Applied Mathematical Sciences, Vol. 3, 2009, no. 54, 2703 - 2706

The Limit of the Statistic R/P in Models of Oil Discovery and Production

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Abstract

When assessing the oil reserves of a given region, often the statistic $S = R/P$ is used, where R denotes the amount of proven reserves in the region and P is the current rate of production. This statistic can be misleading because the rate of production typically varies over time. We investigate a general framework for oil discovery and production and find the limit of S as time tends to ∞ for several selected models.

Mathematics Subject Classification: 86A99, 86A32

Keywords: modeling, oil production

1 Introduction

When assessing the oil reserves of a region, often the statistic $S = R/P$ is used, where R is the amount of known recoverable oil in the region and P is the current rate of production. The S statistic can be misleading because the rate of production is typically not constant but varies over time.

In 1956, M. King Hubbert [4] used curve fitting to predict that the peak of oil production in the U.S.A. would occur between 1965 and 1970. Oil production for the U.S.A actually peaked in 1970. In Hubbert's analysis the profile for oil production is approximated by a logistic curve, however Deffeyes [3] claims that a normal approximation can give a better fit to the data and also considers the Lorentzian curve. It is therefore desirable in our analysis of the S statistic to allow for these or other types of profiles.

It is of vital interest to understand the behaviour of S in the limit as time tends to ∞, because when the statistic is bounded below by a positive number it cannot give a good estimate of the time remaining for which oil can be usefully produced. Broto [2] does some unpublished work in this direction.

2 A preliminary result

We denote the quantity of oil that is ever discovered in a given region by C and suppose that all of it is eventually produced. We suppose that the total amount of oil discovered up to and including time t is given by a nondecreasing continuous function $G_X(t)$ such that $\lim_{t\to-\infty} G_X(t) = 0$ and $\lim_{t\to\infty} G_X(t) = C$. If we define $F_X(t)$ to be $F_X(t) = G_X(t)/C$, then $F_X(t)$ is the cumulative distribution function of a random variable X. We suppose that the probability density function $f_X(t)$ of X exists. Similarly, if $G_Y(t)$ denotes all of the oil produced up to and including time t, then we suppose that $F_Y(t) = G_Y(t)/C$ determines the cumulative distribution function of a continuous random variable Y with corresponding probability density function $f_Y(t)$. We call $F_X(t)$ the *discovery profile* and call $F_Y(t)$ the *production profile*.

The difference $G_X(t) - G_Y(t)$ between the *cumulative oil discovered* $G_X(t)$ and the *cumulative oil produced* $G_Y(t)$ is called the *proven reserves*; see [5], page 59. It equals the amount of discovered oil remaining to be pumped out of the ground at time t. The statistic S is determined by

$$S = \frac{G_X(t) - G_Y(t)}{G'_Y(t)} = \frac{F_X(t) - F_Y(t)}{f_Y(t)}.$$

We now prove a basic proposition about the limit of S as $t \to \infty$.

Proposition 1 *Suppose that*

$$\lim_{t\to\infty} f_Y(t) = 0. \tag{1}$$

Then,

$$\lim_{t\to\infty} S = \lim_{t\to\infty} \frac{f_X(t) - f_Y(t)}{f'_Y(t)}.$$

If, in addition,

$$\lim_{t\to\infty} \frac{f_X(t)}{f_Y(t)} = L < 1, \tag{2}$$

then

$$\lim_{t\to\infty} S = (L-1) \lim_{t\to\infty} \frac{f_Y(t)}{f'_Y(t)}.$$

Proof We have $\lim_{t\to\infty}(F_X(t) - F_Y(t)) = 1 - 1 = 0$. By (1) we may now apply L'Hôspital's Rule and obtain

$$\lim_{t\to\infty} S = \lim_{t\to\infty} \frac{F_X(t) - F_Y(t)}{f_Y(t)} = \lim_{t\to\infty} \frac{f_X(t) - f_Y(t)}{f'_Y(t)}.$$

Under assumption (2), we have

$$\lim_{t\to\infty} S = \lim_{t\to\infty} \frac{f_Y(t)(f_X(t)/f_Y(t) - 1)}{f'_Y(t)} = (L-1) \lim_{t\to\infty} \frac{f_Y(t)}{f'_Y(t)}.$$

■

Note that the conclusion of Proposition 1 is stated in terms of probability density functions, which are usually given more explicitly than are cumulative distribution functions and are therefore easier to use.

3 Models of oil discovery and production.

In this section we specify a framework for oil discovery and production and derived the limit of the S statistic for particular models within that framework.

Note that we must require that

$$F_X(t) \geq F_Y(t) \quad \forall t \in \mathbb{R} \tag{3}$$

because oil can only be produced after it is discovered. In terms of the random variables X and Y, (3) defines the property that Y *stochastically dominates* X.

Suppose that we are given the discovery profile $F_X(t)$ and that $Z = aX + b$ for constants $a \geq 1$, $b > 0$. Then $F_Z(t)$ is a shifted, rescaled version of $F_X(t)$ and so is a reasonable candidate for a production profile. However, it may happen that Z does not stochastically dominate X. For example, it is impossible for one normally distributed random variable to stochastically dominate another normally distributed random variable unless they have the same variance.

If it is not true that Z stochastically dominates X, then $a > 1$ and for the value $t_0 = b/(1-a)$ it will be true that $F_Z(t_0) = F_X(t_0)$. We have required $a \geq 1$ because in that case Z stochastically dominates X for $t \geq t_0$. In case $a > 1$, we may define

$$F_Y(t) = \begin{cases} F_Z(t) & \text{if } t \geq t_0; \\ F_X(t) & \text{if } t < t_0; \end{cases}$$

otherwise, we let $F_Y(t) = F_Z(t)$ $\forall t \in \mathbb{R}$. An interpretation of the preceding definition is that there is some time t_0 in the past up to which all oil discovered was immediately produced and after t_0 some oil that is discovered is not immediately produced. We will assume from now on that $t \geq t_0$, so that

$$f_Y(t) = \frac{1}{a} f_X\left(\frac{t-b}{a}\right). \tag{4}$$

The distributions considered as possible discovery profiles in [3] are Normal, Logistic and Cauchy (which in [3] is called Lorentzian). The corresponding probability density functions are given by

$$f_X(t) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} & \text{if } X \sim \mathrm{N}(\mu, \sigma); \\ \frac{e^{-(t-\mu)/s}}{s(1+e^{-(t-\mu)/s})^2} & \text{if } X \sim \mathrm{L}(\mu, s); \\ \frac{1}{\pi} \frac{r}{(t-\mu)^2 + r^2} & \text{if } X \sim \mathrm{C}(\mu, r). \end{cases} \tag{5}$$

We now find $\lim_{t\to\infty} S$ for these distributions.

Theorem 1 *If $f_Y(t)$ is determined from $f_X(t)$ for constants a, b by (4), then*

$$\lim_{t\to\infty} S = \begin{cases} 0 & \text{if } X \sim \mathrm{N}(\mu,\sigma); \\ s(1-e^{-b/s}) & \text{if } X \sim \mathrm{L}(\mu,s) \text{ and } a = 1; \\ as & \text{if } X \sim \mathrm{L}(\mu,s) \text{ and } a > 1; \\ b & \text{if } X \sim \mathrm{C}(\mu,r) \text{ and } a = 1; \\ \infty & \text{if } X \sim \mathrm{C}(\mu,r) \text{ and } a > 1. \end{cases}$$

Proof The theorem follows from Proposition 1, (4) and (5). For the normal distribution $Y \sim \mathrm{N}(a\mu + b, a\sigma)$ and it is easy to check that $L = 0$ using $a \geq 1$ and $b > 0$. Moreover, $\lim_{t\to\infty} f_Y(t)/f_Y'(t) = 0$. For the logistic distribution $Y \sim \mathrm{L}(a\mu + b, as)$, $L = 0$ if $a > 1$ and $L = e^{-b/s}$ if $a = 1$. Moreover, $\lim_{t\to\infty} f_Y(t)/f_Y'(t) = -as$. For the Cauchy distribution, $Y \sim \mathrm{C}(a\mu + b, ar)$, $L = a^{-1}$ and $\lim_{t\to\infty} f_Y(t)/f_Y'(t) = -\infty$, which proves the theorem when $a > 1$. When $a = 1$,

$$\lim_{t\to\infty} t^3 \left(f_X(t) - f_Y(t)\right) = -\frac{2rb}{\pi}$$

and

$$\lim_{t\to\infty} t^3 f_Y'(t) = -\frac{2r}{\pi}.$$

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Theorem 1 was proved for the logistic distribution and $a = 1$ by a direct approach in unpublished work described in [2].

References

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Received: April, 2009