Peak production in an oil depletion model with triangular field profiles

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Abstract

In this paper a model of oil depletion is analyzed in which the shapes of field profiles are triangular and the sizes of successive fields decrease monotonically. It is shown that the curve representing the rate of production for this model is piecewise linear and concave up to a certain point after which it is monotone decreasing. Equations for natural smooth approximating curves having unique maxima are derived. A related model of natural gas depletion with trapezoidal shaped field profiles is also analyzed.

Keywords: MATHEMATICAL MODELLING, MATHEMATICAL GEOLOGY, CONVEXITY

1 Introduction

M. King Hubbert [5] used curve fitting to predict that the peak of oil production in the U.S.A. would occur between 1965 and 1970. Oil production in the U.S.A. actually peaked in 1970 and has been declining since then. Hubbert used a logistic curve to approximate the rate of oil production. Deffeyes [2] used normal and Lorentzian curves.

As far as the author knows, none of these approximations has ever been given a theoretical justification. The logistic equation justifies the logistic curve in population biology, but no interpretation of the logistic equation to oil production exists. In this paper we analyze a model of oil production introduced by Bentley [1] and give the first theoretical derivation of a Hubbert-type curve.

Although oil depletion is a very serious topic, models of oil depletion have not heretofore been rigorously analyzed. It is to be hoped that eventually, by analyzing models of oil depletion and comparing the results to the data, understanding of the forces that drive actual oil depletion curves will be increased. The models in this paper are rather simple, but as they are apparently the first oil depletion models to be analyzed mathematically, we consider the results of this paper to be important. Some results on the limitations of the use of R/P statistics in assessing oil reserves were derived in [3].

We suppose that in a given region oil is produced from an infinite succession of oil fields of decreasing size. Let $G_i(t)$ be the amount of oil produced from oil field i up to time t. The cumulative amount of oil ever produced from oil field i is $C_i = \lim_{t\to\infty} G_i(t)$. The cumulative amount of oil produced in the region up to time t is $G(t) = \sum_{i=1}^{\infty} G_i(t)$. The rate of oil production of the ith oil field at time t, or *profile* of the ith field, is defined to be $g_i(t) = G_i'(t)$ and the cumulative rate of production of the region at time t is defined to be $g(t) = G'(t)$.

Bentley [1] introduced a model of oil production for which $g_i(t)$ is shown in Figure 1 for positive numbers δ, λ and m_i for $i \geq 1$. More precisely,

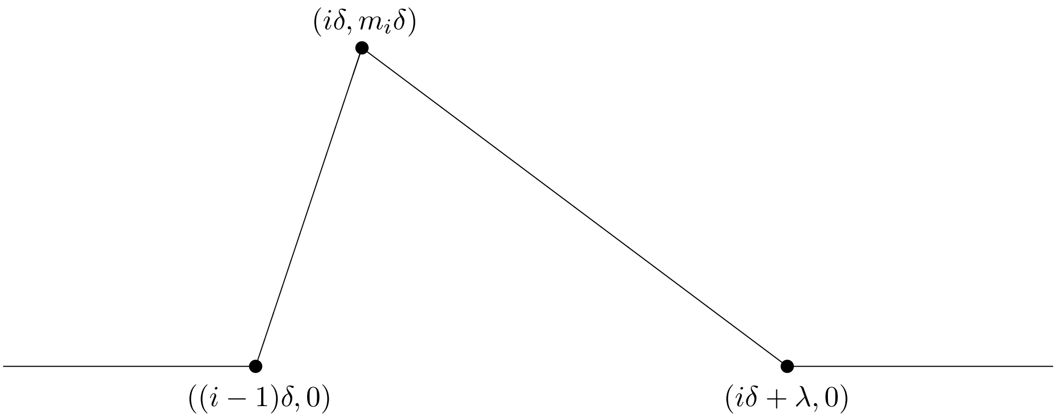

Figure 1: Profile $g_i(t)$ of the ith oil field.

$$C_i = \frac{1}{2} m_i \delta(\lambda + \delta)$$

and

$$G_i(t) = \begin{cases} 0 & \text{if } 0 \leq t < (i-1)\delta; \\ \frac{1}{2} m_i (t - (i-1)\delta)^2 & \text{if } (i-1)\delta \leq t < i\delta; \\ C_i - \frac{m_i \delta}{2\lambda}(i\delta + \lambda - t)^2 & \text{if } i\delta \leq t < i\delta + \lambda; \\ C_i & \text{if } t \geq i\delta + \lambda. \end{cases} \qquad (1)$$

Bentley [1] introduced his model by an example in which $C_i/C_{i+1} = 0.8$ for each i and $\delta/\lambda = 0.1$. The function $g(t)$ was plotted and gave a peak fairly near the center. Bentley's model is discussed in Strahan [4], page 44, where it is suggested that field profiles skewed to the left (δ/λ is small) and falling field sizes (C_i is decreasing) yields a Hubbert-type production curve. In practise, the sharp ascent followed by more gradual decline is often caused by natural gas originally present in the reservoir, which initially helps in pumping out the oil. Later, it may be necessary to pump water into the reservoir to keep up the well pressure as the rate of production declines.

For a fixed t, only a finite number of the field profiles $g_i(t)$ are nonzero. Hence, $g(t)$ exists for all t and is continuous.

Theorem 1 *Given constants $0 < \delta \leq \lambda$ such that λ/δ is an integer and a nonincreasing sequence m_i, the function $g(t)$ is continuous, piecewise linear and concave in the interval $0 \leq t \leq \lambda$. It is monotone decreasing in the interval $t \geq \lambda$. Therefore, $g(t)$ takes on its maximum on a closed interval (which can consist of a single point) intersecting the interval $0 \leq t \leq \lambda$.*

The function $g(t)$ is continuous but piecewise linear. Moreover, it is possible for it to take its maximum on an interval of nonzero length. It may therefore be unsatisfactory as a Hubbert-type curve.

To obtain a smooth Hubbert-type curve, we let consider the sequences of parameters $\delta(n) = \alpha/n$ and $m_i(n) = \phi(i/n)$ for a constant $\alpha > 0$ and a Riemann integrable function $\phi(t)$. The corresponding cumulative amount of oil produced up to time t for these parameters is denoted by $G_n(t)$. We define the limiting curve

$$\tilde{G}(t) = \lim_{n\to\infty} G_n(t) \tag{2}$$

and also define $\tilde{g}(t) = \tilde{G}'(t)$.

Theorem 2 *For a given constant $\alpha > 0$ and a given positive, strictly decreasing, differentiable, function $\phi(t)$ with bounded Riemann integral, the function $\tilde{g}(t)$ is strictly concave for $t \in [0, \lambda]$ and decreasing for $t \in [\lambda, \infty)$. The function $\tilde{g}(t)$ therefore attains a unique maximum which occurs in the interval* $[0, \lambda]$.

Theorem 1 is proven and the original model of Bentley with geometrically decreasing field sizes is considered in Section 2. Theorem 2 is proven and a model with $\phi(x) = Ae^{-\theta x}$ for constants $A, \theta > 0$ is analyzed in Section 3. In Section 4, we analyze a model of natural gas depletion introduced in Bentley [1] along with his oil depletion model. We show that that for the natural gas model, when $\phi(x) = Ae^{-\theta x}$ it is possible that exactly half of the total production occurs at the time of peak production, a statement which is false for the oil model. Some concluding remarks are made in Section 5.

2 Properties of the piecewise linear model

We suppose that we are given constants $0 < \delta < \lambda$ and a decreasing sequence m_i. Proposition 1 provides an explicit formula for $G(t)$. Given a real number x, we let $\lfloor x \rfloor$ be the largest integer less than or equal to x and define $x^+ = \max(x, 0)$.

Proposition 1 *The function $G(t)$ is given by*

$$\begin{aligned} G(t) &= \sum_{i=1}^{\lfloor t/\delta \rfloor} \frac{1}{2} m_i \delta (\lambda + \delta) - \sum_{i=\lfloor (t-\lambda)/\delta \rfloor^+ + 1}^{\lfloor t/\delta \rfloor} \frac{m_i \delta}{2\lambda} (i\delta + \lambda - t)^2 \\ &\quad + \frac{1}{2} m_{\lfloor t/\delta \rfloor + 1} (t - \lfloor t/\delta \rfloor \delta)^2. \end{aligned}$$

Proof We observe that if $1 \leq i \leq \lfloor (t-\lambda)/\delta \rfloor^+$, then $i\delta + \lambda \leq t$; if $\lfloor (t-\lambda)/\delta \rfloor^+ + 1 \leq i \leq \lfloor t/\delta \rfloor$, then $i\delta \leq t$ and $i\delta + \lambda > t$; if $i = \lfloor t/\delta \rfloor + 1$, then $i\delta > t$ and $(i-1)\delta \leq t$; if $i \geq \lfloor t/\delta \rfloor + 2$, then $(i-1)\delta > t$. Therefore, (1) results in

$$\begin{aligned} G(t) &= \sum_{i=1}^{\lfloor (t-\lambda)/\delta \rfloor^+} \frac{1}{2} m_i \delta (\lambda + \delta) + \sum_{i=\lfloor (t-\lambda)/\delta \rfloor^+ + 1}^{\lfloor t/\delta \rfloor} \left\{ \frac{1}{2} m_i \delta (\lambda + \delta) - \frac{m_i \delta}{2\lambda} (i\delta + \lambda - t)^2 \right\} \\ &\quad + \frac{1}{2} m_{\lfloor t/\delta \rfloor + 1} (t - \lfloor t/\delta \rfloor \delta)^2. \end{aligned}$$

■

Proof of Theorem 1 Assume that λ/δ is an integer. If $k\delta \leq t < (k+1)\delta$ for an integer $k \geq 0$, then Proposition 1 produces

$$G(t) = \sum_{i=1}^{k} \frac{1}{2} m_i \delta(\lambda+\delta) - \sum_{i=\lfloor k-\lambda/\delta \rfloor^+ +1}^{k} \frac{m_i \delta}{2\lambda}(i\delta+\lambda-t)^2 + \frac{1}{2} m_{k+1}(t-k\delta)^2. \quad (3)$$

The derivative of the expression (3) for t in the range $k\delta < t < (k+1)\delta$ is

$$g(t) = \sum_{i=\lfloor k-\lambda/\delta \rfloor^+ +1}^{k} \frac{m_i \delta}{\lambda}(i\delta + \lambda - t) + m_{k+1}(t - k\delta).$$

Thus, $g(t)$ is continuous and piecewise linear with slope r_k on the open interval $k\delta < t < (k+1)\delta$ equal to

$$r_k = m_{k+1} - \frac{\delta}{\lambda} \sum_{i=\lfloor k-\lambda/\delta \rfloor^+ +1}^{k} m_i.$$

For $k \geq \lambda/\delta$,

$$r_k = m_{k+1} - \frac{\delta}{\lambda} \sum_{i=k-\lambda/\delta+1}^{k} m_i \leq 0, \quad (4)$$

where the inequality follows from the assumption that the m_i are monotone decreasing. Moreover, for $k \leq \lambda/\delta - 2$,

$$r_{k+1} - r_k = m_{k+2} - (1 + \delta/\lambda) m_{k+1} \leq 0. \quad (5)$$

As $g(t)$ is continuous piecewise linear, (4) shows that it is nonincreasing for $t \geq \lambda$, (5) shows that it is concave for $0 \leq t \leq \lambda$ and therefore the maximum of $g(t)$ must occur in a closed interval intersecting $0 \le t \le \lambda$. ∎

To determine the interval on which $g(t)$ takes its maximum, suppose first that there exists k such that $r_k = 0$ and set $k_1 = \min\{k : r_k = 0\}$ and $k_2 = \max\{k : r_k = 0\}$. Then $g(t)$ takes its maximum on $[k_1\delta, (k_2+1)\delta]$. If there is no k such that $r_k = 0$, then, with k^* defined to be $k^* = \max\{k : r_k > 0\}$, $g(t)$ takes its maximum at $(k^*+1)\delta$.

Example 1 Suppose that $\delta = \lambda$. Then $g(t)$ is maximal over the interval $[\delta, K\delta]$, where $K = \max\{i : m_i = m_1\}$. The amount of oil produced by time $K\delta$ is $G(K\delta) = \sum_{i=1}^{K-1} C_i + \frac{1}{2}C_K$, while the total amount of oil ever produced is $\sum_{i=1}^{\infty} C_i$. Therefore, the proportion of oil produced by the end of the peak interval may be an arbitrarily small proportion of the total oil produced.

Example 2 Consider $m_i = A\rho^i$ for constants $A > 0$ and $\rho \in (0,1)$. Then for $k \ge \lambda/\delta$ we have

$$\begin{aligned} r_{k+1} - r_k &= A\rho^{k+2} - A\rho^{k+1} - \frac{\delta}{\lambda}\left(A\rho^{k+1} - A\rho^{k-\lambda/\delta+1}\right) \\ &= A\rho^{k-\lambda/\delta+1}\left(\rho^{\lambda/\delta+1} - \left(1+\frac{\delta}{\lambda}\right)\rho^{\lambda/\delta} + \frac{\delta}{\lambda}\right) \\ &= A\rho^{k-\lambda/\delta+1}\chi(\rho), \end{aligned}$$

where $\chi(\rho) = \rho^{\lambda/\delta+1} - \left(1+\frac{\delta}{\lambda}\right)\rho^{\lambda/\delta} + \frac{\delta}{\lambda}$. We see that $\psi(0) = \delta/\lambda > 0$ and that $\psi(1) = 0$. Moreover, $\chi'(\rho) = (1+\lambda/\delta)\rho^{\lambda/\delta-1}(\rho-1) \le 0$ and so χ is monotone decreasing in ρ. From this we conclude that $\psi(\rho) \ge 0$ and hence

$g(t)$ is convex for $t \geq \lambda$.

3 Properties of the smooth model

In this section we prove Theorem 2 and consider the example $\phi(x) = Ae^{-\theta x}$ for $A, \theta > 0$, which is the continuous analog of C_i geometrically decreasing.

Proof of Theorem 2 Using Proposition 1, we have

$$\begin{aligned} G_n(t) &= \frac{\alpha(\lambda + \alpha n^{-1})}{2} \sum_{i=1}^{\lfloor nt/\alpha \rfloor} \phi\left(\frac{i}{n}\right) \frac{1}{n} \\ &\quad - \frac{\alpha}{2\lambda} \sum_{i=\lfloor n(t-\lambda)/\alpha \rfloor^+ + 1}^{\lfloor nt/\alpha \rfloor} \phi\left(\frac{i}{n}\right) \left(\frac{\alpha i}{n} + \lambda - t\right)^2 \frac{1}{n} \\ &\quad + \frac{1}{2}\phi\left(\frac{\lfloor nt/\alpha \rfloor + 1}{n}\right)\left(t - \lfloor tn/\alpha \rfloor \frac{\alpha}{n}\right)^2. \end{aligned}$$

Since $\phi(t)$ is bounded, the last term tends to 0 as $n \to \infty$ and since $\phi(t)$ is Riemann integrable, $\tilde{G}(t)$ defined by (2) equals

$$\tilde{G}(t) = \frac{\alpha\lambda}{2} \int_0^{t/\alpha} \phi(x)\, dx - \frac{\alpha}{2\lambda} \int_{(t-\lambda)^+/\alpha}^{t/\alpha} \phi(x)\, (\alpha x + \lambda - t)^2\, dx. \qquad (6)$$

By considering $t < \lambda$ and $t > \lambda$ separately, one may show that

$$\tilde{g}(t) = \frac{\alpha}{\lambda} \int_{(t-\lambda)^+/\alpha}^{t/\alpha} \phi(x)\, (\lambda - t + \alpha x)\, dx.$$

for all $t \in (0, \infty)$, that

$$\tilde{g}'(t) = \phi\left(\frac{t}{\alpha}\right) - \frac{\alpha}{\lambda}\int_{(t-\lambda)^+/\alpha}^{t/\alpha} \phi(x)\,dx, \tag{7}$$

for all $t \in (0, \infty)$, and that

$$\tilde{g}''(t) = \frac{1}{\alpha}\phi'\left(\frac{t}{\alpha}\right) - \frac{1}{\lambda}\phi\left(\frac{t}{\alpha}\right) + \frac{1}{\lambda}\phi\left(\frac{t-\lambda}{\alpha}\right) I[t > \lambda], \tag{8}$$

for all $t \in (0, \lambda) \cup (\lambda, \infty)$, where

$$I[t > \lambda] = \begin{cases} 1 & \text{if } t > \lambda; \\ 0 & \text{if } t \leq \lambda. \end{cases} \tag{9}$$

From (7) it follows that $g(t)$ is decreasing for $t \geq \lambda$ and from (8) and $\phi'(t/\alpha) < 0$ we see that $g(t)$ is concave for $0 \leq t \leq \lambda$. Therefore, $g(t)$ contains a unique maximum which occurs in $0 \leq t \leq \lambda$. ■

We have seen in Example 1 of Section 2 that it is possible that only a small fraction of the oil has been produced at the time of the peak rate of production. Hubbert used symmetric curves in his analysis, for which the ratio of oil produced at the time of peak production compared with the total amount ever produced is 1/2. Although Hubbert said there was no reason to only consider symmetric curves, it is desirable to give examples for which a significant proportion of the total oil ever produced has been produced at the time of peak production.

Example 3 Consider the model defined by $\phi(x) = Ae^{-\theta x}$ for constants $A, \theta > 0$. This is the continuous version of the model of Example 2, which had geometrically decreasing field sizes. The total amount of oil produced in this model is

$$\lim_{t\to\infty} \tilde{G}(t) = \frac{1}{2}\alpha\lambda \int_0^\infty \phi(x)\, dx = \frac{A\alpha\lambda}{2\theta}.$$

To find the value t maximizing $\tilde{g}(t)$, for $t \in [0, \lambda]$, we set $\tilde{g}'(t)$ given by (7) equal to 0, which implies that the maximum rate of oil production occurs at

$$\hat{t} = \frac{\alpha}{\theta} \log\left(1 + \frac{\lambda\theta}{\alpha}\right). \tag{10}$$

The amount of oil that has been produced at the time of peak production is $\tilde{G}(\hat{t})$ and the fraction of oil Φ produced by the time of peak production is given by

$$\Phi = \frac{2\theta}{A\alpha\lambda}\tilde{G}(\hat{t}). \tag{11}$$

For $t \leq \lambda$, (6) gives

$$\tilde{G}(t) = \frac{A\alpha}{\lambda}\left[\left(-\frac{\alpha^2}{\theta^3} - \frac{\alpha\lambda}{\theta^2}\right)\left(1 - e^{-\theta t/\alpha}\right) + \left(\frac{\lambda}{\theta} + \frac{\alpha}{\theta^2}\right)t - \frac{1}{2\theta}t^2\right] \tag{12}$$

and substituting (10) into (12) produces

$$\tilde{G}(\hat{t}) = \frac{A\alpha}{\lambda}\left[-\frac{\alpha\lambda}{\theta^2} + \left(\frac{\alpha\lambda}{\theta^2} + \frac{\alpha^2}{\theta^3}\right)\log\left(1 + \frac{\lambda\theta}{\alpha}\right) - \frac{\alpha^2}{2\theta^3}\log^2\left(1 + \frac{\lambda\theta}{\alpha}\right)\right]. \tag{13}$$

Substituting (13) in (11) and letting

$$y = \lambda\theta/\alpha \tag{14}$$

results in

$$\Phi(y) = -2y^{-1} + 2(y^{-1} + y^{-2})\log(1+y) - y^{-2}\log^2(1+y).$$

For some values of y, a significant proportion of oil is produced before peak in this model. The maximum of Φ occurs at $y_m \doteq 2.7466$ and the maximum is $\Phi(y_m) \doteq 0.3525$. The function $\Phi(y)$ increases steeply to its maximum, after which it decreases rather slowly. As $y \to 0$, $\Phi(y) = \frac{2}{3}y + O(y^2)$. We observe also that for $t > \lambda$, (8) gives

$$\begin{aligned}
\tilde{g}''(t) &= Ae^{-\theta t/\alpha}\left(-\frac{\theta}{\alpha} - \frac{1}{\lambda} + \frac{1}{\lambda}e^{\theta\lambda/\alpha}\right) \\
&> Ae^{-\theta t/\alpha}\left(-\frac{\theta}{\alpha} - \frac{1}{\lambda} + \frac{1}{\lambda}\left(1 + \frac{\theta\lambda}{\alpha}\right)\right) \\
&= 0,
\end{aligned}$$

proving that $g(t)$ is strictly convex for $t > \lambda$ (as opposed to strictly concave for $t \leq \lambda$, as was shown in Theorem 2).

4 Models for natural gas depletion

In this section we analyze a model for natural gas depletion introduced by Bentley [1] which is similar to the model for oil depletion excepting that the field profile is now trapezoidal as opposed to triangular. The reason is that production of natural gas fields is often constrained by pipe size rather than internal pressure and production typically tales off more suddenly for gas fields than it does for oil fields.

The notations $H_i(t)$, $h_i(t)$ and D_i have definitions for the natural gas model similar to the definitions of $G_i(t)$, $g_i(t)$, and C_i with respect to the oil model, with the essential difference being the definition of $h_i(t)$ for natural gas field i depicted in Figure 2 for positive numbers δ, λ and m_i. We have

$$D_i = m_i\delta(\lambda + \delta)$$

and

$$H_i(t) = \begin{cases} 0 & \text{if } 0 \le t < (i-1)\delta; \\ \frac{1}{2}m_i(t-(i-1)\delta)^2 & \text{if } (i-1)\delta \le t < i\delta; \\ D_i - \frac{1}{2}m_i\delta^2 - m_i\delta(i\delta + \lambda - t) & \text{if } i\delta \le t < i\delta + \lambda; \\ D_i - \frac{1}{2}m_i((i+1)\delta + \lambda - t)^2 & \text{if } i\delta + \lambda \le t < (i+1)\delta + \lambda; \\ D_i & \text{if } t \ge (i+1)\delta + \lambda. \end{cases}$$

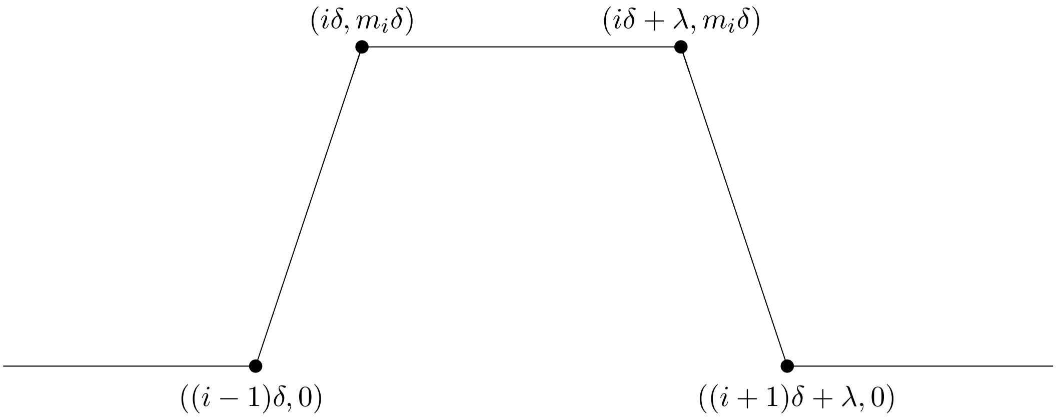

Figure 2: Profile $h_i(t)$ of the ith natural gas field.

Define $H(t) = \sum_{i=1}^{\infty} H_i(t)$ and $h(t) = H'(t)$. The proof of Proposition 2 is similar to the proof of Proposition 1. We define $I[t \geq \lambda + \delta]$ analogously to (9).

Proposition 2 *The function $H(t)$ is given by*

$$\begin{aligned} H(t) &= \sum_{i=1}^{\lfloor t/\delta \rfloor} m_i\delta(\lambda+\delta) - \sum_{i=\lfloor (t-\lambda)/\delta \rfloor^+ + 1}^{\lfloor t/\delta \rfloor} \left\{ \frac{1}{2} m_i \delta^2 + m_i\delta(i\delta + \lambda - t) \right\} \\ & \quad - \frac{1}{2} m_{\lfloor (t-\lambda-\delta)/\delta \rfloor + 1} \left(\lfloor (t-\lambda-\delta)/\delta \rfloor + 2)\delta + \lambda - t \right)^2 I[t \geq \lambda + \delta] \\ & \quad + \frac{1}{2} m_{\lfloor t/\delta \rfloor + 1} (t - \lfloor t/\delta \rfloor \delta)^2. \end{aligned}$$

We next show that $h(t)$ is increasing and concave for $t \leq \lambda$ and decreasing for $t \geq \lambda$.

Theorem 3 *For given constants $0 < \delta \leq \lambda$ such that λ/δ is an integer and a nonincreasing sequence m_i, the function $h(t)$ is continuous, piecewise linear, concave and monotone increasing in the interval $0 \leq t \leq \lambda + \delta$.*

It is monotone decreasing in the interval $t \geq \lambda + \delta$. Therefore, $h(t)$ takes on its maximum on a closed interval (which can consist of a single point) intersecting the interval $0 \leq t \leq \lambda + \delta$.

Proof If $k\delta < t < (k+1)\delta$ for an integer $k \geq 0$, then

$$\begin{aligned} H(t) &= \sum_{i=1}^{k} m_i \delta(\lambda+\delta) - \sum_{i=\lfloor k-\lambda/\delta \rfloor^+ + 1}^{k} \left(\frac{1}{2} m_i \delta^2 + m_i \delta(\lambda + i\delta - t) \right) \\ &\quad - \frac{1}{2} m_{k-\lambda/\delta} \left((k+1)\delta - t \right)^2 I[k \geq \lambda/\delta + 1] \\ &\quad + \frac{1}{2} m_{k+1} (t - k\delta)^2, \end{aligned}$$

hence

$$\begin{aligned} h(t) &= \sum_{i=\lfloor k-\lambda/\delta \rfloor^+ + 1}^{k} m_i \delta + m_{k+1}(t - k\delta) \\ &\quad + m_{k-\lambda/\delta} \left((k+1)\delta - t \right) I[k \geq \lambda/\delta + 1] \end{aligned}$$

and therefore $h(t)$ is continuous piecewise linear with slope on the interval $k\delta < t < (k+1)\delta$ equal to

$$s_k = m_{k+1} - m_{k-\lambda/\delta} I[k \geq \lambda/\delta + 1].$$

For $k \leq \lambda/\delta$, $s_k = m_k \geq 0$ and for $k \leq \lambda/\delta - 1$, $s_{k+1} - s_k = m_{k+2} - m_{k+1} \leq 0$, Therefore, $h(t)$ is concave and nondecreasing for $t \leq \lambda + \delta$. For $k \geq \lambda/\delta + 1$, we have $s_k = m_{k+1} - m_{k-\lambda/\delta} \leq 0$ and so $h(t)$ is decreasing on the interval $t \geq \lambda + \delta$. ∎

Example 4 Suppose that $m_i = A\rho^i$, $0 < \rho < 1$. Then, for $k \geq \lambda/\delta + 1$,

$$\begin{aligned} s_{k+1} - s_k &= A\rho^{k+2} - A\rho^{k+1-\lambda/\delta} - A\rho^{k+1} + A\rho^{k-\lambda/\delta} \\ &= A\rho^{k+1}(1-\rho)(\rho^{-\lambda/\delta-1} - 1) \\ &> 0. \end{aligned}$$

and so $h(t)$ is concave for $t \geq \lambda + \delta$.

Let $H_n(t)$ denote $H(t)$ for the sequence of parameters $\delta(n) = \alpha/n$ and $m_i(n) = \phi(i/n)$ for a constant $\alpha > 0$ and a positive, differentiable, strictly decreasing function $\phi(t)$ with bounded Riemann integral. Define $\tilde{H}(t) = \lim_{n\to\infty} H_n(t)$ and $\tilde{h}(t) = \tilde{H}'(t)$.

Theorem 4 *For a given constant $\alpha > 0$ and a given positive, strictly decreasing, differentiable, function $\phi(t)$ with bounded Riemann integral, the function $\tilde{h}(t)$ is strictly concave and is increasing for $t \in [0, \lambda]$ and decreasing for $t \in [\lambda, \infty)$. Therefore, $\tilde{h}(t)$ attains a unique maximum at λ.*

Proof By taking the limits of Riemann sums obtained from Proposition 2, we obtain

$$\tilde{H}(t) = \alpha\lambda \int_0^{t/\alpha} \phi(x)\,dx - \alpha \int_{(t-\lambda)^+/\alpha}^{t/\alpha} \phi(x)(\alpha x + \lambda - t)\,dx,$$

and therefore

$$\tilde{h}(t) = \alpha \int_{(t-\lambda)^+/\alpha}^{t/\alpha} \phi(x)\,dx,$$

for $t \in (0,\infty)$,

$$\tilde{h}'(t) = \phi\left(\frac{t}{\alpha}\right) - \phi\left(\frac{t-\lambda}{\alpha}\right) I[t > \lambda]$$

for $t \in (0,\lambda) \cup (\lambda,\infty)$, and

$$\tilde{h}''(t) = \alpha^{-1}\phi'\left(\frac{t}{\alpha}\right) - \alpha^{-1}\phi'\left(\frac{t-\lambda}{\alpha}\right) I[t > \lambda]$$

for $t \in (0,\lambda) \cup (\lambda,\infty)$. It follows that $\tilde{h}'(t) > 0$ for $t < \lambda$, that $\tilde{h}'(t) < 0$ for $t > \lambda$, and that $\tilde{h}''(t) < 0$ for $t < \lambda$. Therefore $\tilde{h}(t)$ is strictly concave and strictly increasing on the interval $(0,\lambda)$ and is strictly decreasing on the interval (λ,∞). Hence, $\tilde{h}(t)$ takes its maximum at $t = \lambda$. ■

Example 5 Suppose that $\phi(x) = Ae^{-\theta x}$ for $A, \theta > 0$. Then, for $t > \lambda$,

$$\tilde{h}''(t) = (e^{\theta\lambda/\alpha} - 1)A\theta\alpha^{-1}e^{-\theta t/\alpha} > 0$$

and so $\tilde{h}(t)$ is convex for $t > \lambda$. The total amount of oil ever produced is

$$\lim_{t\to\infty} \tilde{H}(t) = \alpha\lambda \int_0^\infty \phi(x)\,dx = \frac{A\alpha\lambda}{\theta}.$$

The amount of gas produced at time $t = \lambda$ is

$$\begin{aligned}\tilde{H}(\lambda) &= \alpha\lambda \int_0^{\lambda/\alpha} Ae^{-\theta x}\,dx - \alpha^2 \int_0^{\lambda/\alpha} Axe^{-\theta x}\,dx \\ &= \frac{A\alpha\lambda}{\theta} - \frac{A\alpha^2}{\theta^2}\left(1 - e^{-\theta\lambda/\alpha}\right).\end{aligned}$$

The fraction of oil produced at time λ, given by

$$\Psi = \frac{\tilde{H}(\lambda)\theta}{A\alpha\lambda},$$

equals

$$\Psi(y) = 1 - y^{-1}(1 - e^{-y})$$

where y is defined by (14). As $\Psi'(y) = y^{-2}\left(1 - e^{-y}(1+y)\right) > 0$, the function $\Psi(y)$ is increasing and, since $\lim_{y\to 0}\Psi(y) = 0$ and $\lim_{y\to\infty}\Psi(y) = 1$, there is a unique value $y_h \doteq 1.5936$ for which $\Psi(y_h) = 1/2$. For $y = y_h$, exactly half of the cumulative gas production occurs at the time of peak production. However, $\tilde{h}(t)$ is not symmetric, as it is concave for $t < \lambda$ and convex for $t > \lambda$.

5 Conclusion

We have demonstrated that Hubbert-type curves can be derived from simple models. Unfortunately, none of the curves produced here are symmetric. Another feature of our analysis is that the function $\phi(x)$ must have bounded Riemann integral, excluding the possibility that $\phi(x)$ decreases according to Zipf's law, which seems to describe the distributions of some real oil field sizes; see [2]. Future research in this area could include other fixed field profiles, random field profiles or functions $\phi(x)$ with unbounded Riemann integral.

References

[1] Roger W. Bentley, Global Oil and Gas Depletion: An Overview, *Energy Policy*, 30, 189–205, 2002.

[2] K. A. Deffeyes, Hubbert's Peak, Princeton University Press, 2001.

[3] Dudley Stark, The limit of the statistic R/P in models of oil discovery and production, Accepted to appear in *International Journal of Pure and Applied Mathematical Sciences*.

[4] David Strahan, The Last Oil Shock, John Murray, 2007.

[5] M. K. Hubbert, 1956, Nuclear energy and the fossil fuels, American Petroleum Institute Drilling and Production Practise, Proceedings of Spring Meeting, San Antonio.