ARITHMETIC EXPONENTS IN PIECEWISE-AFFINE PLANAR MAPS

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Abstract. We consider the growth of some indicators of arithmetical complexity of rational orbits of (piecewise) affine maps of the plane, with rational parameters. The exponential growth rates are expressed by a set of exponents; one exponent describes the growth rate of the so-called logarithmic height of the points of an orbit, while the others describe the growth rate of the size of such points, measured with respect to the p-adic metric. Here p is any prime number which divides the parameters of the map. We show that almost all the points in a domain of linearity (such as an elliptic island in an area-preserving map) have the same set of exponents. We also show that the convergence of the p-adic exponents may be non-uniform, with arbitrarily large fluctuations occurring arbitrarily close to any point. We explore numerically the behaviour of these quantities in the chaotic regions, in both area-preserving and dissipative systems. In the former case, we conjecture that wherever the Lyapounov exponent is zero, the arithmetical exponents achieve a local maximum.

1. Introduction

This paper is concerned with the analysis of indicators of arithmetical complexity of points of rational orbits of affine and piecewise affine planar maps. We present a combination of rigorous results and numerical experiments connecting the exponential growth rate of certain arithmetical functions to the dynamics on a divided phase space, where regular and irregular motions co-exist (see figure 1). Our aim is to complement existing quantitative measures of irregularity of motion —Lyapounov exponents, entropies— with measures of arithmetical complexity. Quantities of this type —the so-called heights— are well-established in diophantine geometry, and recently similar constructs (algebraic and arithmetical entropies) have been introduced in dynamics in the context of rational maps —see [27, 28] and references therein. In particular, there is a fairly complete theory of heights for polynomial automorphisms [27, section 7.1]. On a similar vein, the integrability criteria for discrete-time dynamical systems [14] have been extended to include tests of algebraic and arithmetical origin. Among the latter, the notion of diophantine integrability has recently been suggested, based on the slow (sub-exponential) growth of heights [15].

We are interested in monitoring the arithmetical complexity of the points of orbits of piecewise affine maps $F : \mathbb{Q}^2 \to \mathbb{Q}^2$ (notated $F : \mathbb{A}^2(\mathbb{Q}) \to \mathbb{A}^2(\mathbb{Q})$ when the distinction between affine and projective phase space is important). These maps feature highly complex dynamics from minimal ingredients, and the literature devoted to them is substantial, see, e.g., [11, 6, 4, 1, 23, 24, 25, 12, 9]. The increase in complexity of the iterates of a piecewise affine map derives solely from the growth of the coefficients, since the degree remains the same; these are the growth rates of interest to us. By contrast, for the iterates of polynomials and rational functions of degree greater than one, the growth of the degree is a preferred indicator of complexity —see for example [2, 8, 18, 5, 28].

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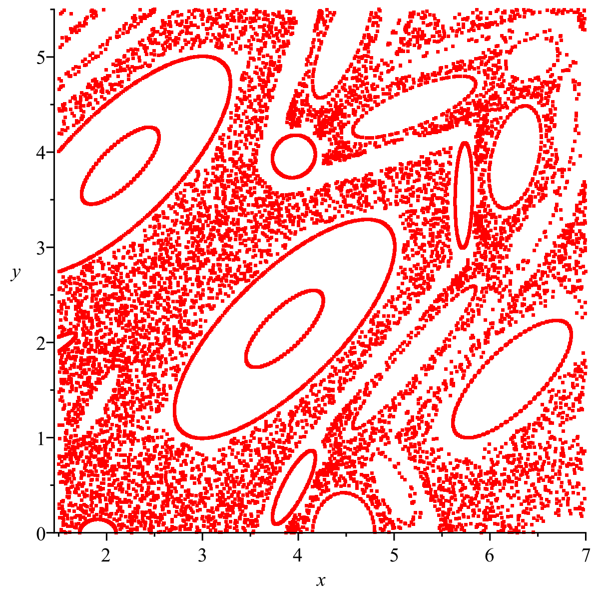

Figure 1: Phase portrait of the area-preserving map F defined in equation (11), with f given in (12) and $d = 1$, showing a mixture of regular orbits on island chains and chaotic orbits.

The simplest measure of the complexity of a rational number $x = m/n$ is its *height* $H(x)$, defined as [27, chapter 3]

$$H(m/n) = \max(|m|, |n|) \qquad \gcd(m,n) = 1. \tag{1}$$

The notions of size and height are extended to two dimensions as follows

$$\|z\| = \max(|x|, |y|) \qquad H(z) = \max(H(x), H(y)) \qquad z = (x, y). \tag{2}$$

The height will typically grow exponentially along orbits, so we define an allied quantity, the *arithmetic exponent* of the point z:

$$\lambda(z) = \lim_{t\to\infty} \frac{1}{t} \log H(F^t(z)) \tag{3}$$

if the limit exists. We see that the arithmetic exponent of a point z is the average order of the so-called logarithmic height $\log(H(z))$ of the images of this point. Since $\lambda(z) = \lambda(F(z))$, the arithmetic exponent is a property of an orbit. If z is a (pre)-periodic point, then $H(F^t(z))$ is bounded, so that $\lambda(z) = 0$ (as long as the orbit of z doesn't go through the origin).

The quantity $\lambda(z)$ is closely related to the *arithmetic entropy* introduced by Silverman [28], where the limit (3) is replaced by a lim sup.

Further indicators of complexity are defined by means of the p-adic absolute value $|\cdot|_p$, where p is a prime number. (For background reference on p-adic numbers, see [13].) Let the *order* $\nu_p(m)$ of an integer m be the largest non-negative integer k such that p^k divides m, with $\nu(0) = \infty$. This definition is extended to the rational numbers $r = m/n$ by letting $\nu_p(r) = \nu_p(m) - \nu_p(n)$ (the value of this expression doesn't depend on m and n being co-prime). Finally, we define

$$|\, r\,|_p = p^{-\nu_p(r)}.$$

The function $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}$ has the properties of the ordinary absolute value, with the triangular inequality replaced by the stronger ultrametric inequality

$$|x+y|_p \leqslant \max(|x|_p, |y|_p) \qquad \text{or} \qquad \nu_p(x+y) \geqslant \min(\nu_p(x), \nu_p(y)) \tag{4}$$

where equality holds if $|x|_p \neq |y|_p$ (or $\nu_p(x) \neq \nu_p(y)$). We shall be using the estimate

$$\nu_p(n) \leqslant \frac{\log n}{\log p} \qquad n \geqslant 1. \tag{5}$$

The following identity connects the various absolute values over $\mathbb{Q}$:

$$\forall x \in \mathbb{Q} \setminus \{0\}, \qquad |x| \prod_p |x|_p = 1 \tag{6}$$

where the product is taken over all primes. Only finitely many terms of this product are different from 1; they correspond to the prime divisors of the numerator and the denominator of x.

In two dimensions we use the quantities

$$\|z\|_p = \max(|x|_p, |y|_p) \qquad \nu_p(z) = \min(\nu_p(x), \nu_p(y)). \tag{7}$$

The norm $\|\cdot\|_p$ and valuation ν_p can be shown to satisfy the ultrametric inequalities analogous to (4), respectively, with equality holding if the two terms have distinct size. Next we define the analogue of (3), namely the *p-adic* (*arithmetic*) *exponent* $\lambda_p(z)$ of the initial point z of an orbit:

$$\lambda_p(z) = \lim_{t\to\infty} -\frac{1}{t}\, \nu_p(F^t(z)). \tag{8}$$

Comparing (8) with (3), we note that the function ν_p is already logarithmic, and that there is no need of considering separately numerator and denominator, since the prime p will appear only in one of them.

The functions λ and λ_p should be compared with the so-called *canonical height* defined for morphisms of degree greater than one [27, chapter 3]. In this case, in place of (1) one defines

$$\hat{H}(m/n) = \max(|m|, |n|) \prod_p \max(|m|_p, |n|_p)$$

and then one lets

$$\hat{h}(x) = \lim_{t\to\infty} \frac{1}{\deg(F)^t} \log H(F^t(x))$$

where $\deg(F) > 1$ is the degree of F. The height $\hat{h}$ behaves nicely under iteration: $\hat{h}(F(x)) = \deg(F)\hat{h}(x)$. It measures the average rate of growth of the degree of F, collecting contributions from all absolute values. In our case, we have kept the contributions from the various primes separate (as in the so-called *local canonical heights*) because they contain valuable information about the dynamics.

The height may be used to characterize generic properties of rational points. To this end, we consider the set $\mathcal{B}_N$ of points in $\mathbb{Q}^2$ whose height is at most N:

$$\mathcal{B}_N = \{z \in \mathbb{Q}^2 \,:\, H(z) \leqslant N\}. \tag{9}$$

This set is finite. Indeed if $H(m/n) \leqslant N$, then $H(-m/n), H(\pm n/m) \leqslant N$, and we deduce that

$$\#\mathcal{B}_N = \big(3 + 4\sum_{k=2}^{N} \phi(k)\big)^2 \sim \frac{12^2}{\pi^4} N^4 \qquad (N \to \infty)$$

where ϕ is Euler's function [16, section 5.5] and where we have used the estimate $\sum_{k=1}^{N} \phi(k) \sim 3N^2/\pi^2$ (see [16, theorem 330] and also [27, p 135]). Half of the elements of B_N lie within the square $\|z\| \leqslant 1$, where they approach a uniform distribution (because the Farey sequence has that property [26, 10]); the other half lie outside the square, and they are obtained from the points inside the square by an inversion. Thus the limiting distribution of points of bounded height approaches a smooth limit on sufficiently regular bounded sets.

Let us now consider a set A such that $A \subset X \subset \mathbb{Q}^2$, where X is some ambient set (possibly the whole of $\mathbb{Q}^2$). The density $\mu(A)$ of A (in X) with respect to $\mathcal{B}_N$ is given by

$$\mu(A) = \lim_{N\to\infty} \frac{\#(A \cap \mathcal{B}_N)}{\#(X \cap \mathcal{B}_N)} \tag{10}$$

if the limit exists[1]. If $\mu(A) = 1$, then we say that A is 'generic', or that the defining property of A holds 'almost everywhere' (in X). For example, the rational points on a smooth curve on the plane have zero density and hence are non-generic.

For the numerical experiments reported in section 5 we have chosen maps F of the form

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad (x, y) \mapsto (f(x) - y, dx) \tag{11}$$

where f is a piecewise-affine real function and d is a real number (the Jacobian determinant of F). More precisely, we have a set I of indices (possibly infinite), a partition $\{\Delta_i\}_{i\in I}$ of the real line into intervals, and a collection $\{f_i\}_{i\in I}$ of real affine functions

$$f_i : \mathbb{R} \to \mathbb{R} \qquad x \mapsto a_i x + b_i \qquad a_i, b_i \in \mathbb{R}$$

such that

$$f(x) = f_i(x) \qquad x \in \Delta_i.$$

If $d = 1$, then for any choice of f the map F is area-preserving (see section 4).

Let now $a_i, b_i, d \in \mathbb{Q}$. Then the set $\mathbb{Q}^2$ is invariant under F, and it makes sense to restrict the dynamics to rational points. (In fact one can restrict the space further —see the appendix.)

The 2-adic exponent for some orbits of the map F given by

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{3}{2} & x < -1 \\ 0 & -1 \leqslant x \leqslant 1 \\ \frac{3}{2}x - \frac{3}{2} & x > 1 \end{cases} \tag{12}$$

with $d = 1$ is shown in figure 2. The initial conditions are evenly spaced rational points on the positive x-axis. The alternation of constancy and fluctuations is a distinctive feature of the arithmetic exponents along smooth curves in phase space, which reflects to the co-existence of regular and irregular motions. (To wit, compare figures 1 and 2.)

The plan of this paper is the following. In section 2 we compute the p-adic exponents in affine maps, and show that, generically, all rational points have the same exponent (theorem 1). We identify the conditions under which convergence of the exponent is non-uniform, but also show that the set of points having slow convergence have an exponentially large height. We then obtain explicit formulae for the valuation function ν_p along orbits in terms of Lucas polynomials; this gives us an alternative proof of theorem 1. In section 3 we determine the arithmetic exponent of an affine map, and show that, generically, all rational points have the same exponent (theorem 3). Theorems 1 and 3 are quite natural, but to the best of our

[1] For this it suffices to require that the closure of the boundary of A has zero measure (Jordan measurability)

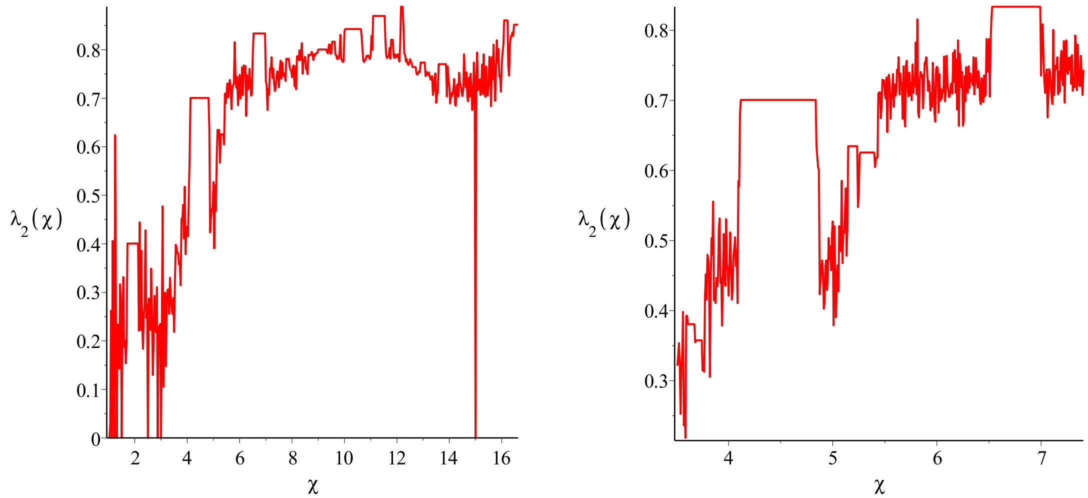

Figure 2: Behaviour of the exponent $\lambda_2(x)$ for the map F defined in equation (12), with initial conditions $z_0 = (x, 0)$. The plot on the right shows a detail of that on the left.

knowledge they have not appeared in the literature (in particular, that of affine automorphisms in two dimensions [27, section 7.1]). In section 4 we consider piecewise-affine maps F of $\mathbb{Q}^2$ (which include maps of the type (11)), and their islands, which are bounded invariant domains where the motion is locally linear. In the islands the results of the previous sections apply and all exponents are constants, which explains the plateaus in figure 2 (theorem 4).

In section 5 we explore numerically the convergence of arithmetical exponents in the chaotic regions, and also consider briefly the exponents of quasi-periodic points. As a result, we formulate two conjectures. In the appendix we construct a set $\mathbb{L}^2$, where $\mathbb{L}$ is a module over a certain sub-ring of $\mathbb{Q}$ depending on the map's parameters, which serves as a natural minimal phase space of a piecewise affine map. This is the set relevant to our numerical experiments.

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2. p-ADIC EXPONENTS IN AFFINE MAPS

We consider the behaviour of p-adic exponents (8) of the rational points for the affine map:

$$F : \mathbb{Q}^2 \to \mathbb{Q}^2 \qquad z = (x, y) \mapsto \mathrm{M}\, z + s \tag{13}$$

where $\mathrm{M} \in \mathrm{GL}(2, \mathbb{Q})$ is a non-singular matrix with rational entries, and $s \in \mathbb{Q}^2$. (For notational ease, we do not use transpose symbols where it is clear by context, e.g., for z and s above.)

The map F has a single rational fixed point

$$z^* = (x^*, y^*) = -(\mathrm{M} - \mathbb{1})^{-1}\, s,$$

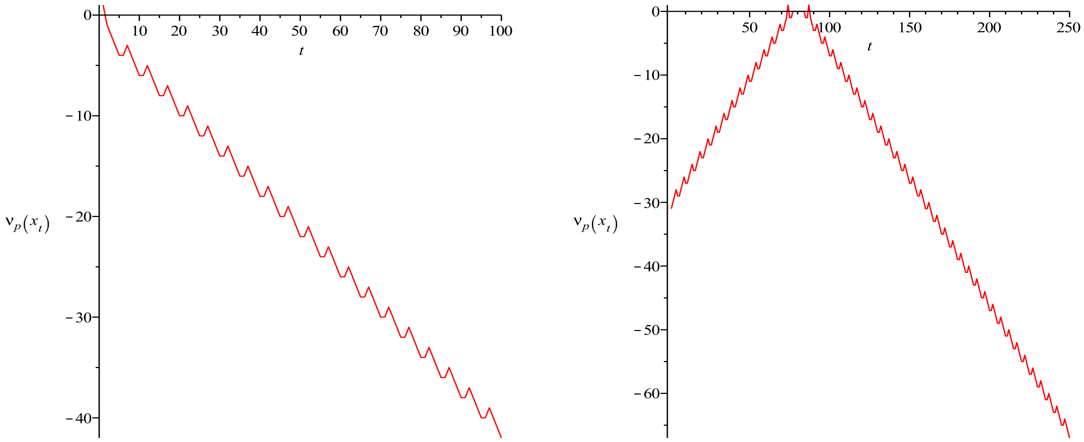

Figure 3: Time-dependence of $\nu_2(x_t)$ for two rational orbits of the map (12), with very close initial conditions inside the same island with elliptic periodic point $z^* = (21/11, 0)$. Left: typical behaviour, for $z_0 = (2, 0)$. Right: anomalous behaviour, for $z_0 = (2, 0) + z'$ with $\|z'\| < 10^{-8}$. In this case the point z_0 lies in the vicinity of the stable manifold of z^* in $\mathbb{Q}_2^2$.

and if $z_0 = z^* + z_0'$, then

$$z_t = F^t(z_0) = \mathrm{M}^t\, z_0' + z^*. \tag{14}$$

We define

$$T = \mathrm{tr}(\mathrm{M}), \qquad D = \det(\mathrm{M}), \tag{15}$$

and we let $q(x) = x^2 - Tx + D$ be the characteristic polynomial of M, with roots α and β.

The computation of p-adic exponent is an eigenvalue problem analogous to the computation of the ordinary Lyapunov exponent. Further insight is obtained by studying the detailed behaviour of the sequence $(\nu_p(z_t))$ (see figure 3), which will be considered in section 2.2.

Theorem 1. *Let F be the affine map (13) with T, D, s as above. If $s = (0, 0)$, then for almost all $z \in \mathbb{Q}^2$ we have:*

i) if $\nu_p(D) > 2\nu_p(T)$ then $\lambda_p(z) = -\nu_p(T)$;
ii) if $\nu_p(D) \leqslant 2\nu_p(T)$ then $\lambda_p(z) = -\nu_p(D)/2$.

If $s \neq (0, 0)$ then the above expressions for λ_p must be replaced by $\max(-\nu_p(T), 0)$ and $\max(-\nu_p(D)/2, 0)$, respectively.

Proof. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$. The eigenvalues α, β of M lie in a field K which is either $\mathbb{Q}_p$ or a quadratic extension of $\mathbb{Q}_p$. In K there is a prime element π (either p or $\sqrt{p}$) and a valuation ν_π, which is either ν_p or is an extension of ν_p which agrees with ν_p on $\mathbb{Q}$. Let α be a largest eigenvalue, that is, $\nu_\pi(\alpha) \leqslant \nu_\pi(\beta)$. Let

$$u = \nu_p(D), \qquad v = \nu_p(T) \tag{16}$$

and let Π be the Newton polygon of $q(x)$, namely the convex hull of the points $(0, \infty)$, $(0, u)$, $(1, v)$, $(2, 0)$, $(2, \infty)$. If $u > 2v$ then Π has two finite sides with distinct slopes $v - u$ and $-v$, of which the latter is the largest. Hence by [13, Theorem 6.4.7] we have $\nu_\pi(\beta) = u - v$

and $\nu_\pi(\alpha) = v$. Likewise, if $u \leqslant 2v$ then Π has one finite side of slope $-u/2$. Hence $\nu_\pi(\beta) = \nu_\pi(\alpha) = u/2$.

First we consider the parameter $s = (0, 0)$. We begin with the case $|\alpha|_\pi > |\beta|_\pi$, which is case i). We have $K = \mathbb{Q}_p$ (see section 2.1), and we write

$$z_t = \alpha^t c_1 \mathbf{w}_1 + \beta^t c_2 \mathbf{w}_2 \tag{17}$$

where the $\mathbf{w}_i$ are linearly independent eigenvectors of M in $\mathbb{Q}_p^2$ and the coefficients c_i are in $\mathbb{Q}_p$. For generic initial conditions $c_1 \neq 0$ (i.e., z_0 does not lie in the eigenspace generated by $\mathbf{w}_2$), the p-adic exponent in a linear system is determined by the eigenvalue with largest p-adic absolute value, which is α. Specifically, for all large enough t, the two terms in (17) have distinct size, and hence from (7) and following comments we see that

$$\|z_t\|_p = |\alpha|_p^t \|c_1 \mathbf{w}_1\|_p, \tag{18}$$

from which $\nu_p(z_t) \sim t\nu_p(\alpha)$ and the result follows.

Let us now deal with case ii). If $|\alpha|_\pi = |\beta|_\pi$, but $\alpha \neq \beta$, we rewrite (17) as

$$z_t = \alpha^t u_t \qquad\qquad u_t = c_1 \mathbf{w}_1 + (\beta/\alpha)^t c_2 \mathbf{w}_2,$$

noting that α is non-zero. Then $\mu = \beta/\alpha$ is a p-adic unit, and hence there exits a smallest positive integer n such that $\mu^n = \overline{\mu} = 1 + \gamma$ with $|\gamma|_\pi < 1$. If $\gamma = 0$, that is, μ is a root of unity, then u_t is periodic, and hence $\lambda_p(z) = -\nu_p(\alpha)$, as desired.

If $\gamma \neq 0$, then the sequence $(\overline{\mu}^t)$ is dense in a disc (see [3] and [17, chapter 5]), and hence (μ_t) is dense in the union of n discs. Thus each component of $z_t = (x_t, y_t)$ is also dense in a finite union of discs. If none of these discs contains the origin, then $\|u_t\|_p$ assumes finitely many values, and the result follows. Otherwise $\|u_t\|_p$ is bounded above but not bounded away from zero, and the rate at which $\|u_t\|_\pi$ approaches zero is the same as the rate at which $\overline{\mu}^t$ approaches 1. From the binomial theorem we obtain $|\overline{\mu}^t - 1|_\pi = p^{\nu_\pi(t)}$ and hence the quantity $\max_{t<T}\{\nu_\pi(z_t')\}$ grows logarithmically, from (5). It follows that $\nu_p(z_t) \sim t\nu_\pi(\alpha)$, as desired.

Finally, if the Jordan form of M is not diagonal, then the sequence (z_t) contains a term affine in t. The contribution of this term is logarithmic, again due to (5). Hence, in all cases, $\lambda_p = -\nu_\pi(\alpha)$.

If $s \neq (0, 0)$ then from (7) and (14) we find $\nu_p(z_t) \geqslant \min(\nu_p(M^t z_0'), \nu_p(z^*))$. In case i), if $v < 0$, then, for all sufficiently large t the first term is the largest, that is, $\nu_p(z_t) = \nu_p(M^t z_0')$, and the previous analysis applies. Likewise, if $v > 0$, then eventually the second term becomes the largest, and since this term is constant, we get $\lambda_p = 0$. If $v = 0$, then the inequality remains such, but the first term grows at most logarithmically, and so $\lambda_p = 0$. Case ii) is treated similarly. □

2.1. **p-adic eigenspaces.** We look more closely at the p-adic dynamics of a linear map with eigenvalues α, β of distinct magnitude, which is case i) of theorem 1. Using the notation (16), we see that if $u > 2v$, then, necessarily, $v \neq +\infty$ ($T \neq 0$). Letting

$$T = T' p^{\nu(T)} \qquad\qquad D = D' p^{\nu(D)}$$

we have $T' \neq 0$. Let now $\theta = p^{-v}\alpha$, where θ is a root of the polynomial

$$s(x) = x^2 - T'x + D'p^{u-2v} \qquad \text{with} \qquad \frac{ds(x)}{dx} = 2x - T'. \tag{19}$$

We have the factorisation:

$$s(x) \equiv x(x - T') \pmod{p}.$$

Then $s(x)$ has two distinct roots modulo p, congruent to 0 and T', respectively, and at these roots $s(x)$ is equal to $\pm T' \not\equiv 0$ from (19). From Hensel's lemma [13, section 3.4], we have that $s(x)$ has two distinct roots in $\mathbb{Z}_p$, which we denote by α', β', of which the largest, α', is a unit. Hence $\nu_p(\alpha) = \nu_p(T)$, in agreement with theorem 1

Now, the polynomial $s(x)$ is irreducible over $\mathbb{Q}$ if and only if $q(x)$ is irreducible, since their roots differ by a rational factor. If $q(x)$ is reducible, then these eigenspaces have infinitely many rational points; if $q(x)$ is irreducible, then these eigenspaces have no rational points, apart from the origin.

In the first case there will be a non-generic (zero-density) set of rational points with exponent $\nu_p(T) - \nu_p(D)$, lying on the eigenspace corresponding to the smallest eigenvalue. Thus a sufficient condition for all non-zero rational points to have the same p-adic exponent is $1 = v \leqslant u$, for in this case $q(x)$ is irreducible by Eisenstein's criterion [13, Proposition 5.3.11].

In the second case all points have the same exponent $-\nu_p(D)$, apart from the origin. Rational approximants for the roots of $q(x)$ may be constructed by iterating Newton's map for $q(x)$ sufficiently many times, with an appropriate initial condition [13, section 3.4]. The components of an eigenvector of M may be chosen to be linear expression in such eigenvalues, with rational coefficients.

We are interested in motion in the p-adic vicinity of the eigenspace W_p^β corresponding to the smaller eigenvalue. We begin with a general lemma.

Lemma 2. *Let p be a prime number. For any $z \in \mathbb{Q}^2$, any $\zeta \in \mathbb{Q}_p^2$, and any $\epsilon > 0$, there is $z' \in \mathbb{Q}^2$ such that*

$$\|z' - z\| + \|z' - \zeta\|_p < \epsilon$$

with the norms (2) and (7), respectively.

Proof. The rational sequence

$$r_k = \frac{1}{1 + p^k} \qquad k = 1, 2, \dots \tag{20}$$

has the property that, as $k \to \infty$, $r_k \to 0$ in $\mathbb{Q}$, while $r_k \to 1$ in $\mathbb{Q}_p$. Let now $z = (x, y) \in \mathbb{Q}^2$ and $\epsilon > 0$ be given. For any $(a, b) \in \mathbb{Q}^2$, the sequence

$$z^{(k)} = z + r_k(a, b) \qquad k = 1, 2, \dots \tag{21}$$

converges to z in $\|\cdot\|$. We choose K_1 such that, for all $k > K_1$, we have $\|z^{(k)} - z\| < \epsilon/2$.

Let $\zeta = (\zeta_1, \zeta_2)$. We will show that a, b in (21) may be chosen so that $\|z^{(k)} - \zeta\|_p \to 0$. We find

$$z^{(k)} - \zeta = (x + ar_k - \zeta_1,\ y + br_k - \zeta_2).$$

Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, we can find $s = (s_1, s_2) \in \mathbb{Q}^2$ such that $\|\zeta - s\|_p < \epsilon/2$. Let $a = s_1 - x$. Then there is K_2 such that for all $k > K_2$ we have $|x + ar_k - s_1|_p < \epsilon/2$. Similarly, let $b = s_2 - y$. Then there is K_3 such that for all $k > K_3$ we have $|y + br_k - s_3|_p < \epsilon/2$.

Let now $K = \max(K_1, K_2, K_3)$. For all $k > K$, the ultrametric inequality (4) gives

$$\begin{aligned} |x + ar_k - \zeta_1|_p &= |x + ar_k - s + s - \zeta_1|_p \\ &\leqslant \max(|x + ar_k - s|_p, |s - \zeta_1|_p) \\ &\leqslant \max(\epsilon/2, \epsilon/2) = \epsilon/2. \end{aligned}$$

Similarly, $|y + br_k - \zeta_2|_p \leqslant \epsilon/2$. In the same k-range, we obtain

$$\|z^{(k)} - \zeta\|_p = \max(|x + ar_k - \zeta_1|_p, |y + br_k - \zeta_2|_p) < \max(\frac{\epsilon}{2}, \frac{\epsilon}{2}) = \frac{\epsilon}{2}.$$

We have shown that for all $k > K$, the point $z' = z^{(k)}$ lies within an $\epsilon/2$-neighbourhood of z in the ordinary norm, and within an $\epsilon/2$-neighbourhood of ζ in the p-adic norm. The lemma follows. $\square$

Now choose $\zeta \in W_p^\beta \subset \mathbb{Q}_p^2$. The lemma states that arbitrarily close to any rational point we can find another rational point as close as we please to an eigenvector ζ of M. Thus, irrespective of the rationality of the eigenvalues, there always will be a dense set of initial conditions that are to close to the eigenspace W_p^β to cause the second term in (17) to dominate for small values of t. For these orbits the convergence of λ_p will be slow. The sequence $(\nu_p(z_t))$ will feature two distinct affine regimes, with slopes $\nu_p(T) - \nu_p(D)$ and $-\nu_p(T)$, respectively. If the slopes have different sign and $z^* \neq (0,0)$, then these regimes may be separated by a third regime, determined by a constant lower bound —see figure 3.

We want to justify the statement that the exponent of a 'typical' rational point converges rapidly to its asymptotic value $-\nu_p(T)$, in apparent defiance of the pathologies exposed by lemma 2 above. We will show that points for which the non-archimedean exponent has anomalous time-dependence must also have a large height. For brevity, we consider only the linear case.

Let $\mathcal{E} = \mathbb{Q}^2 \setminus W_p^\beta$. Then, in the regime in which equation (18) holds, we have that $\|z_{t+1}\|_p = |\alpha|_p \|z_t\|_p$. Now we define the *lag time* $\tau(z)$ to be the time at which this asymptotic regime sets in, namely,

$$\tau : \mathcal{E} \to \mathbb{N} \qquad \tau(z) = \min\{t \in \mathbb{N} \,:\, \forall s \geqslant t,\ \|z_{s+1}\|_p = |\alpha|_p \|z_s\|_p\}. \tag{22}$$

Because the eigenspace W_p^β of β has been excluded, the function τ is well-defined. The larger the value of $\tau(z)$, the slower the convergence of the p-adic exponent $\lambda_p(z)$.

From equations (17) and (22) and the ultrametric inequality, we find that

$$\left|\frac{\alpha}{\beta}\right|^{\tau(z)} = \left|\frac{c_2(z)}{c_1(z)}\right|_p \frac{\|\mathbf{w}_2\|_p}{\|\mathbf{w}_1\|_p}\, \gamma(z)$$

where the quantity $\gamma \in (|\beta/\alpha|_p, 1]$ ensures that τ is an integer. Hence, as $\tau \to \infty$ we must have $|c_2/c_1|_p \to \infty$. Now, for any non-zero rational number r and any prime p, we have the estimate $H(r) \geqslant p^{|\nu_p(r)|}$. Hence for large enough τ there is a constant κ independent of z such that

$$\kappa \left|\frac{\alpha}{\beta}\right|^{\tau(z)} \leqslant \left|\frac{c_2(z)}{c_1(z)}\right|_p = p^{\nu_p(c_1(z)/c_2(z))} \leqslant H(c_1'(z)/c_2'(z)),$$

where c_1' and c_2' are any rational approximants of c_1 and c_2 such that $\nu_p(c_1'/c_2') = \nu_p(c_1/c_2)$. Thus the height of the ratio of the coefficients of z_t in the representation (17) grows at least exponentially in the lag time τ.

2.2. **Explicit formulae.** In this section we derive explicit formulae for z_t and $\nu_p(z_t)$, which will give us an alternative, more direct proof of theorem 1, with the exclusion of some special cases.

From (14), we need the powers of the rational matrix M. Using the Cayley-Hamilton theorem, one proves by induction (e.g., [7, Lemma 1]) that for $t \in \mathbb{Z}$, the following relation holds

$$\mathrm{M}^t = U_t\,\mathrm{M} - D\,U_{t-1}\,\mathbb{1}, \tag{23}$$

where the sequence of rational numbers $U_t = U_t(T, D)$ obeys the recursion

$$U_0 = 0, \quad U_1 = 1, \qquad U_{t+1}(T,D) = T U_t(T,D) - D U_{t-1}(T,D), \quad t \geqslant 1. \tag{24}$$

If T and D are integers, then U_t is an integer sequence, known as the *Lucas sequence of the first kind*. In a slight abuse of notation, we will use the same symbol U_t for our case of a rational sequence generated by (24) because many of the properties of Lucas sequences are independent of whether T and D are integers. It follows by iteration of (24) that $U_t(T, D)$ is a polynomial in T and D with integer coefficients. Its general form [21] is

$$U_t(T,D) = \sum_{k=0}^{\lfloor (t-1)/2 \rfloor} c_k^{(t)}\, T^{t-2k-1}\,(-D)^k \tag{25}$$

where

$$c_k^{(t)} = \binom{t-k-1}{k}.$$

We note that

$$0 \leqslant \nu_p\left(\binom{n}{m}\right) \leqslant \left\lfloor \frac{\log n}{\log p} \right\rfloor - \nu_p(m). \tag{26}$$

From (25), we have that, for all $t \geqslant 1$:

- The polynomial $U_t(T, D^2)$ is homogeneous of degree $t-1$.
- The leading term of U_t is T^{t-1} (i.e., U_t is monic) while the term of lowest total degree is $(-D)^{\frac{t-1}{2}}$ if t is odd and $\frac{t}{2}\,T\,(-D)^{\frac{t}{2}-1}$ if t is even.

From (14) with (23), we see that

$$z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = U_t(T,D) \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} - D\,U_{t-1}(T,D) \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \begin{pmatrix} x^* \\ y^* \end{pmatrix} \tag{27}$$

where

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \mathrm{M} \begin{pmatrix} x_0' \\ y_0' \end{pmatrix}.$$

Let us now consider the first component of z_t in (27). Using (25) we rewrite it as follows:

$$x_t = \mathcal{T}_t^{(1)} + \mathcal{T}_t^{(0)} + x^* \tag{28}$$

where

$$\mathcal{T}_t^{(1)} = x_1' \sum_{i_1=0}^{\lfloor (t-1)/2 \rfloor} c_{i_1}^{(t)}\, T^{t-2i_1-1}\,(-D)^{i_1}, \qquad \mathcal{T}_t^{(0)} = x_0' \sum_{i_0=1}^{\lfloor t/2 \rfloor} c_{i_0-1}^{(t-1)}\, T^{t-2i_0}\,(-D)^{i_0-1}. \tag{29}$$

The greatest value of the summation indices is given by:

$$\begin{array}{rlll} t \text{ odd}: & i_1^{max} := & \lfloor (t-1)/2 \rfloor = (t-1)/2 & i_0^{max} := \lfloor t/2 \rfloor = (t-1)/2 \\ t \text{ even}: & i_1^{max} := & \lfloor (t-1)/2 \rfloor = t/2 - 1 & i_0^{max} := \lfloor t/2 \rfloor = t/2. \end{array}$$

From (28) and the ultrametric inequality (4) it follows that

$$\nu_p(x_t) \geqslant \min(\nu_p(\mathcal{T}_t^{(1)}), \nu_p(\mathcal{T}_t^{(0)}), \nu_p(x^*)). \tag{30}$$

For the order of the first term, using (29) gives

$$\nu_p(\mathcal{T}_t^{(1)}) \geqslant \nu_p(x_1') + \min_{i_1}(\nu_p(c_{i_1}^{(t)}) + i_1\,(\nu_p(D) - 2\nu_p(T)) + (t-1)\nu_p(T)).$$

We have three cases:

i) $\nu_p(D) > 2\nu_p(T)$. Using (26), we see that the unique minimum is achieved at $i_1 = 0$ with $c_0^{(t)} = 1$, giving

$$\nu_p(\mathcal{T}_t^{(1)}) = \nu_p(x_1') + (t-1)\nu_p(T).$$

ii) $\nu_p(D) < 2\nu_p(T)$. The unique minimum is achieved at $i = i_1^{max}$, where $c_{i_1^{max}}^{(t)}$ is equal to 1 when t is odd and to $t/2$ when t is even. Thus

$$\nu_p(\mathcal{T}_t^{(1)}) = \nu_p(x_1') + \begin{cases} \dfrac{t-1}{2}\,\nu_p(D) & t\,\text{odd} \\ \dfrac{t-2}{2}\,\nu_p(D) + \nu_p\big(\dfrac{t}{2}\big) + \nu_p(T) & t\,\text{even.} \end{cases}$$

iii) $\nu_p(D) = 2\nu_p(T)$. A minimum is achieved at $i_1 = 0$, with $c_0^{(t)} = 1$, giving

$$\nu_p(\mathcal{T}_t^{(1)}) \geqslant \nu_p(x_1') + (t-1)\nu_p(T).$$

A very similar analysis for the order $\nu_p(\mathcal{T}_t^{(0)})$ in (30) gives

i) $\nu_p(D) > 2\nu_p(T)$.

$$\nu_p(\mathcal{T}_t^{(0)}) = \nu_p(x_0') + (t-2)\,\nu_p(T) + \nu_p(D).$$

ii) $\nu_p(D) < 2\nu_p(T)$.

$$\nu_p(\mathcal{T}_t^{(0)}) = \nu_p(x_0') + \begin{cases} \dfrac{t-1}{2}\,\nu_p(D) + \nu_p\big(\dfrac{t-1}{2}\big) + \nu_p(T) & t\,\text{odd} \\ \dfrac{t}{2}\,\nu_p(D) & t\,\text{even.} \end{cases}$$

iii) $\nu_p(D) = 2\nu_p(T)$.

$$\nu_p(\mathcal{T}_t^{(0)}) \geqslant \nu_p(x_0') + t\nu_p(T).$$

The analysis for the second component y_t in (27) is identical.

From the above and (30), we have:

i) $\nu_p(D) > 2\nu_p(T)$:

$$\nu_p(x_t) \geqslant \min\{\nu_p(x_1') + (t-1)\nu_p(T), \nu_p(x_0') + (t-2)\,\nu_p(T) + \nu_p(D), \nu_p(x^*)\}. \tag{31}$$

If $\nu_p(T) \geqslant 0$, then the linear terms are increasing, and we have two possibilities. If $x^* \neq 0$, then eventually we have $\nu_p(x_t) = \nu_p(x^*)$. If, $x^* = 0$, then eventually, under the non-degeneracy condition

$$\nu_p(x_1') + \nu_p(T) \neq \nu_p(x_0') + \nu_p(D) \tag{32}$$

a unique minimum emerges in (31), and $\nu_p(x_t)$ becomes affine. In the degenerate case, the inequality (31) remains such.

If $\nu_p(T)<0$, then $\nu_p(x_t)$ is initially bounded below by a constant. If (32) holds, then the minimum is achieved by a single affine term, and (31) becomes an equality.

Given a similar analysis for $\nu_p(y_t)$, we have thus proved part $i)$ of theorem 1, under the restriction (32) or the corresponding restriction for y (a single restriction will suffice). Such a restriction avoids the pathologies described in section 2.1.

$ii)$ $\nu_p(D)<2\nu_p(T)$:

$$
\begin{aligned}
\nu_p(x_t) \;&\geqslant\; \min\Big\{\nu_p(x_1')+\frac{t-1}{2}\nu_p(D),\\
&\qquad \nu_p(x_0')+\frac{t-1}{2}\,\nu_p(D)+\nu_p\big(\frac{t-1}{2}\big)+\nu_p(T),\nu_p(x^*)\Big\} \qquad t\,\text{odd}\\
\nu_p(x_t) \;&\geqslant\; \min\Big\{\nu_p(x_1')+\frac{t-2}{2}\nu_p(D)+\nu_p\big(\frac{t}{2}\big)+\nu_p(T),\\
&\qquad \nu_p(x_0')+\frac{t}{2}\,\nu_p(D),\nu_p(x^*)\Big\} \qquad t\,\text{even.}
\end{aligned}
\tag{33}
$$

If $\nu_p(D)\geqslant 0$, then the linear terms are increasing, and we have two possibilities. If $x^*\neq 0$, then eventually we have $\nu_p(x_t)=\nu_p(x^*)$. If $x^*=0$, then (33) becomes an equality provided that (here for odd t)

$$
\nu_p(x_1')-\nu_p(x_0')-\nu_p(T)\neq\nu_p\big(\frac{t-1}{2}\big)
\tag{34}
$$

and similarly for even t. The right-hand side of (34) is non-negative and grows without bounds but at most logarithmically, due to (5). If $\nu_p(x_1')=\nu_p(x_0')$, then the left-hand side of (34) is negative, so this condition always holds and we have

$$
\lim_{t\to\infty}\frac{\nu_p(x_t)}{t}=\frac{\nu_p(D)}{2}.
$$

If $\nu_p(x_1')\neq\nu_p(x_0')$, then the left-hand side of (34) can be made negative by multiplying the initial conditions by a suitable power of p. Thus there is a rescaled sequence for which the above limit holds. The linearity of the system ensures that the same limit holds for the original sequence.

If $\nu_p(D)<0$, then the linear terms decrease, and hence become dominant. There is a condition analogous to (34), and we reach an analogous result. This establishes the strict inequality in part $ii)$ of theorem 1.

$iii)$ $\nu_p(D)=2\nu_p(T)$:

$$
\nu_p(x_t)\geqslant\min\{\nu_p(x_1')+(t-1)\nu_p(T),\nu_p(x_0')+t\nu_p(T),\nu_p(x^*)\}.
$$

In this case we only obtain a lower bound for $\nu_p(x_t)$, and analogously for $\nu_p(y_t)$, leading to an upper bound for λ_p. One verifies that the latter agrees with the value of λ_p given in theorem 1 for this case.

3. Arithmetic exponent

In this section we determine the arithmetical exponent(3) for the rational points of the affine map F given in (13). The dynamics of F on $\mathbb{R}^2$ is standard [22, section 1.2].

Let T, D and $q(x)$ be as in section 2. For a rational number x we shall adopt the notation

$$x = \frac{\overline{x}}{\underline{x}} \qquad \overline{x}, \underline{x} \in \mathbb{Z}, \quad \gcd(\overline{x}, \underline{x}) = 1. \tag{35}$$

As before, the eigenvalues of M are α and β with $|\alpha| \geqslant |\beta|$.

We consider the prime divisors of the denominators of T and/or D, and split them into two disjoint families:

$$\begin{aligned} P_1 &= \{p : \nu_p(\underline{D}) < 2\nu_p(\underline{T})\} \\ P_2 &= \{p : \nu_p(\underline{D}) \geqslant 2\nu_p(\underline{T}),\ \nu_p(\underline{D}) \neq 0\}\,. \end{aligned}$$

Then we define

$$\lambda^* = \sum_{p \in P_1} \nu_p(\underline{T}) \log(p) + \frac{1}{2} \sum_{p \in P_2} \nu_p(\underline{D}) \log(p) \tag{36}$$

where the sum is zero if the corresponding set of primes is empty.

Theorem 3. *Let F and* M *be as in (13). Then for almost all rational initial conditions z, the arithmetical exponent $\lambda(z)$ defined in (3) is given by*

$$\lambda(z) = \max(0, \log|\alpha|) + \lambda^*$$

where α is a largest eigenvalue of M *and λ^* is as in* (36)*.*

Proof. We determine the height (1) of each component x_t and y_t of z_t. In each case, this means considering their numerator and denominator *after* cancelling common factors between them, so a given prime appears in only one of $\overline{x}_t$, $\underline{x}_t$ if it appears at all. From (6), we can write:

$$|\underline{x}_t| \prod_p p^{-\nu_p(\underline{x}_t)} = 1,$$

where the nontrivial terms in the product correspond to the prime divisors of $\underline{x}_t$.

We begin with the parameter value $s = (0, 0)$. From theorem 1 we have

$$\nu_p(z_t) \sim \begin{cases} -t\nu_p(T) & \text{if } \nu_p(D) > 2\nu_p(T) \\ -t\nu_p(D)/2 & \text{if } \nu_p(D) \leqslant 2\nu_p(T). \end{cases}$$

The only primes which will contribute to the height of $\underline{x}_t$ are the divisors of $\underline{T}$ or $\underline{D}$. The contribution of the primes which divide the denominator of the initial conditions is asymptotically zero. As a result, we have

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\log|\underline{x}_t| &= \sum_p \lim_{t\to\infty}\frac{\nu_p(\underline{x}_t)}{t}\log p\\
&= \sum_{p\in P_1}\lim_{t\to\infty}\frac{\nu_p(\underline{x}_t)}{t}\log p + \sum_{p\in P_2}\lim_{t\to\infty}\frac{\nu_p(\underline{x}_t)}{t}\log p\\
&= \sum_{p\in P_1}\lim_{t\to\infty}\frac{\nu_p(x_t)}{t}\log p + \sum_{p\in P_2}\lim_{t\to\infty}\frac{\nu_p(x_t)}{t}\log p\\
&= -\sum_{p\in P_2}\nu_p(T)\log p - \frac{1}{2}\sum_{p\in P_1}\nu_p(D)\log p\\
&= \sum_{p\in P_2}\nu_p(\underline{T})\log p + \frac{1}{2}\sum_{p\in P_1}\nu_p(\underline{D})\log p = \lambda^*.
\end{aligned} \tag{37}$$

The analogous calculation for $y_t = \overline{y}_t/\underline{y}_t$ means we have established

$$\lim_{t\to\infty}\frac{1}{t}\log|\underline{x}_t| = \lim_{t\to\infty}\frac{1}{t}\log|\underline{y}_t| = \lambda^*. \tag{38}$$

Now we consider the arithmetic exponent (3). As $\overline{x} = x\underline{x}$, we can write

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\log|\overline{x}_t| &= \lim_{t\to\infty}\frac{1}{t}\log|x_t\ \underline{x}_t|\\
&= \lim_{t\to\infty}\frac{1}{t}\log|x_t| + \lim_{t\to\infty}\frac{1}{t}\log|\underline{x}_t|,
\end{aligned} \tag{39}$$

provided the separate limits exist. To learn about the nature of x_t in the argument of the first logarithm on the right, we need to inject information on the archimedean dynamics of M on $\mathbb{Q}^2$.

We begin by assuming that M has diagonal Jordan form. If $|\alpha| \leqslant 1$, then all orbits are bounded, i.e., $|x_t|, |y_t| < C$ for some real number C independent of t. We have

$$0 < |\underline{x}_t| \leqslant H(x_t) = \max(|\underline{x}_t|, |\overline{x}_t|) \leqslant C\,|\underline{x}_t|$$

so that

$$\lim_{t\to\infty}\frac{1}{t}\log H(x_t) = \lim_{t\to\infty}\frac{1}{t}\log|\underline{x}_t|,$$

and similarly for $H(y_t)$ and since $\log|\alpha| \leqslant 0$, we recover (36) via (38).

If $|\alpha| > 1$, then (almost) all orbits in forward time escape to infinity at the rate

$$x_t^2 + y_t^2 \sim |\alpha|^{2t}(x_0^2 + y_0^2).$$

Because

$$\frac{1}{2}(x^2+y^2) \leqslant \max(|x|^2, |y|^2) \leqslant x^2 + y^2,$$

it follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\max(|x_t|, |y_t|) = \log|\alpha|. \tag{40}$$

We have

$$\begin{aligned}\frac{1}{t}\,\log\max(|x_t|,|y_t|) &= \frac{1}{t}\,\max(\log|x_t|,\log|y_t|)\\ &= \max\left(\frac{1}{t}\,(\,\log|\overline{x}_t|-\log|\underline{x}_t|\,),\frac{1}{t}\,(\,\log|\overline{y}_t|-\log|\underline{y}_t|\,)\right).\end{aligned}$$

From (40) and the known limits (38), we learn

$$\lambda(z_0)=\lim_{t\to\infty}\frac{1}{t}\,\log\max(|\overline{x}_t|,|\overline{y}_t|)=\log(|\alpha|)+\lambda^*$$

as desired.

If the Jordan form of M is not diagonal, then $\|z_t\|$ contains an affine term which grows sub-exponentially, and the exponential terms dominate. If $|\alpha|=1$, then $h(z)=0$. In this case $\lambda^*=0$ (both P_1 and P_2 are empty) and $\log|\alpha|=0$, as desired.

It remains to consider the case $s\neq(0,0)$, corresponding to a non-zero fixed point z^*. If $|\alpha|<1$, then all orbits are asymptotic to the fixed point z^*, so $x_t\to x^*$ and the first term on the RHS of (39) vanishes, while the case $|\alpha|\geqslant 1$ is dealt with by the previous analysis. Thus, asymptotically, the logarithmic height of $\overline{x}_t$ and $\underline{x}_t$ is the same, similarly for $\overline{y}_t$ and $\underline{y}_t$. From (38) we see that (3) has the value λ^*. □

The previous theorem shows that the arithmetic exponent depends only on T and D for the matrix M as these determine the eigenvalues. Thus this exponent is preserved by conjugacy in $\mathrm{GL}(2,\mathbb{Q})$. Related to M is its associated companion matrix C, also with rational entries:

$$C=\begin{pmatrix}T & -D\\ 1 & 0\end{pmatrix}. \tag{41}$$

It is well-known that provided M is not a rational multiple of the identity matrix, then M is conjugate to C over $\mathbb{Q}$.

4. Piecewise affine maps

We consider now two-dimensional piecewise-affine maps over the rationals, defined as follows. Given a finite or countable set I of indices, we choose a partition of $\mathbb{Q}^2$ into domains Ω_i, with $i\in I$. Typically, each Ω_i will be a convex (finite or infinite) polygon. For each $i\in I$, we choose $\mathrm{M}_i\in\mathrm{GL}_2(\mathbb{Q})$ and $s_i\in\mathbb{Q}^2$, to obtain the map $F_i:\mathbb{Q}^2\to\mathbb{Q}^2$ given by $z\mapsto \mathrm{M}_i z+s_i$. The mapping F is then defined by the rule

$$F:\mathbb{Q}^2\to\mathbb{Q}^2 \qquad z\mapsto F_i(z),\quad z\in\Omega_i. \tag{42}$$

We shall assume that the partition $\{\Omega_i\}$ is irreducible, namely that F is not differentiable on the boundaries of the domains Ω_i.

To every orbit (z_t) of F we associate a doubly-infinite sequence $\sigma=(\sigma_t)\in I^{\mathbb{Z}}$ via the rule

$$\sigma_t=i \quad\Leftrightarrow\quad z_t\in\Omega_i. \tag{43}$$

The maps (11) are of the type (42), with $\Omega_i=\Delta_i\times\mathbb{R}$. Their symbolic dynamics (43) is determed by the simpler condition

$$\sigma_t=i \quad\Leftrightarrow\quad x_t\in\Delta_i.$$

The function $z_0\mapsto\sigma(z_0)$ is not injective, and we are interested in the structure of the sets of points which share the same code. The map F fails to be differentiable on the set of lines and segments $\partial\Omega$, where $\partial\Omega$ is the union of the boundaries of the domains Ω_i. By forming all pre-images of these lines we obtain the discontinuity set X of the map:

$$X = \bigcup_{t \geqslant 0} F^{-t}(\partial\Omega) \qquad \partial\Omega = \bigcup_{i \in I} \partial\Omega_i. \tag{44}$$

The set X is a union of segments, lines, and rays. Now consider the complement of the closure of X in $\mathbb{R}^2$. This is an open set, which decomposes as the union of connected components. By construction, all points of each connected component have the same code.

The bounded connected components with a periodic code are called *islands*, denoted by $\mathcal{E}$. (This terminology is normally reserved for the area-preserving case, for which $\mathcal{E}$ is also periodic.) If n is the period of the code, then F^n is affine and the results of the previous section apply. The Jacobian J of F^n is the same at every point of the island, since it depends only on the code. Since $\mathcal{E}$ is bounded, the eigenvalues of J are necessarily in the closed unit disc in $\mathbb{C}$.

Let P be the set of prime divisors of the denominator of the trace or the determinant of the matrices M_i. This is the set of primes of interest to us (see also the appendix). Now fix $p \in P$ and embed the rational points of an island $\mathcal{E}$ in the space $\mathbb{Q}_p^2$. The following result justifies the presence of plateaus in the graph of λ_p displayed in figure 2.

Theorem 4. *Almost all points of a rational island have the same exponents λ and λ_p for all primes p. The latter are rational numbers.*

Proof. Let n be the period of the island. If the restriction of F^n to $\mathcal{E}$ has finite order, then all points in $\mathcal{E}$ are periodic, and their exponents is zero. Let us thus assume that F^n has infinite order and let J be the Jacobian of F^n on $\mathcal{E}$. The result follows by applying Theorems 1 and 3, respectively, to the affine map F^n, noting that T and D of (15) now refer to the trace and determinant of J, plus the respective results λ_p and λ^* of these theorems should be divided by n to account for the different time scale of the return map to the island. So the p-adic exponents are rationals, in general. $\square$

Let us now consider the behaviour of $\nu_p(z_t)$ for points in an island (figure 3). This is the case *i*) of theorem 1, where $\mathrm{M} = \mathrm{J}$ is the Jacobian of the return map to the island. The conditions of lemma 2 are satisfied by J. Hence, by choosing $\zeta \in W_p^\beta$, we can find near every point of the island initial conditions for orbits which perform rotations in $\mathbb{Q}^2$, while they simultaneously approach the unstable fixed point z^* in $\mathbb{Q}_p^2$ as close as we please.

5. Numerical experiments

In this section we explore the convergence of p-adic exponents for rational orbits in chaotic regions and their boundaries. Two such regions are displayed in figure 4, where in each case we have plotted a large number of points of a single rational orbit. These plots suggest that the closure of these orbits is a bounded subset of the plane, with positive Lebesgue measure.

At present, statements on this kind can only be established in very special cases. For piecewise affine symplectic maps, our knowledge of the boundary of chaotic regions is inadequate, and proofs of global stability have relied on the presence of piecewise-smooth bounding invariant curves, which is a non-generic situation [11, 4, 24]. In the present examples there are no such curves, and we can only establish boundedness inside island chains. Thus any consideration on convergence of the arithmetic exponents along other types of non-periodic orbits will necessarily be speculative.

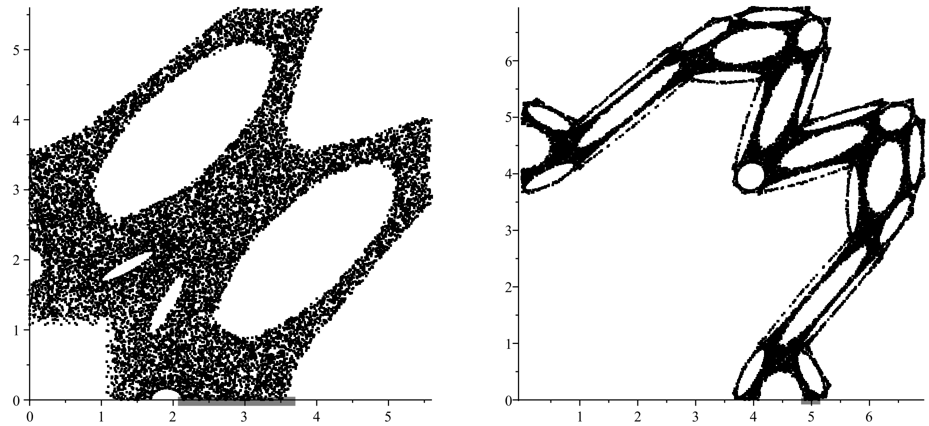

Figure 4: Chaotic regions of the map (12). We display the first 50000 iterates of the point $z_0 = (7/3, 0)$ (left) and $z_0 = (5, 0)$ (right) within the first quadrant.

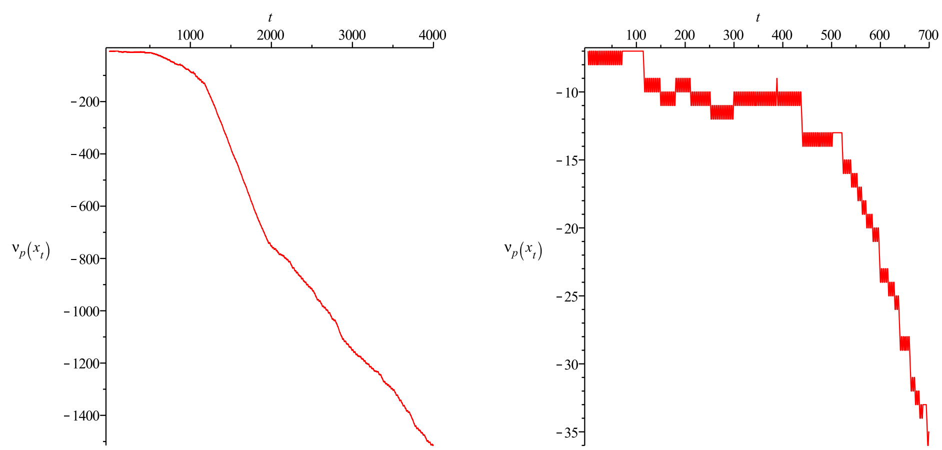

Figure 5: Time-dependence of $\nu_2(x_t)$ for one rational orbit of the map (12). Left: typical behaviour, showing transitions between four different regimes. Right: detail of the first plateau and the beginning of the drop.

We begin to examine the behaviour of $\nu_p(x_t)$ along an individual orbit of the area-preserving map (11), with f given by (12). There is only one prime in P, namely $p = 2$ (the set P was defined in section 4). We choose the initial condition z_0 near the boundary of the square stable region containing the origin in figure 4, left. The time-dependence of ν_2, shown in figure 5, features a concatenation of distinct regimes, in which the rate of change of ν_2 remains approximately constant.

Each regime has a dynamical signature. In figure 6 we plot the orbit that generated the data of figure 5. The initial plateau corresponds to the neighbourhood of the square island mentioned above. After a transitional phase, the orbit migrates to a neighbourhood of the large island chain visible in the middle of the chaotic sea, where the local value of the height

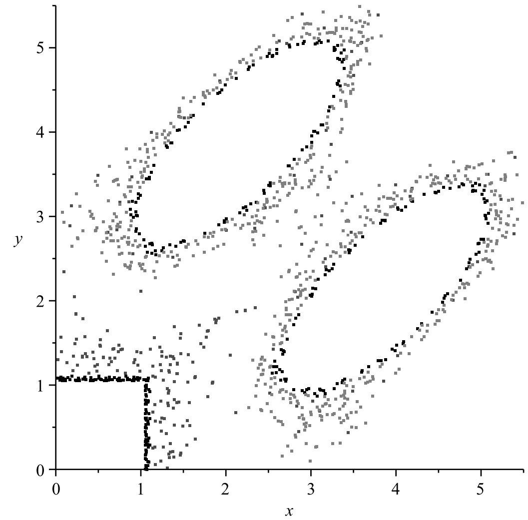

Figure 6: Phase plot of the orbit of figure 5. The points corresponding to the four different sections of the left diagram are plotted in different shades of grey.

(the slope of the curve) remains approximately constant. Then the orbit leaves this region, and the height decreases.

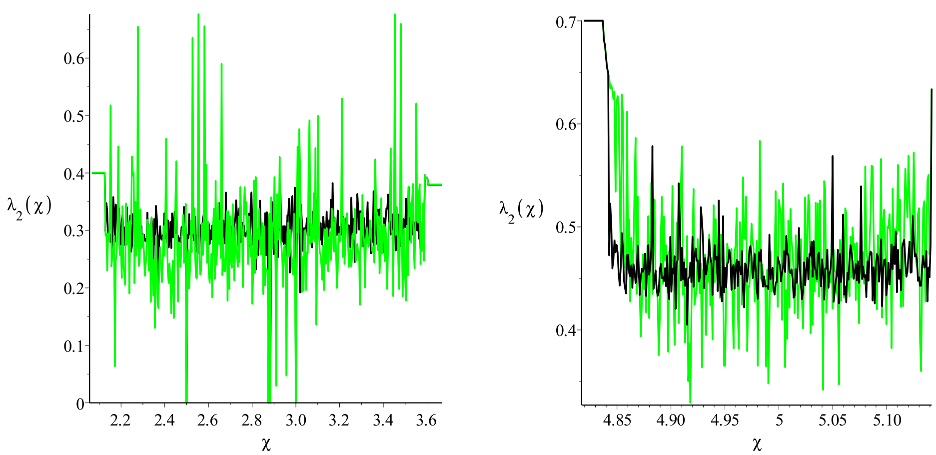

Figure 7: Value of $-\nu_2(x_T)/T$ for approximately 300 orbits with initial conditions $z_0^{(i)} = (x^{(i)}, 0), i = 1, 2, \ldots$ evenly spaced along a segment in phase space. The end-points of the segment lie inside islands, where the height is constant. In both figures the black and green curves correxpond to to $T = 4000$ and $T = 64000$, respectively, indicating slowly decreasing fluctuations.

To shed light on the global picture, we have computed the approximate value of the height for some 300 distinct orbits. The initial conditions are points equally spaced on a segment connecting two islands, but otherwise lying in the chaotic sea. These segments are placed along the x-axis, and are visualized as grey strips in figure 4. A numerical approximation for the p-adic exponent, given by

$$\lambda_p(z_0, T) = \frac{\nu_p(z_0) - \nu_p(z_T)}{T} \approx \lambda_p(z_0) \tag{45}$$

is computed for each orbit at several values of T: $T = 4000, 8000, 16000, 32000, 64000$. (The value $T = 64000$ yields rational numbers with over 5000 decimal digits at numerator and denominator.) The data for $T = 4000$ and $T = 64000$ are displayed in figure 7.

The fluctuations appear to decrease, albeit slowly, with T. To quantify this phenomenon we have computed the normalised total variation V of the 2-adic exponent

$$V_N(T) = \frac{1}{N-1} \sum_{i=1}^{N-1} \left| \lambda_2(z_0^{(i+1)}, T) - \lambda_2(z_0^{(i)}, T) \right| \tag{46}$$

where N is the number of orbits, and $z_0^{(i)}$ is the initial condition of the ith orbit. The behaviour of $V_N(T)$ for both cases is shown in figure 8 in doubly logarithmic scale. The data suggest a regular decrease of the total variation of the numerical height with the time T, and are consistent with a slow convergence to a value which is constant almost everywhere in a chaotic region. Clearly there will be exceptional orbits where the height assumes a different value, such as unstable periodic orbits.

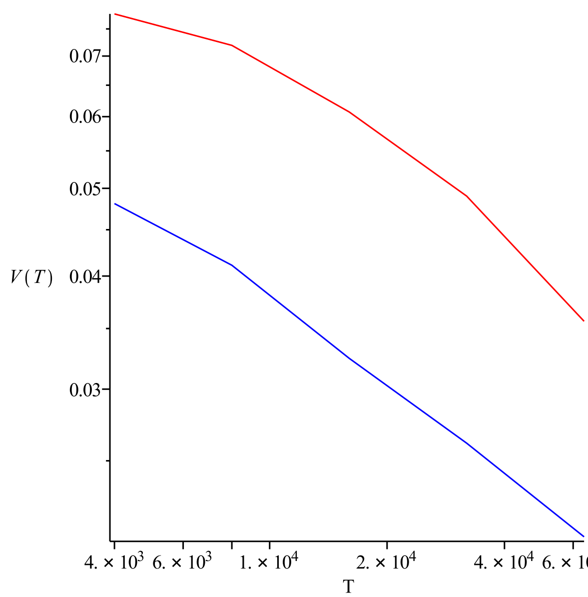

Figure 8: Plot of $V_N(T)$ defined in (46) versus the number T of iterations for the data of figure 7 (the red and blue curves correspond to the left and right plots in the figure, respectively).

The scenario for dissipative maps is simpler; the orbits, after a transient, relax to a small number of point attractors (figure 9, left). In figure 9, right, we plot the approximate height λ_2 for initial conditions of the type $z_0 = (x, 0)$ with x in an interval which crosses an island. Outside the islands the height jumps wildly between few values, presumably due to the very complicated boundaries of the basins of the various attractors.

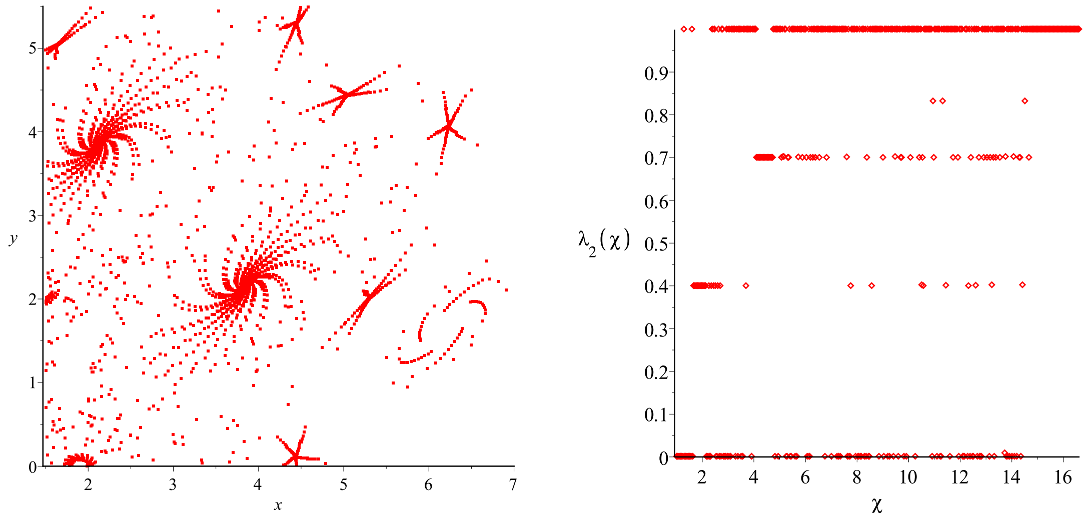

Figure 9: The dissipative map F given in (11), with f as in (12) (the same as in figure 1) and $d = 497/499$. Left: phase portrait, with orbits spiralling towards the centres of the islands. Right: the 2-adic exponent $\lambda_2(x)$ for $z_0 = (x, 0)$ (to be compared with figure 2, right). The limited set of values it assumes (four, in total) reflects the existence of a limited number of attractors. The absence of fluctuations suggests that these attractors have a simple structure.

In the present context, we have identified regular orbits with linear bounded orbits within islands, which either foliate the island into invariant ellipses or spiral towards the fixed point in the centre. No analysis of planar maps would be complete without some reference to more general types of regular orbits, namely quasi-periodic orbits on invariant curves (not necessarily smooth) which are topologically conjugate to irrational rotations. It has long been known that non-smooth symplectic maps may support isolated invariant curves [19], and even foliations of non-smooth curves, see figure 10. Unfortunately the existence of such curves —isolated or not— for non-smooth maps cannot be established in general, and this limitation applies to piecewise affine maps with rational parameters considered here.

There are however important results for specific models. These include a specific two-parameter family of piecewise-linear mappings of the type (11), where a foliation of the plane into invariant curves has been proved (or can reasonably be conjectured) to exist [4, 23, 24, 25]. These are maps or type (11), with the piecewise linear functions

$$f(x) = \begin{cases} a_1 x & x < 0 \\ a_2 x & x \geqslant 0. \end{cases} \tag{47}$$

Due to local linearity, these maps transform the lines through the origin into themselves while preserving their order, thereby inducing a circle map with a well-defined rotation number.

The existence of piecewise-smooth invariant curves has been established for some parameter values given by algebraic numbers of degree 2 [24, theorem 2.2]. The situation for rational parameters less clear. If the rotation number is irrational with bounded partial quotients, then an early result by M. Herman [20, theorem VIII.5.1] implies that (47) is topologically

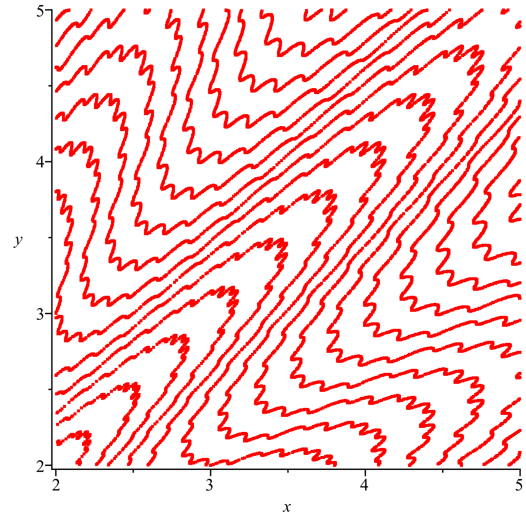

Figure 10: Foliation of the plane into non-smooth invariant curves for the map (47) for $a_1 = 2/3, a_2 = 3/2$.

conjugate to a planar rotation. To the authors' knowledge, the required diophantine condition have not been established in the case of rational parameter a_1 and a_2 in (47).

Numerical experiment suggest that for rational parameters $a_1 \neq a_2$, if the orbits of the map f are bounded, then the plane foliates into invariant curves which typically are non-smooth. Under such a circumstance, we found that all exponents are constant over the entire plane.

We synthesise our findings with two conjectures.

Conjecture 1. *The arithmetic exponents of a piecewise affine map of $\mathbb{Q}^2$ exist for almost all points with bounded orbit.*

The expression 'almost all' means full density with respect to the set of rational points in an appropriate open domain —see (10). Note that all but finitely many p-adic exponents (those with $p \in P$) will be zero.

Conjecture 2. *Let f be an area-preserving piecewise affine map of $\mathbb{Q}^2$. Then any arithmetic exponent has a (non-strict) local maximum at almost all points $z \in \mathbb{Q}^2$ for which the Lyapounov exponent is zero.*

Informally, this conjecture states that rational points of chaotic orbits feature a greater degree of cancellation between numerator and denominator than the points of neighbouring regular orbits. The rational points that need to be excluded are those that belong to an elliptic island and simultaneously lie on the stable manifold of a p-adic hyperbolic fixed point, for some p. This phenomenon can happen only if the characteristic polynomial of the Jacobian matrix of the island is reducible over $\mathbb{Q}$. Conjecture 2 implies that any invariant curve which does not foliate the space must be isolated.

Appendix

We define a module $\mathbb{L}$ with the property that $\mathbb{L}^2$ serves as a minimal phase space for piecewise-affine maps F of the form (42) with $F_i(z) = \mathrm{M}_i z + s_i$. Let $\mathrm{M}_i = (m_{j,k})$ and let P be the (possibly empty, or infinite) set of primes which divide the denominator of $m_{j,k}$ for some j, k. If P is empty, then we let $\mathbb{K} = \mathbb{Z}$; otherwise we let

$$\mathbb{K} = \prod_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right] \tag{48}$$

where the product denotes the algebraic (Minkowski) product of sets. The set $\mathbb{K}$ is the subring of $\mathbb{Q}$ consisting of all the rationals whose denominator is divisible only by primes in P. The module $\mathbb{L}$ of the map F is defined as

$$\mathbb{L} = \mathbb{K} + \sum_{\substack{i \in I \\ j=1,2}} \{s_j^{(i)}\} \tag{49}$$

where $s_i = (s_1^{(i)}, s_2^{(i)})$ and the sum denotes algebraic sum of sets. The set $\mathbb{L}$ is a $\mathbb{K}$-module (a group under addition, with a multiplication by elements of $\mathbb{K}$).

If I is finite, then there is an integer N such that

$$\mathbb{L} = \frac{1}{N}\,\mathbb{K}.$$

To compute N, we let d_i be the least common multiple of the denominators of $s_1^{(i)}$ and $s_2^{(i)}$ and let

$$d_i' = d_i \prod_{p \in P} p^{-\nu_p(d_i)} \qquad i \in I. \tag{50}$$

(This product is finite.) Thus d_i' is the largest divisor of d_i which is co-prime to all primes in P. Then N is the least common multiple of the d_i's, for $i \in I$.

If I is infinite, then the integer N defined above need not exist.

By construction, we have that $F_i(\mathbb{L}^2) \subset \mathbb{L}^2$ for all $i \in I$. Hence $F(\mathbb{L}^2) \subset \mathbb{L}^2$ and $\mathbb{L}^2$ is a natural minimal phase space for F.

The set $\mathbb{L}$ may be embedded in $\mathbb{Q}_p$ for any prime p (the field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$). If $p \in P$, then $\mathbb{L}$ is an unbounded dense subset, and so even if the $\mathbb{Q}^2$ motion is bounded, the p-adic dynamics may be unbounded. If $p \notin P$ and the set I of indices is finite, then $\mathbb{L}$ is bounded in $\mathbb{Q}_p$, and if p does not divide any of the d_i' (see (50)), then $\mathbb{L}$ lies within the unit disc in $\mathbb{Q}_p$. If I is infinite, then $\mathbb{L}$ may still be unbounded even if $p \notin P$, that is, the p-adic height may grow entirely due to the additive action of F (the translations s_i).

References

[1] D. Aharonov, R. L. Devaney and U Elias, The dynamics of a piecewise linear map and its smooth approximation, *Int. J. of Bifurcation and Chaos* **7** (1997) 351–372.

[2] V. I. Arnold, Dynamics of complexity of intersections, *Bol. Soc. Brazil. Math.* **21** (1990) 1–10.

[3] D. K. Arrowsmith and F. Vivaldi, Geometry of p-adic Siegel discs, *Physica D* **71** (1994) 222–236.

[4] A. F. Beardon, S. R. Bullett, P. J. Rippon, Periodic orbits of difference equations, *Proc. Roy. Soc. Edinburgh* **125** (1995) 657–674.

[5] E. Bedford and K. Kim, Linear recurrences in the degree sequences of monomial mappings, *Ergodic Theory & Dyn. Sys.* **28** (2008) 1369–1375.
[6] N. Bird and F. Vivaldi, Periodic orbits of the sawtooth maps, *Physica* **30***D* (1988) 164–176.
[7] M. Baake, J. A. G. Roberts, and A. Weiss, Periodic orbits of linear endomorphisms of the 2-torus and its lattices, *Nonlinearity* **21** (2008) 2427–2446.
[8] M. P. Bellon and C.-M. Viallet, Algebraic entropy, *Comm. Math. Phys.* **204** (1999) 425–237.
[9] V. Botella-Soler, J.A. Oteo, J. Ros and P. Glendinning, Lyapunov exponent and topological entropy plateaus in piecewise linear maps, *J. Phys. A: Math. Theor.* **46** (2013) 125101.
[10] P. Codec and A. Perelli, On the uniform distribution (mod 1) of the Farey fractions and l^p spaces, *Math. Ann.* **279** (1988) 413–422.
[11] R. L. Devaney A Piecewise linear model for the zones of instability of an area preserving map, *Physica D* **10** (1984) 387–393.
[12] Z. Elhadj and J.C. Sprott, A new simple 2-D piecewise linear map, *J. Syst. Sci. Complex* **23** (2010) 379–389.
[13] F. Q. Gouvêa, *p-adic numbers: An introduction*, Springer-Verlag, Berlin, second edition (1997).
[14] B. Grammaticos, R. G. Halburd, A. Ramani and C.-M. Viallet, How to detect the integrability of discrete systems, *J. Phys. A: Math. Theor.* **42** (2009) 454002.
[15] R. G. Halburd, Diophantine integrability, *J. Phys. A: Math. Gen.* **38** (2005) L263–L269.
[16] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers,* Oxford University Press, Oxford (1979).
[17] H. Hasse, *Number Theory*, Springer-Verlag, Berlin (2002).
[18] B. Hasselblatt and J. Propp, *Degree growth in monomial maps*, *Ergodic Theory & Dyn. Sys.* (2007) 1375–1397.
[19] M. Hénon and J. Wisdom, The Benettin-Strelcyn oval billiard revisited *Physica D* **8** (1983) 157–169.
[20] M. Herman, Sur al conjugasion diffenrentiable des disséomorphismes de l'Anneau, Vol 2 *Astérisque* **144** Soc Math. de France, Paris (1986).
[21] V.E. Hoggatt and C.T. Long, Divisibility properties of generalized Fibonacci polynomials, *Fibonacci Quarterly* **12** (1974) 113-120.
[22] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge (1995).
[23] J. C. Lagarias and E. Rains, Dynamics of a family of piecewise-linear area-preserving plane maps I. Rational Rotation Number, *J. Difference Eqns. Appl.* **11** (2005) 1089–1108.
[24] J. C. Lagarias and E. Rains, Dynamics of a family of piecewise-linear area-preserving plane maps II. Invariant Circles, *J. Difference Eqns. Appl.* **11** (2005) 1137–1163.
[25] J. C. Lagarias and E. Rains, Dynamics of a family of piecewise-linear area-preserving plane maps III. Cantor set spectra, *J. Difference Eqns. Appl.* **11** (2005) 1205–1224.
[26] M. Mikolás, On a theorem of J. Frankel, *Kgl. Norske Videnskabers Selskabs Forhandlinger* **21** (1948) 98–101.
[27] J. H. Silverman, *The arithmetic of dynamical systems*, Springer-Verlag, New York (2007).
[28] J. H. Silverman, *Dynamical degree, arithmetic entropy, and canonical heights for dominant rational self maps of projective space*, Ergodic Theory and Dyn. Sys. **34** (2014) 647–678.

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