My research in plain English: Arithmetic Dynamics

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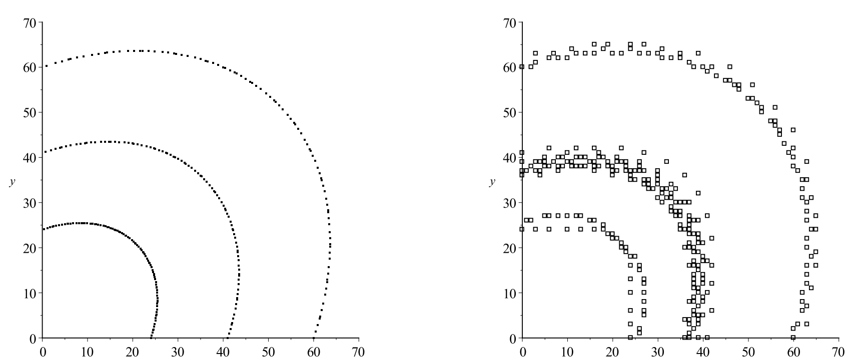
Predicting the future is one of man's greatest desires. In principle, we have a well-developed science for doing this —the discipline of **dynamical systems**— and powerful computers to carry out the necessary calculations. In practice, predictions become impossible in the presence of **chaos**, a phenomenon discovered a few decades ago and subsequently found to be very common. Knowledge of the present state of a system is essential for predictions, but this information cannot be known with infinite accuracy. A small inaccuracy does not normally matter, but if a system is chaotic, then errors get amplified at an exponential rate, quickly wiping out all reliable information.

This mechanism —popularised as the '**butterfly effect**'— is not confined to the physical world. It also operates at a very fundamental level within mathematics, where it is responsible, among other things, for the intrinsic unpredictability found in the numbers themselves. For instance, the digits of a fraction

$$\frac{1}{97} = 0.0103092783505154639175257731958762886597938144\\ 32989690721649484536082474226804123711340206185567\ldots$$

show an evident, surprising complexity, which computer scientists exploit in applications. The computation that generates these digits (first considered by the great mathematician Gauss over 200 years ago) can be thought of as a chaotic dynamical system in arithmetic. Here predicting the future corresponds to predicting the sequence of digits from the input datum $1/97$.

My research area **—arithmetic dynamics—** is concerned with chaos and unpredictability in arithmetical processes. Unlike in more general number systems, the numbers used in arithmetic (of which integers and fractions are the simplest examples) can be represented and manipulated exactly in a computer. This leads to a computer-friendly mathematical theory.



Here's an example of my research. In the figure on the left, you can see a set of points on the plane, neatly arranged along curves. They represents regular, predictable motions, like the motions of planets around the sun. On the right there is an extreme magnification of the same object, showing how this information is stored in a computer. The computer's world is discrete, like the pixels of a digital camera. At microscopic level there are no smooth curves, only irregular clouds of points scattered along them. These small-scale, unpredictable fluctuations, called **round-off errors**, are both unavoidable and mysterious. To explain their origin, I had to considered an esoteric number system —the ***p*-adic numbers**— introduced in arithmetic in the early 20th Century, long before the age of computers.

One wouldn't have guessed that round-off errors in computer arithmetic are a manifestation of chaos over the *p*-adic numbers.