal form: the dynamics

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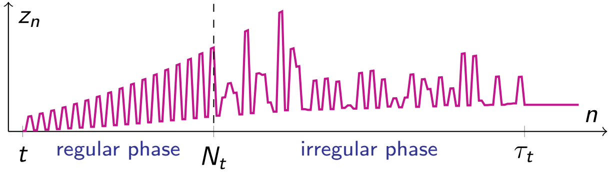
- The orbit $(0, 1, z_t, z_{t+1}, \ldots)$ for $t = 55$.



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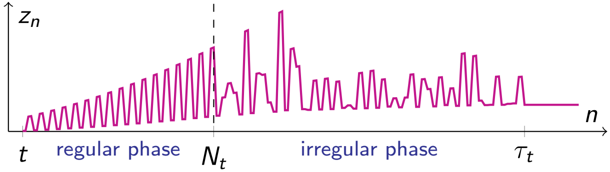

- **Regular phase:** $\left(t \leqslant n \leqslant N_t \sim t\sqrt{5}\right)$

$$z_{t+4\ell+k} = \begin{cases} \frac{\ell+1}{2}t + \ell^2 + \ell & \text{if } k = 0 \\ \frac{\ell+1}{2}t + \ell^2 + \ell + 1 & \text{if } k = 1 \\ \frac{t^2}{4} + \frac{5}{2}\ell t + 5\ell^2 - \ell & \text{if } k = 2 \\ \frac{t^2}{4} + \frac{5\ell+1}{2}t + 5\ell^2 + \ell - 1 & \text{if } k = 3. \end{cases}$$

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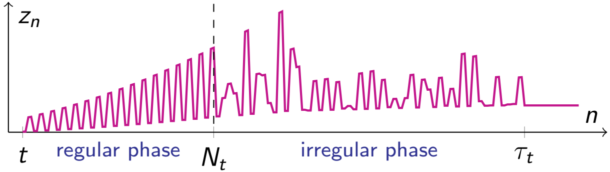

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Slow growth of denominators: $\{z_t, \ldots, z_{N_t}\} \subseteq \frac{1}{2^2}\mathbb{Z}$.

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Conjecture

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Theorem

There is a family ξ_i of rational initial sequences of increasing length, whose orbits have stabilisation time $\sim |\xi_i|^2/2$.

Normal form: improved computations

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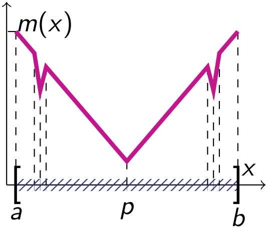

a regular domain

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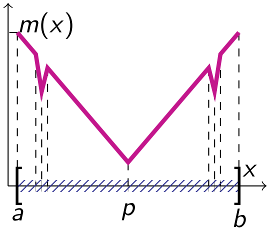

a regular domain

- **The computed regular domains only cover 13.1% of the measure**.

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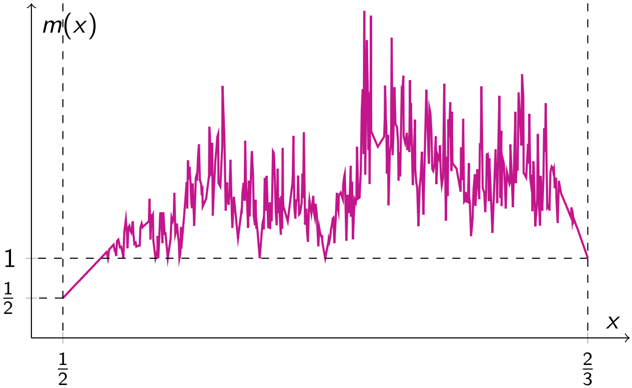

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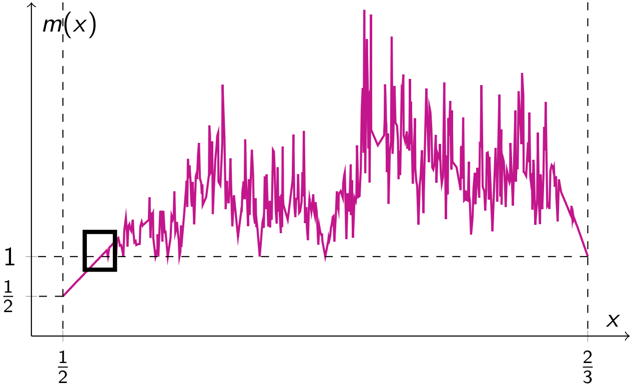

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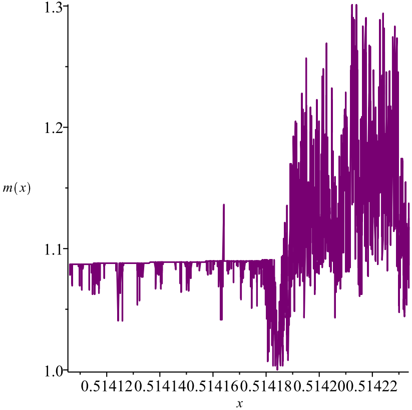

Total variation V of the limit function

sampled over the qth Farey fractions $\mathcal{F}_q$

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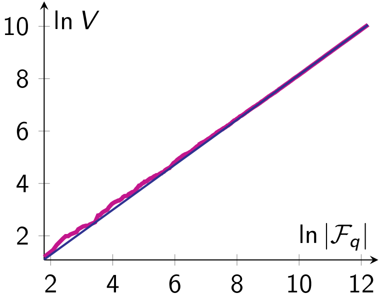

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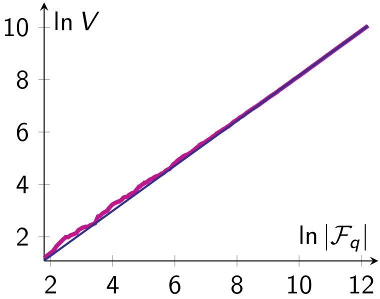

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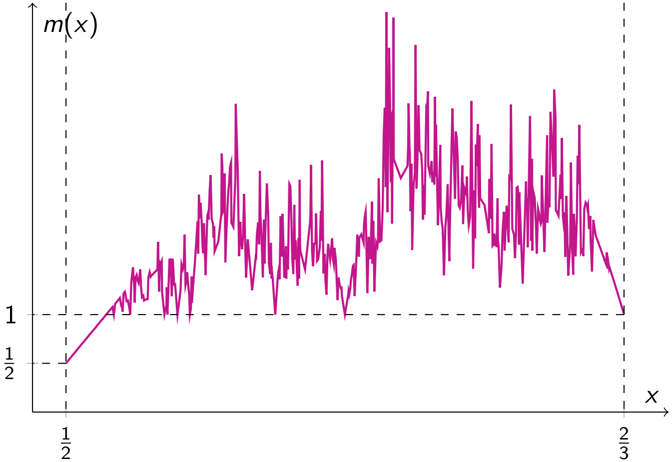
Conjecture

The Hausdorff dimension of the graph of the limit function is greater than 1.

A hierarchy of rationals



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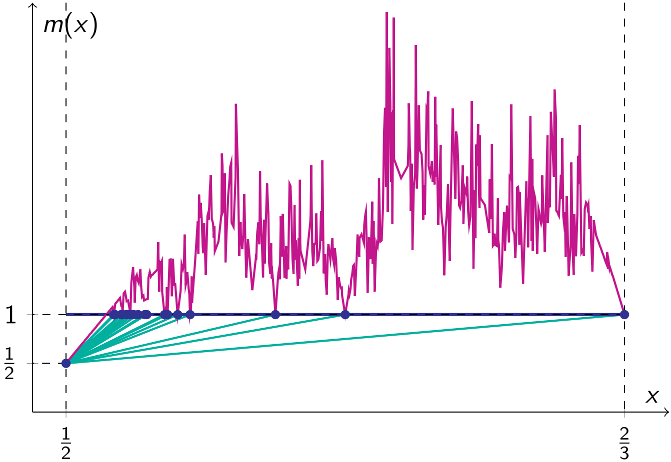

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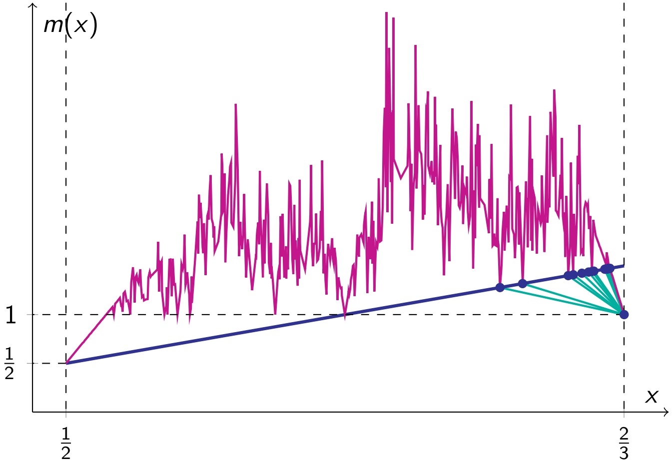

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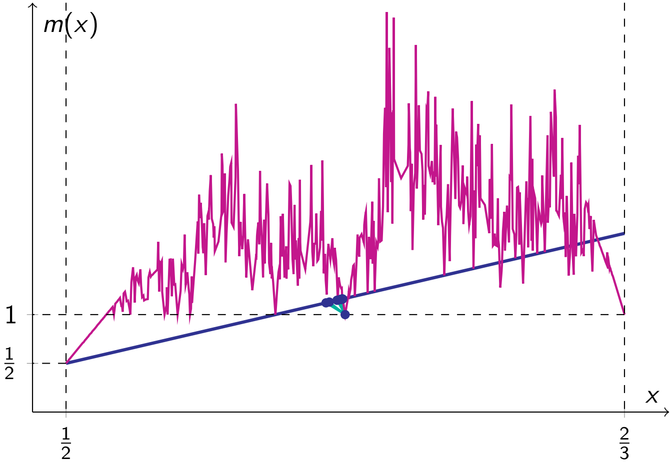

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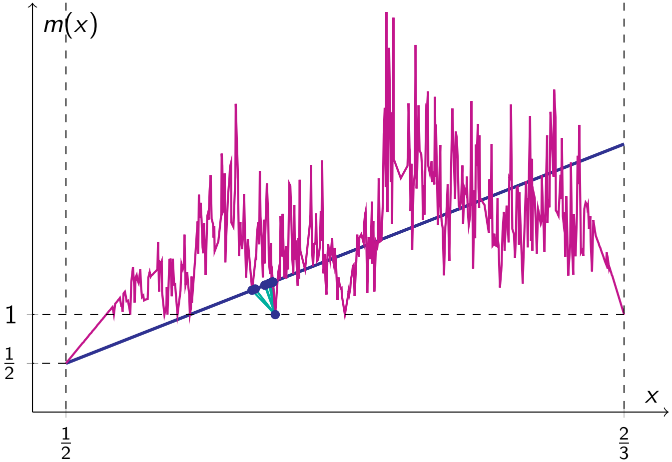

Thank you for your attention

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