Flow Polynomials, Schur Polynomials and Discrete Orthogonal Polynomials

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Abstract. We consider the integer flow polynomials on directed planar graphs. A summation formula of the flow polynomials over possible strength and spatial distribution of sinks of flow reported by Arrowsmith, Mason and Essam (AME) in 1991 implies the identity

$$\sum_{0 \leq \lambda_t \leq \lambda_{t-1} \leq \cdots \leq \lambda_1 \leq n} \prod_{1 \leq i < j \leq t} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \leq i \leq j \leq t} \frac{n + i + j - 1}{i + j - 1}$$

for $n = 0, 1, 2, \cdots$ and $t = 1, 2, 3, \cdots$. Motivated by the product forms of this formula we discuss combinatorial and determinantal identities of flow polynomials and related other polynomials in the present paper. First we show that the Bender-Knuth formula of plane partition can be regarded as a q-analogue of the above AME formula and these formulae are derived from Macdonald's summation formula of Schur polynomials. We define the q-flow polynomials and its multi-variable generalization and discuss the relations between the flow polynomials and the Schur polynomials. Two different proofs of the Macdonald formula recently given by Bressoud and by Okada, respectively, are reviewed from the viewpoint of flow polynomials. Recent work by Johansson on a random growth model implies an interesting similarity between the AME formula and the Gaussian orthogonal ensemble of random matrices and calculation of physical quantities in it. Following the standard argument of the random matrix theory we give determinantal expressions to the flow polynomials, in which matrix elements are given by the discrete orthogonal polynomials. Probability measures of ensembles of flow polynomials having determinantal expressions are introduced and it is shown that the Macdonald formula provides probability laws on these probability measures.

Keywords. Flow polynomials, directed graphs, Schur polynomials, Macdonald formula, q-analogue, random matrix theory, orthogonal polynomials, determinantal probability measures

1 Introduction

1.1 Integer Flows on Directed Graphs

Consider a graph $G = (V, E)$, where the set of vertices (resp. edges) of G is denoted by V (resp. E). Let $\mathcal{D}(G)$ be the set of directed graphs obtained by directing the edge set E of G in all possible ways. Then each directed graphs $H \in \mathcal{D}(G)$ consists of a set of vertices V and a set of arcs (oriented edges) A.

Fix an integer $n \in \mathbf{Z}^+ \equiv \{0, 1, 2, \cdots\}$. An improper $(n+1)$-flow on the directed graph $H = (V, A) \in \mathcal{D}(G)$ is a map $\phi : A \to \mathbf{Z}_{n+1} \equiv \{0, 1, 2, \cdots, n\}$ such that

$$\sum_{a \in A_v^+} \phi(a) = \sum_{a \in A_v^-} \phi(a) \qquad (1.1)$$

for every vertex $v \in V$, where A_v^+, A_v^- are respectively the sets of arcs oriented in and out of the vertex v. The number of such flow is given by a polynomials of n, which we write as $F(n, H)$.

The situation, which we consider, is generalized by introducing a source and sinks of flow as follows. In $V(H)$ a vertex O is chosen as a source of strength $m(\leq n)$ and a set of vertices $\{s_1, s_2, \cdots, s_k\}$ are considered as sinks of strength $m_1, m_2, \cdots, m_k$, respectively, with the condition $\sum_{i=1}^k m_i = m$. That is, we consider a map $\phi : A \to \mathbf{Z}_{n+1}$ such that (1.1) holds for any vertices $v \in V \setminus \{O, s_1, \cdots, s_k\}$ and that

$$\sum_{a \in A_O^+} \phi(a) + m = \sum_{a \in A_O^-} \phi(a), \qquad (1.2)$$

$$\sum_{a \in A_{s_i}^+} \phi(a) = \sum_{a \in A_{s_i}^-} \phi(a) + m_i, \quad i = 1, 2, \cdots, k. \qquad (1.3)$$

The number of such flows with a source and sinks may be a polynomial of $m_1, m_2, \cdots, m_k$. By definition, if $m_i = 0 \quad \forall i$, this polynomial is reduced to be $F(n, H)$. The conditions (1.1)-(1.3) ensure the conservation of flow in the system.

Arrowsmith, Mason and Essam [1] studied the integer flow polynomials on the following special cases on directed planar graphs. Let $t \in \mathbf{N} \equiv \{1, 2, 3, \cdots\}$ and define a subset of $\mathbf{Z}^2$

$$V_t = \{(x, y) \in \mathbf{Z}^2 : x + y = \text{even}, \ 0 \leq y \leq t, \ -y \leq x \leq y\}$$

and E_t be the set of all edges which connect the nearest-neighbour pairs of vertices in V_t. Arrowsmith *et al.* considered such a directing A_t that all edges are oriented in the positive direction of the y-axis. An example of the directed graph $H_t = (V_t, A_t)$ with $t = 5$ is shown in Figure 1. The source is on the origin $O = (0, 0)$ and the sinks $\{s_1, \cdots, s_k\}$ are chosen from the vertices on the upper ends of V_t:

$$\bar{V}_t = \{(-t, t), (-t+2, t), \cdots, (t-2, t), (t, t)\}.$$

If we think that the strength of sink can take a value 0, all of $t+1$ vertices on $\bar{V}_t$ can be regarded as sinks. We specify the strength of the sink at $(-t+2k, t)$ as $n(-t+2k) \geq 0$ for $k =$

$0, 1, \cdots, t$ and the total number of possible flows is denoted by $F_t(n(-t), n(-t+2), \cdots, n(t))$, which is a polynomial of $n(-t), \cdots, n(t)$ (see (1.7) with (1.6) below). By the conservation of flow, the strength of the source O should be $n = \sum_{k=0}^{t} n(-t+2k)$.

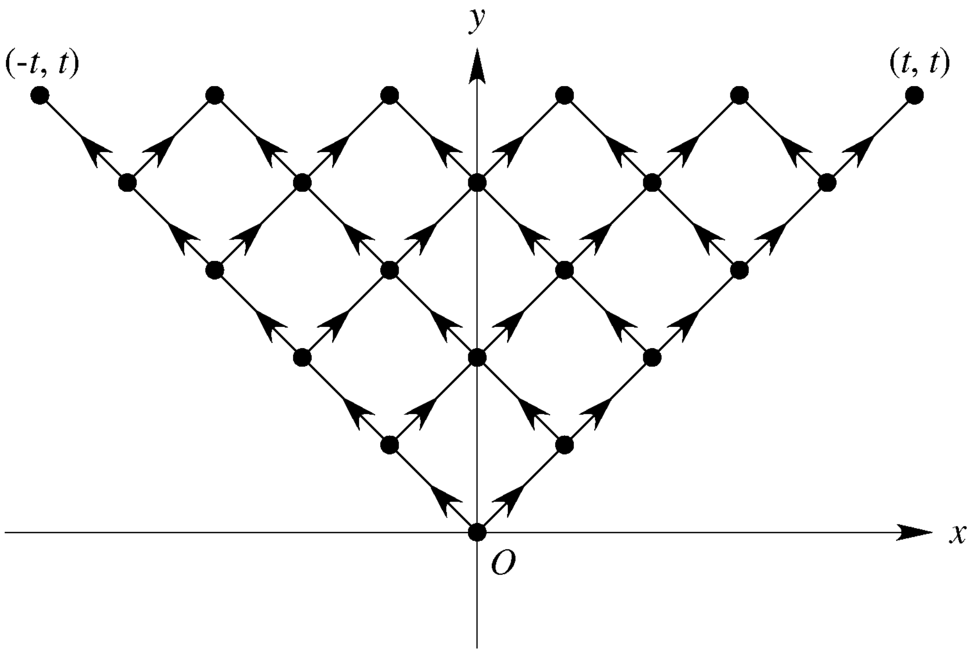

Figure 1: The directed graph $H_t = (V_t, A_t)$ with $t = 5$.

The reason why such special directings of edges with a source and sinks was chosen was clearly explained in [1]; there is established a bijection between the set of such integer flows and trajectories of an interacting particle system called vicious walkers with fixed start-points and end-points on a spatio-temporal plane (see Section 3.1 below). Many applications of vicious walker models in the statistical mechanics and the condensed matter physics are described by Fisher in his Boltzmann medal lecture [10]. See also [9] and references therein for the relation between vicious walker problems and polymer physics. It should be noted that the flow polynomial $F_t(n(-t), n(-t+2), \cdots, n(t))$ gives the special value at a parameter $p = 1$ of the partition function of the friendly walkers [28, 7, 16].

1.2 Arrowsmith-Mason-Essam Formula

The following formulae were conjectured by Arrowsmith *et al.* [1]:

$$F_t(0, \cdots, 0, n(-t+2k) = n, 0, \cdots, 0) = \prod_{i=1}^{k} \frac{(n+i)_{t-2i+1}}{(i)_{t-2i+1}} \quad \text{for } 1 \leq k \leq t-1, \tag{1.4}$$

and

$$\sum_{n(-t)\geq 0} \sum_{n(-t+2)\geq 0} \cdots \sum_{n(t)\geq 0} \mathbf{1}\left(\sum_{k=0}^{t} n(-t+2k) = n\right) F_t(n(-t), n(-t+2), \cdots, n(t))$$

$$= \prod_{i=1}^{[(t+1)/2]} \frac{(n+2i-1)_{2t-4i+3}}{(2i-1)_{2t-4i+3}}, \tag{1.5}$$

where $(a)_k$ is the Pochhammer-symbol; $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ for $k = 1, 2, 3, \cdots$, $\mathbf{1}(\omega)$ is an indicator function such as $\mathbf{1}(\omega) = 1$ if the condition ω is satisfied and $\mathbf{1}(\omega) = 0$ otherwise, and $[a]$ denotes the largest integer not greater than the number a.

Here we define the variables $\{\lambda_i\}_{i=1}^t$ as functions of $\{n(-t+2k)\}_{k=0}^{t-1}$ by

$$\lambda_i = \sum_{k=0}^{t-i} n(-t+2k), \quad i = 1, 2, \cdots, t, \tag{1.6}$$

that is,

$$\begin{aligned}
\lambda_1 &= n(-t) + n(-t+2) + \cdots + n(t-6) + n(t-4) + n(t-2) \\
\lambda_2 &= n(-t) + n(-t+2) + \cdots + n(t-6) + n(t-4) \\
\lambda_3 &= n(-t) + n(-t+2) + \cdots + n(t-6) \\
&\cdots \\
\lambda_{t-1} &= n(-t) + n(-t+2) \\
\lambda_t &= n(-t).
\end{aligned}$$

Then the formula (1.4) will be generalized and simply expressed as [18]

$$F_t(n(-t), n(-t+2), \cdots, n(t)) = \prod_{1 \le i < j \le t} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{1.7}$$

Assuming (1.7), (1.5) can be written as the following equality, which we call the Arrowsmith-Mason-Essam formula in the present paper.

Formula 1.1 (Arrowsmith-Mason-Essam)

$$\sum_{0 \le \lambda_t \le \lambda_{t-1} \le \cdots \le \lambda_1 \le n} \prod_{1 \le i < j \le t} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{1 \le i \le j \le t} \frac{n + i + j - 1}{i + j - 1}.$$

1.3 Bender-Knuth Formula and Macdonald Formula: A Hierarchy

Guttmann, Owczarek and Viennot [13] noticed a simple bijection between vicious walker trajectories (*i.e.* flows on the directed graph) and semi-standard Young tableaux. This observation led them to regard the Arrowsmith-Mason-Essam formula as a $q \to 1$ limit of the following formula known as the Bender-Knuth conjecture [6, 12].

Formula 1.2 (Bender-Knuth)

$$\sum_{0 \le \lambda_t \le \lambda_{t-1} \le \cdots \le \lambda_1 \le n} q^{\sum_{i=1}^t i\lambda_i} \prod_{1 \le i < j \le t} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{1 - q^{j-i}} = \prod_{1 \le i \le j \le t} \frac{1 - q^{n+i+j-1}}{1 - q^{i+j-1}}.$$

The bijection found by Guttmann *et al.* implies that the number of flow with fixed strength of sinks, $F_t(n(-t), n(-t+2), \cdots, n(t))$, is equal to the total number of distinct semi-standard Young tableaux on a given Young (Ferrers) diagram. The Young diagram is specified by a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$ of an integer $N = \sum_{i=1}^{t} \lambda_i$. The relation between $\{n(-t+2k)\}_{k=0}^{t-1}$ and λ is given by (1.6) and $N = \sum_{i=1}^{t} \sum_{k=0}^{t-1} n(-t+2k) = \sum_{k=0}^{t-1} (t-k) n(-t+2k)$. Krattenthaler, Guttmann and Viennot [18] claimed that $F_t(n(-t), n(-t+2), \cdots, n(t))$ is thus obtained as a value at $x_1 = x_2 = \cdots = x_t = 1$ of a function $s_\lambda(x_1, x_2, \cdots, x_t)$ of t variables $\{x_i\}_{i=1}^{t}$, which is specified by a partition λ and called the Schur polynomial;

$$F_t(n(-t), n(-t+2), \cdots, n(t)) = s_\lambda(x_1, x_2, \cdots, x_t)\big|_{x_1 = x_2 = \cdots = x_t = 1}. \tag{1.8}$$

It was also claimed by Krattenthaler *et al.* [18] that the Bender-Knuth formula is a corollary of a summation formula of $\sum_{\lambda \in \{n^t\}} s_\lambda(x_1, \cdots, x_t)$. This summation formula, which we call the Macdonald formula (eq. (2′) on p.84 in [20]), can be written as the following form.

Formula 1.3 (Macdonald)

$$\sum_{0 \leq \lambda_t \leq \lambda_{t-1} \leq \cdots \leq \lambda_1 \leq n} \frac{\det(x_i^{\lambda_j + t - j})_{1 \leq i,j \leq t}}{\det(x_i^{t-j})_{1 \leq i,j \leq t}} = \frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1 \leq i,j \leq t}}{\det(x_i^{j-1} - x_i^{2t-j})_{1 \leq i,j \leq t}}.$$

Thus we have the hierarchy of formulae.

Macdonald Formula (on polynomials of $\{x_i\}$)
|
Bender-Knuth Formula (on polynomials of q)
|
Arrowsmith-Mason-Essam Formula (on polynomials of $\{n(-t+2k)\}$)

One of the purpose of the present paper is to discuss this hierarchy using flow polynomials.

In Section 2 we generalize the flow polynomials by introducing variables $x_1, \cdots, x_t$ and define the polynomials of them, $F_t(n(-t), n(-t+2), \cdots, n(t); x_1, \cdots, x_t)$, so that

$$F_t(n(-t), n(-t+2), \cdots, n(t); x_1, \cdots, x_t) = s_\lambda(x_1, \cdots, x_t),$$

and

$$F_t(n(-t), n(-t+2), \cdots, n(t); x_1 \cdots, x_t)\big|_{x_1 = x_2 = \cdots = x_t = 1} = F_t(n(-t), n(-t+2), \cdots, n(t)).$$

This gives another proof of the formulae (1.7) and (1.8) given by Krattenthaler *et al.* [18], where in our proof we do not need to use the bijection between flows and Young tableaux but we show that the generalized flow polynomial and the Schur polynomial satisfy the same recurrence equation. There in order to take the limit $x_i \to 1$, $1 \leq \forall i \leq t$, we introduce the q-flow polynomials, $F_t(n(-t), \cdots, n(t); q)$. We also show in Section 3 that the equivalence between the generalized flow polynomial and the Schur polynomial can be proved by applying the Gessel-Viennot theorem to enumeration problem of nonintersecting paths. We notice that by considering the $x_1 = x_2 = \cdots = x_t = 1$ case of the Gessel-Viennot determinant (binomial determinant) Essam and Guttmann [9] derived an expression for the flow polynomial (3.5), which is different from (1.7). We explain that the expressions (1.7) and (3.5) can be considered to be conjugate to each other in the sense of the conjugate of Young diagrams representing partitions.

In Section 4 we explain how to derive the Bender-Knuth formula from the Macdonald formula and the Arrowsmith-Mason-Essam formula from the Bender-Knuth formula.

1.4 Proofs of Macdonald Formula

In the present paper we review two different proofs of Macdonald formula in Section 5 and Section 6. The former proof is essentially equivalent with the "elementary proof" given by Bressoud [5], but there we put an emphasis on the fact that the LHS of Formula 1.3, the summation of ratios of determinants, satisfies a recurrence equation and that the RHS of the formula gives a compact solution for it. The latter proof given in Section 6 is owing to Okada [24]. The Macdonald formula states that the summation of determinants is expressed by single determinant with an appropriate prefactor. Okada considered such a summation of determinants as a minor summation of a larger-sized determinant and gave a general formula to express it by a Pfaffian [23]. Then the problem is how can we express the Pfaffian by a determinant [27]. Pfaffian identities given by Okada [24] enable us to perform this transformation and the Macdonald formula is concluded.

1.5 Random Matrix Theory and Orthogonal Polynomials

The LHS of the Arrowsmith-Mason-Essam formula (Formula 1.1) can be written as follows. Let $h_j = \lambda_j + t - j, 1 \leq j \leq t$, and then

$$(\text{LHS}) = \frac{1}{\prod_{i=1}^{t-1} i!} \times \frac{1}{t!} \sum_{h_1 \in \mathbf{Z}^+} \cdots \sum_{h_t \in \mathbf{Z}^+} \mathbf{1}\left(\max_{1\leq i\leq t}\{h_i\} \leq t+n-1\right) \prod_{1\leq i<j\leq t} |h_i - h_j|. \tag{1.9}$$

The random matrix theory treats the distribution functions of eigenvalues $\{y_i\}_{i=1}^N$ of $N \times N$ random Hermitian matrices in the form

$$P_{N\beta}(y_1, \cdots, y_N) = \text{const.} \times \mathrm{e}^{-\frac{\beta}{2}\sum_{j=1}^N y_j^2} \prod_{1\leq i<j\leq N} |y_i - y_j|^\beta, \tag{1.10}$$

where the index β takes the values 1, 2 and 4 depending to the constraints on the ensemble of random matrices and each cases are called the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), and Gaussian symplectic ensemble (GSE), respectively [21, 8]. Under these distributions (1.10), a function $A_\beta(\theta)$ called the level spacing is calculated as

$$A_\beta(\theta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{1}\left(\min_{1\leq i\leq N}\{|y_i|\} \geq \theta\right) P_{N\beta}(y_1, \cdots, y_N) dy_1 \cdots dy_N. \tag{1.11}$$

Johansson [17] studied the similarity and difference between (1.9) and (1.11) with (1.10). (Strictly speaking, (1.9) corresponds to the $\beta = 1$ (GOE) case of (1.10), but Johansson considered the $\beta = 2$ (GUE) case [17]. See also [2, 3].) The main difference is that the eigenvalues y_i's are continuous variables (real numbers), while h_i's are discrete variables (nonnegative integers). The Gaussian kernel $e^{-\beta/2 \sum_{j=1}^{N} y_j^2}$ is missing in (1.9) and some other kernel should be introduced in (1.9). In spite of these differences, the similarity between the functional structures of measures and the quantities which we calculate implies the possible applications of the techniques developed in the random matrix theory to the present flow polynomial problem.

Orthogonal polynomials play important roles in the random matrix theory, since the standard calculation is started by giving determinantal expressions to the distribution functions $P_{N\beta}(y_1, \cdots, y_N)$, in which the matrix elements are given by the orthogonal polynomials. In Section 7, after giving a brief review of such a method of random matrices, we introduce two sets of discrete orthogonal polynomials, which are defined using the Meixner polynomials [19, 17] and the little q-Jacobi polynomials [19], respectively. Using these discrete orthogonal polynomials in matrix elements, the determinantal expressions are given to the anisotropic-flow polynomials and the q-flow polynomials.

Motivated by the random matrix theory we consider ensembles of these flow polynomials and define the probability measures having the determinantal expressions. On these measures the Macdonald formula provides probability laws.

2 Flow Polynomials with Fixed Sinks and Schur Polynomials

2.1 Recurrence Equation of Flow Polynomials

By definition of the flow polynomials $F_t(n(-t), n(-t+2), \cdots, n(t))$ on $H_t = (V_t, A_t)$ given in Section 1.1, it is easy to confirm that they satisfy the recurrence equation

$$\begin{aligned} & F_t(n(-t), n(-t+2), \cdots, n(t)) \\ = & \sum_{f(-t+2)=0}^{n(-t+2)} \sum_{f(-t+4)=0}^{n(-t+4)} \cdots \sum_{f(t-2)=0}^{n(t-2)} F_{t-1}(n(-t) + (n(-t+2) - f(-t+2)), \\ & f(-t+2) + (n(-t+4) - f(-t+4)), \cdots f(t-4) + (n(t-2) - f(t-2)), f(t-2) + n(t)) \end{aligned} \tag{2.1}$$

for $t = 2, 3, \cdots$ with the initial condition

$$F_1(n(-1), n(1)) = 1. \tag{2.2}$$

As illustrated by Figure 2, (2.1) constructs $F_t(n(-t), n(-t+2), \cdots, n(t))$ from $F_{t-1}(n'(-t+1), \cdots, n'(t-1))$ by summing up all possible ways of flows $\{f(-t+2k)\}_{k=1}^{t-1}$ from the vertices with $y = t-1$ to those with $y = t$.

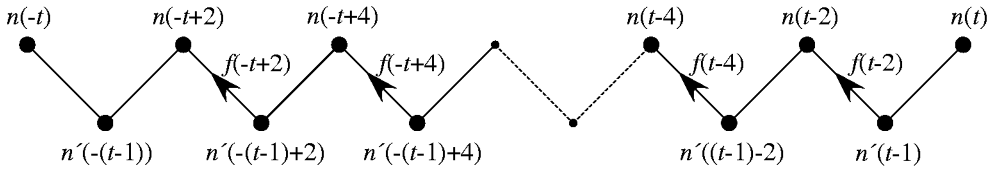

Figure 2: Construction of $F_t(n(-t), \cdots, n(t))$ from $F_{t-1}(n'(-t+1), \cdots, n'(t-1))$.

Performing the first and the second recurrence of (2.1) starting from (2.2) gives

$$F_2(n(-2), n(0), n(2)) = \sum_{f(0)=0}^{n(0)} 1 = 1 + n(0) \qquad \text{and}$$

$$\begin{aligned}
&F_3(n(-3), n(-1), n(1), n(3)) \\
&= \sum_{f(-1)=0}^{n(-1)} \sum_{f(1)=0}^{n(1)} F_2(n(-3) + (n(-1) - f(-1)), f(-1) + (n(1) - f(1)), f(1) + n(3)) \\
&= \sum_{f(-1)=0}^{n(-1)} \sum_{f(1)=0}^{n(1)} [1 + \{f(-1) + (n(1) - f(1))\}] \\
&= (1 + n(1))(1 + n(-1)) \left(1 + \frac{1}{2}(n(-1) + n(1))\right).
\end{aligned}$$

Solving (1.6) for $\{n(-t+2k)\}_{k=0}^{t-1}$ with $t = 2$ and $t = 3$ gives $n(-2) = \lambda_2, n(0) = \lambda_1 - \lambda_2$ and $n(-3) = \lambda_3, n(-1) = \lambda_2 - \lambda_3, n(1) = \lambda_1 - \lambda_2$, respectively, and substituting them in above results gives

$$\begin{aligned}
F_2(n(-2), n(0), n(2)) &= 1 + (\lambda_1 - \lambda_2) = \frac{\lambda_1 - \lambda_2 + 2 - 1}{2 - 1} \qquad \text{and} \\
F_3(n(-3), n(-1), n(1), n(3)) &= \{1 + (\lambda_1 - \lambda_2)\}\{1 + (\lambda_2 - \lambda_3)\} \left(1 + \frac{1}{2}(\lambda_1 - \lambda_3)\right) \\
&= \frac{\lambda_1 - \lambda_2 + 2 - 1}{2 - 1} \times \frac{\lambda_2 - \lambda_3 + 3 - 2}{3 - 2} \times \frac{\lambda_3 - \lambda_1 + 3 - 1}{3 - 1} \\
&= \prod_{1 \le i < j \le 3} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\end{aligned}$$

In order to prove that the solution of (2.1) with (2.2) is generally given by (1.7) for an arbitrary t, we generalize the flow polynomials and the recurrence equations as follows.

2.2 Generalized Flow Polynomials

Definition 2.1 *Let $t \in \mathbf{N}$ and $x_i \in \mathbf{C}, i = 1, 2, \cdots, t$. Define $F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)$ as a unique solution of the recurrence equation*

$$F_t(n(-t), n(-t+2), \cdots, n(t); x_1, \cdots, x_t)$$

$$= \sum_{f(-t+2)=0}^{n(-t+2)} \sum_{f(-t+4)=0}^{n(-t+4)} \cdots \sum_{f(t-2)=0}^{n(t-2)} F_{t-1}(n(-t) + (n(-t+2) - f(-t+2)),$$
$$f(-t+2) + (n(-t+4) - f(-t+4)), \cdots, f(t-4) + (n(t-2) - f(t-2)),$$
$$f(t-2) + n(t); x_1, \cdots, x_{t-1}) \times x_t^{n(-t)+\sum_{k=1}^{t-1} f(-t+2k)} \tag{2.3}$$

for $t = 2, 3, \cdots$ *with the initial condition*

$$F_1(n(-1), n(1); x_1) = x_1^{n(-1)} \tag{2.4}$$

By this definition the following properties are observed.

Lemma 2.2 *(i)* $F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)$ *is a homogeneous polynomials of degree* $\sum_{k=0}^{t-1}(t-k)n(-t+2k)$ *in* $x_1, \cdots, x_t$.

(ii) $F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)|_{x_1=\cdots=x_t=1} = F_t(n(-t), \cdots, n(t))$.

As we have done in Section 2.1, we can solve the recurrence equation for $t = 2$ and 3 as

$$F_2(n(-2), n(0), n(2); x_1, x_2) = \sum_{f(0)=0}^{n(0)} F_1(n(-2) + (n(0) - f(0)), f(0) + n(2); x_1)\ x_2^{n(-2)+f(0)}$$
$$= \sum_{f(0)=0}^{n(0)} x_1^{n(-2)+(n(0)-f(0))} x_2^{n(-2)+f(0)} = (x_1 x_2)^{n(-2)} x_1^{n(0)} \frac{(x_2/x_1)^{n(0)+1} - 1}{x_2/x_1 - 1}$$
$$= \frac{\begin{vmatrix} x_1^{n(-2)+n(0)+1} & x_1^{n(-2)} \\ x_2^{n(-2)+n(0)+1} & x_2^{n(-2)} \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}, \qquad \text{and}$$

$$F_3(n(-3), n(-1), n(1), n(3); x_1, x_2, x_3)$$
$$= \sum_{f(-1)=0}^{n(-1)} \sum_{f(1)=0}^{n(1)} F_2(n(-3) + (n(-1) - f(-1)), f(-1) + (n(1) - f(1)), f(1) + n(3);$$
$$x_1, x_2) \times x_3^{n(-3)+f(-1)+f(1)}$$
$$= \sum_{f(-1)=0}^{n(-1)} \sum_{f(1)=0}^{n(1)} (x_1 x_2)^{n(-3)+(n(-1)-f(-1))} x_1^{f(-1)+(n(1)-f(1))}$$
$$\times \frac{(x_2/x_1)^{f(-1)+(n(1)-f(1))+1} - 1}{x_2/x_1 - 1} \times x_3^{n(-3)+f(-1)+f(1)}$$
$$= \frac{(x_1 x_2 x_3)^{n(-3)} (x_1 x_2)^{n(-1)} x_1^{n(1)}}{(x_2/x_1 - 1)(x_3/x_1 - 1)(x_3/x_2 - 1)} \times \left[\left(\frac{x_3}{x_1}\right)^{n(-1)+n(1)+2} - \left(\frac{x_3}{x_1}\right)^{n(-1)+1} \left(\frac{x_2}{x_1}\right)^{n(1)+1} \right.$$
$$\left. + \left(\frac{x_2}{x_1}\right)^{n(1)+1} - \left(\frac{x_3}{x_2}\right)^{n(-1)+1} \left(\frac{x_3}{x_1}\right)^{n(1)+1} + \left(\frac{x_3}{x_2}\right)^{n(-1)+1} - 1 \right]$$

$$= \frac{\begin{vmatrix} x_1^{n(-3)+n(-1)+n(1)+2} & x_1^{n(-3)+n(-1)+1} & x_1^{n(-3)} \\ x_2^{n(-3)+n(-1)+n(1)+2} & x_2^{n(-3)+n(-1)+1} & x_2^{n(-3)} \\ x_3^{n(-3)+n(-1)+n(1)+2} & x_3^{n(-3)+n(-1)+1} & x_3^{n(-3)} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}.$$

2.3 Schur Polynomials and Proctor's Lemma

Let $t \in \mathbf{N}$ and $\lambda_i \in \mathbf{Z}^+$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0$. Define

$$\begin{aligned} \Delta_\lambda(x_1, \cdots, x_t) &= \det(x_i^{\lambda_j+t-j})_{1\leq i,j\leq t} \\ &= \begin{vmatrix} x_1^{\lambda_1+t-1} & x_1^{\lambda_2+t-2} & \cdots & x_1^{\lambda_t} \\ x_2^{\lambda_1+t-1} & x_2^{\lambda_2+t-2} & \cdots & x_2^{\lambda_t} \\ & \cdots & \cdots & \\ x_t^{\lambda_1+t-1} & x_t^{\lambda_2+t-2} & \cdots & x_t^{\lambda_t} \end{vmatrix}. \end{aligned}$$

If we set $x_k = x_\ell$ for $1 \leq k < \ell \leq t$, then $\Delta_\lambda = 0$, since then the k-th row is equal to the ℓ-th row. Then Δ_λ is divisible by each of the differences $x_k - x_\ell, 1 \leq k < \ell \leq t$, and hence by their product $\prod_{1\leq i<j\leq t}(x_i - x_j)$. This product of all differences is known as the Vandermonde determinant, which is nothing but Δ_0;

$$\Delta_0(x_1, \cdots, x_t) = \det(x_i^{t-j})_{1\leq i,j\leq t} = \prod_{1\leq i<j\leq t}(x_i - x_j). \tag{2.5}$$

Therefore it is concluded that the ratio of two determinant Δ_λ/Δ_0 is a polynomial in $x_1, \cdots, x_t$. Moreover, we can see that it is a homogeneous polynomial of degree $\sum_{i=1}^t \lambda_i$ in $x_1, \cdots, x_t$.

Only using the fundamental properties of determinant, Proctor proved a lemma in [25] that the polynomials $\{\Delta_\lambda(x_1, \cdots, x_t)/\Delta_0(x_1, \cdots, x_t)\}_t$ satisfy the recurrence equation

$$\frac{\Delta_\lambda(x_1, \cdots, x_t)}{\Delta_0(x_1, \cdots, x_t)} = \sum_{\lambda_i \geq \mu_i \geq \lambda_{i+1}, i=1,2,\cdots,t-1} \frac{\Delta_\mu(x_1, \cdots, x_{t-1})}{\Delta_0(x_1, \cdots, x_{t-1})} \times x_t^{\sum_{i=1}^t \lambda_i - \sum_{i=1}^{t-1} \mu_i}$$

for $t = 2, 3, \cdots$. For $t = 1$ we have $\Delta_{\lambda_1}(x_1)/\Delta_0(x_1) = x_1^{\lambda_1}$. The polynomial Δ_λ/Δ_0 is called the Schur polynomial [20, 11, 26].

Definition 2.3 (Jacobi-Trudi) *Fix $t \in \mathbf{N}$ and $\lambda_i \in \mathbf{Z}^+$ with $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_t \geq 0$. For $x_i \in \mathbf{C}, i = 1, \cdots, t$, the Schur polynomial $s_\lambda(x_1, \cdots, x_t)$ is defined by*

$$s_\lambda(x_1, \cdots, x_t) = \frac{\det(x_i^{\lambda_j+t-j})_{1\leq i,j\leq t}}{\det(x_i^{t-j})_{1\leq i,j\leq t}}.$$

Then we have the following lemma.

Lemma 2.4 *The Schur polynomial $s_\lambda(x_1, \cdots, x_t)$ satisfies the recurrence relation*

$$s_\lambda(x_1, \cdots, x_t) = \sum_{\lambda_i \geq \mu_i \geq \lambda_{i+1}, i=1,2\cdots,t-1} s_\mu(x_1, \cdots, x_{t-1}) \quad x_t^{\sum_{i=1}^t \lambda_i - \sum_{i=1}^{t-1} \mu_i}.$$

2.4 Equivalence of Generalized Flow Polynomial and Schur polynomial

The initial condition (2.4), which we put, does not depend on the variable $n(1)$. Assume that $F_{t-1}(n'(-t+1),\cdots,n'(t-1);x_1,\cdots,x_{t-1})$ does not depend on $n'(t-1)$, then the independence of $F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t)$ on $n(t)$ is concluded by the recurrence equation (2.3). This induction proves that $F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t)$ is independent of $n(t)$, which is already observed in the explicit solutions of recurrence equations for $t=2$ and 3 given in Section 2.2. Under this observation, we define $\hat{F}_t$ by

$$F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t)=\hat{F}_t\left(\sum_{k=0}^{t-1}n(-t+2k),\sum_{k=0}^{t-2}n(-t+2k),\cdots,n(-t);x_1,\cdots,x_t\right).$$

Then the recurrence equation (2.3) is written as

$$\begin{aligned}&\hat{F}_t\left(\sum_{k=0}^{t-1}n(-t+2k),\sum_{k=0}^{t-2}n(-t+2k),\cdots,n(-t);x_1,\cdots,x_t\right)\\ =&\sum_{f(-t+2)=0}^{n(-t+2)}\sum_{f(-t+4)=0}^{n(-t+4)}\cdots\sum_{f(t-2)=0}^{n(t-2)}\hat{F}_{t-1}\left(\sum_{k=0}^{t-1}n(-t+2k)-f(t-2),\sum_{k=0}^{t-2}n(-t+2k)-f(t-4),\right.\\ &\left.\cdots,\sum_{k=0}^{1}n(-t+2k)-f(-t+2);x_1,\cdots,x_{t-1}\right)\ x_t^{n(-t)+\sum_{k=1}^{t-1}f(-t+2k)}.\end{aligned}$$

Let $\lambda_s=\sum_{k=0}^{t-s}n(-t+2k)$ and $\mu_s=\lambda_s-f(t-2s)$ for $s=1,2,\cdots,t-1$. Then the summation with respect to $f(-t+2k)$ from 0 to $n(-t+2k)$ is realized by the summation with respect to μ_{t-k} from λ_{t-k+1} to λ_{t-k} for each $k=1,\cdots,t-1$ and thus the recurrence equation becomes

$$\begin{aligned}&\hat{F}_t(\lambda_1,\cdots,\lambda_t;x_1,\cdots,x_t)\\ &=\sum_{\lambda_i\geq\mu_i\geq\lambda_{i+1},i=1,2,\cdots,t-1}\hat{F}_{t-1}(\mu_1,\cdots,\mu_{t-1};x_1,\cdots,x_{t-1})\ x_t^{\sum_{k=1}^{t}\lambda_k-\sum_{k=1}^{t-1}\mu_k}.\end{aligned}$$

Lemma 2.4 shows that the above generalized flow polynomials and the corresponding Schur polynomials related by the variable transformation (1.6) satisfy exactly the same recurrence equation. Since the initial conditions are the same,

$$F_t(n(-1);x_1)=x_1^{n(-1)}=x_1^{\lambda_1}=s_{\lambda_1}(x_1),$$

the following equivalence is proved.

Theorem 2.5 *The generalized flow polynomial $F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t)$ is independent of the value $n(t)$ and, if $\lambda=(\lambda_1,\cdots,\lambda_t)$ is defined by $\{n(-t+2k)\}_{k=0}^{t-1}$ through (1.6), then*

$$F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t)=s_\lambda(x_1,\cdots,x_t).$$

That is,

$$
\begin{aligned}
F_t(n(-t),\cdots,n(t);x_1,\cdots,x_t) &= \frac{\det(x_i^{\lambda_j+t-j})_{1\le i,j\le t}}{\det(x_i^{t-j})_{1\le i,j\le t}} \\
&= \frac{\det\left(x_i^{\sum_{k=0}^{t-j} n(-t+2k)+t-j}\right)_{1\le i,j\le t}}{\det(x_i^{t-j})_{1\le i,j\le t}}. \qquad (2.6)
\end{aligned}
$$

2.5 q-Flow Polynomials and their $q\to 1$ Limits

Let $q\in\mathbf{C}$ and define the q-flow polynomials as follows.

Definition 2.6

$$
F_t(n(-t),n(-t+2),\cdots,n(t);q) = F_t(n(-t),n(-t+2),\cdots,n(t);q^t,q^{t-1},\cdots,q).
$$

Theorem 2.5 gives

$$
F_t(n(-t),\cdots,n(t);q) = \frac{\det(q^{(t-i+1)(\lambda_j+t-j)})_{1\le i,j\le t}}{\det(q^{(t-i+1)(t-j)})_{1\le i,j\le t}}.
$$

Let $z_i = q^{\lambda_i+t-i}, 1\le i\le t$. Then

$$
\det(q^{(t-i+1)(\lambda_j+t-j)})_{1\le i,j\le t} = \prod_{i=1}^t z_i \times \Delta_0(z_1,\cdots,z_t),
$$

where Δ_0 is the Vandermonde determinant (2.5), and we have

$$
\begin{aligned}
\det(q^{(t-i+1)(\lambda_j+t-j)})_{1\le i,j\le t} &= \prod_{i=1}^t z_i \times \prod_{1\le i<j\le t}(z_i-z_j) \\
&= \prod_{i=1}^t q^{\lambda_i+t-i} \times \prod_{1\le i<j\le t} q^{\lambda_j+t-j} \times \prod_{1\le i<j\le t}(q^{\lambda_i-\lambda_j+j-i}-1) \\
&= q^{\sum_{i=1}^t i\lambda_i+\frac{1}{6}(t-1)t(t+1)} \prod_{1\le i<j\le t}(q^{\lambda_i-\lambda_j+j-i}-1).
\end{aligned}
$$

The product formula for the q-flow polynomials is thus obtained as follows.

Corollary 2.7 *Assume the relation (1.6), then*

$$
F_t(n(-t),\cdots,n(t);q) = q^{\sum_{i=1}^t i\lambda_i} \prod_{1\le i<j\le t} \frac{1-q^{\lambda_i-\lambda_j+j-i}}{1-q^{j-i}}.
$$

The formula (1.7) is immediately follows as the $q\to 1$ limit of this result.

Corollary 2.8

$$\begin{aligned}
F_t(n(-t), n(-t+2), \cdots, n(t)) &= \lim_{q\to 1} F_t(n(-t), n(-t+2), \cdots, n(t); q) \\
&= \prod_{1\le i<j\le t} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\
&= \prod_{i=0}^{t-2}\prod_{k=1}^{t-1}\left(1 + \frac{1}{i+1}\sum_{j=0}^{i} n(-t+2k+2j)\right).
\end{aligned}$$

where (1.6) is assumed.

As a special case of Corollary 2.8, we can consider the case such that, for a fixed k, $1 \le k \le t-1$, $n(-t+2\ell)$ is n if $\ell = k$ and is zero otherwise. Since (1.6) gives $\lambda_i = n$ for $1 \le i \le t-k$ and $\lambda_i = 0$ for $t-k+1 \le i \le t$ and thus $\lambda_i - \lambda_j$ is n if $1 \le i \le t-k < j \le t$ and is zero otherwise, we have

$$\begin{aligned}
F_t(0, \cdots, 0, n(-t+2k) = n, 0 \cdots, 0) &= \prod_{i=1}^{t-k}\prod_{j=t-k+1}^{t} \frac{n+j-i}{j-i} = \prod_{i=1}^{k}\prod_{j=k+1}^{t} \frac{n+j-i}{j-i} \\
&= \prod_{i=1}^{k} \frac{(n+i)_{t-2i+1}}{(i)_{t-2i+1}}.
\end{aligned}$$

In the second equality we have used the symmetry between k and $t-k$ and the last equality gives (1.4). This special case of flow polynomial is well studied as the number of *watermelon* configurations with deviation k for the vicious walker problem and the above formula was first derived by Essam and Guttmann [9] (see Section 3.4 below). For the number of watermelon configurations with fixed deviation see [13], where an interesting relation with the MacMahon formula of plane partition is discussed.

3 The Gessel-Viennot Determinant and Conjugate Expression of Essam and Guttmann

3.1 Nonintersecting Paths

Let $t \in \mathbf{N}$ and define a subset of $\mathbf{Z}^2$ as

$$V'_t = \{(x,y) \in \mathbf{Z}^2 : x+y = \text{even},\ 0 \le y \le t\}$$

and A'_t as the set of all arcs which connect the nearest-neighbour pairs of vertices in V'_t and all of which are oriented in the positive direction of the y-axis. The present graph $H'_t \equiv (V'_t, A'_t)$ is an infinite strip with width t and the triangle-shaped graph $H_t = (V_t, A_t)$ introduced in Section 1.1 is a proper subset of H'_t. For any pair of vertices $u, v \in V'_t$, we say that there is a path P from u to v, if we can take a sequence of successive arcs $\in A'_t$ oriented from u to v.

(Each path is regarded as a set of these arcs; $P = \{a_1, a_2, \cdots, a_\ell\} \subset A'_t$.) Let $\mathcal{P}(u,v)$ denote the set of all paths from u to v.

Let $n \in \mathbf{N}$ and choose a set of n integers $r_i \in \mathbf{Z}$ such that $r_1 \leq r_2 \leq \cdots \leq r_n$. Then let $u_i = (2(i-1), 0), v_i(r_i) = (r_i + 2(i-1), t)$ and let $P(r_1, \cdots, r_n)$ denote the set of n paths $P^{(n)} \equiv (P_1, \cdots, P_n)$ with $P_i \in \mathcal{P}(u_i, v_i(r_i)), 1 \leq i \leq n$. Two paths P and P' are said to intersect if they share a common vertex and let $\mathcal{P}_0(r_1, \cdots, r_n) \subset P(r_1, \cdots, r_n)$ be a set of all nonintersecting n paths [27].

The n vicious walkers with fixed starting and ending points on $\mathbf{Z}$ are defined as the interacting random walkers, whose trajectory during time interval $[0,t]$ is given by an element of $\mathcal{P}_0(r_1, \cdots, r_n)$ on the spatio-temporal plane H'_t. The restriction of nonintersecting on paths realizes the vicious property of walkers [10, 1, 9].

Let
$$\mu_i = \frac{1}{2}(t - r_i), \quad i = 1, 2, \cdots, n \tag{3.1}$$
which denotes the number of left steps by the i-th walker during time interval $[0,t]$. The bijection between the integer flows and the trajectories of vicious walkers was established by Arrowsmith *et al.* [1], which gives the following identity.

Lemma 3.1 (Arrowsmith-Mason-Essam) *Assume that $\{\mu_i\}_{i=1}^n$ is given by (3.1) and let*
$$n(-t+2k) = |\{i : \mu_i = t-k\}|, \quad k = 0, 1, \cdots, t. \tag{3.2}$$
Then
$$F_t(n(-t), \cdots, n(t)) = |\mathcal{P}_0(r_1, \cdots, r_n)|.$$

3.2 Gessel-Viennot Determinant and Elementary Symmetric Polynomials

Let $x_i \in \mathbf{C}, i = 1, 2, \cdots, t$, and consider a set of polynomials in $x_1, \cdots, x_t$ with integer coefficients $R_t = \mathbf{Z}[[x_i : i = 1, 2, \cdots, t]]$. A weight function is a map $w : A'_t \to R_t$ that assigns values in R_t to each arc $\in A'_t$. For any path P the weight is defined as $w(P) = \prod_{a \in P} w(a)$, and for each $P^{(n)} = (P_1, \cdots, P_n) \in \mathcal{P}_0(r_1, \cdots, r_n)$ the weight is given as $w(P^{(n)}) = \prod_{i=1}^n w(P_i)$. Define
$$W(u,v) = \sum_{P \in \mathcal{P}(u,v)} w(P).$$
Then the Gessel-Viennot theorem [14] gives the following lemma [27] (see also [4]).

Theorem 3.2 (Gessel-Viennot) *Define*
$$W(r_1, \cdots, r_n; x_1, \cdots, x_t) = \sum_{P^{(n)} \in \mathcal{P}_0(r_1, \cdots, r_n)} w(P^{(n)}).$$
Then
$$W(r_1, \cdots, r_n; x_1, \cdots, x_t) = \det \left(W(u_i, v_i)\right)_{1 \leq i,j \leq n}.$$

Hereafter we specify the weight function as follows. For each $s = 1, 2, \cdots, t$, the arc a from a vertex $(r, s-1) \in V'_t$ to a vertex $(r', s) \in V'_t$ has the weight

$$w(a) = \begin{cases} x_s & \text{if } r' = r-1 \\ 1 & \text{if } r' = r+1. \end{cases}$$

(Remark that we have assumed that there are at most two arcs oriented out of each vertex in H'_t.) We can see that if $x_1 = x_2 = \cdots = x_t = 1$, then the weight function is a constant 1 and

$$|\mathcal{P}_0(r_1, \cdots, r_n)| = W(r_1, \cdots, r_n; x_1, \cdots, x_t)|_{x_1=x_2=\cdots=x_t=1}.$$

Moreover Definition 2.1 and the above choice of weight function imply that Lemma 3.1 can be generalized as

$$F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t) = W_t(r_1, \cdots, r_n; x_1, \cdots, x_t)$$

by using the same bijection of Arrowsmith *et al.*

Let $e_k(x_1, \cdots, x_t)$ be the sum of all monomials in the form $x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k}$ for all strictly increasing sequences $1 \leq i_1 < \cdots < i_k \leq t$. In other words, $e_k(x_1, \cdots, x_t)$'s are defined as the coefficients in the expansion

$$\prod_{i=1}^{t}(1 + x_i z) = \sum_{k=0}^{t} e_k(x_1, \cdots, x_t) z^k. \tag{3.3}$$

By definition $e_k(x_1, \cdots, x_t)$ is a symmetric polynomial in $x_1, \cdots, x_t$ called the k-th elementary symmetric polynomial [11, 20]. Then it is easy to see that

$$W(u_i, v_j) = e_{\mu_j+i-j}(x_1, \cdots, x_t).$$

Applying the Gessel-Viennot theorem (Theorem 3.2) we have the following determinantal expression.

Theorem 3.3 *Assume (3.1) and (3.2). Then*

$$F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t) = \det\left(e_{\mu_j+i-j}(x_1, \cdots, x_t)\right)_{1 \leq i,j \leq n}.$$

3.3 Conjugate of Young Diagrams

Now we have two distinct determinantal expressions for the generalized flow polynomials; (i) a ratio of two determinants of $t \times t$ matrices (Theorem 2.5) and (ii) a determinant of $n \times n$ matrix (Theorem 3.3). Here we show that this duality of expressions for n walkers in time period t can be understood well by using the knowledge of Young diagrams for partitions [11, 20].

The flow polynomials are specified by a set of integers $\{n(-t+2k)\}_{k=0}^{t-1}$ which represent the strengths of sinks. By the transformation (1.6) we have a set of integers $\{\lambda_i\}_{i=1}^{t}$ with the property $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0$. Such a sequence of non-negative integers in decreasing order $\lambda = (\lambda_1, \cdots, \lambda_t)$ is called a partition and each partition λ is represented by a Young diagram. The conjugate of a partition λ is defined as the partition λ' whose diagram is the transpose of the diagram of λ. The number $m_i(\lambda) = |\{j : \lambda_j = i\}|$ is called the multiplicity of i in λ and the above definition of conjugate gives the relation

$$\lambda_i = \sum_{k \geq i} m_k(\lambda').$$

For nonintersecting paths (vicious walkers) we assume the inequalities $r_1 \leq r_2 \leq \cdots \leq r_n$ and thus $\mu = (\mu_1, \cdots, \mu_t)$ given by (3.1) is a partition. The relation (3.2) means $n(t - 2k) = m_k(\mu), k = 1, 2, \cdots, t$. Then we have

$$\sum_{k=i}^{t} m_k(\mu) = \sum_{k=0}^{t-i} n(-t + 2k).$$

Since (1.6) holds, we can conclude that $\mu = (\mu_1, \cdots, \mu_n)$ is the conjugate of $\lambda = (\lambda_1, \cdots, \lambda_t)$.

It is known that the Schur polynomial $s_\lambda(x_1, \cdots, x_t)$ can be expressed as a polynomial in the elementary symmetry polynomials $e_k(x_1, \cdots, x_t)$ as [11, 20]

$$s_\lambda(x_1, \cdots, x_t) = \det \left(e_{\lambda'_j + i - j}(x_1, \cdots, x_t) \right)_{1 \leq i,j \leq n}, \tag{3.4}$$

when $\lambda_1 \leq n$. The expression given by Theorem 3.3 is equivalent with the identity (3.4) for we have proved $\mu = \lambda'$. Since we proved the equivalence between the generalized flow polynomials and the Schur functions (Theorem 2.5), we can say that the identity (3.4) was derived by applying the Gessel-Viennot theorem to the flow polynomials (*i.e.* nonintersecting paths/vicious walkers). Inversely speaking, the identity (3.4) with Theorem 3.3 gives another proof of Theorem 2.5.

3.4 Essam-Guttmann's Conjugate Expression

If we set $x_1 = x_2 = \cdots = x_t = x$ in (3.3), we have

$$(1 + xz)^t = \sum_{k=0}^{t} \binom{t}{k} x^k z^k = \sum_{k=0}^{t} e_k(x, \cdots, x) z^k.$$

Therefore

$$W(u_i, v_j)|_{x_1 = x_2 = \cdots = x_t = 1} = \binom{t}{\lambda'_j + i - j}$$

and Theorem 3.3 gives

$$F_t(n(-t), \cdots, n(t)) = \det \left(\binom{t}{\lambda'_j + i - j} \right)_{1 \leq i,j \leq n}.$$

Essam and Guttmann [9] proved that this binomial determinant can be reduced to the following product formula (see also [13, 18]).

Theorem 3.4 (Essam-Guttmann) *Let $n \in \mathbf{N}$ and assume (1.6) with $\sum_{k=0}^{t} n(-t+2k) = n$. And let $\lambda' = (\lambda'_1, \cdots, \lambda'_n)$ be the conjugate partition of λ. Then*

$$F_t(n(-t), \cdots, n(t)) = \prod_{i}^{n} \frac{(t+n-i)!}{(\lambda'_i+n-i)!(t-\lambda'_i+i-1)!} \prod_{1\le i<j\le n} (\lambda'_i - \lambda'_j + j - i). \qquad (3.5)$$

This expression may be regarded as the conjugate expression of (1.7) (*i.e.* the identities given in Corollary 2.8). As remarked after Corollary 2.8, Essam and Guttmann derived the exact formula (1.4) for the number of watermelon configurations as a special case of Theorem 3.4.

4 Summation of Flow Polynomials and Macdonald Formula

In the previous section we studied the generalized flow polynomials with a fixed set of strengths of sinks $\{n(-t+2k)\}_{k=0}^{t}$. There are many possible choices of the set $\{n(-t+2k)\}_{k=0}^{t}$ which are compatible to a given strength of the source at $O = (0,0)$. In other words, all flow polynomials, which satisfy the condition $\sum_{k=0}^{t} n(-t+2k) = n$ have the same strength n of the source. Here we consider the summation of all such flow polynomials with a fixed n defined as follows.

Definition 4.1 *For $t \in \mathbf{N}$ and $n \in \mathbf{Z}^+$, define*

$$Z_{t,n}(x_1, \cdots, x_t) = \sum_{n(-t)\ge 0} \cdots \sum_{n(t)\ge 0} \mathbf{1}\left(\sum_{k=0}^{t} n(-t+2k) = n\right) F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t).$$

The corresponding function for the q-flow polynomials is defined by

$$Z_{t,n}(q) = Z_{t,n}(q^t, q^{t-1}, \cdots, q).$$

The Macdonald formula (Formula 1.3) is stated as the following theorem.

Theorem 4.2 (Macdonald)

$$\begin{aligned} Z_{t,n}(x_1, \cdots, x_t) &= \frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1\le i,j\le t}}{\det(x_i^{j-1} - x_i^{2t-j})_{1\le i,j\le t}} \\ &= \frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1\le i,j\le t}}{\prod_{i=1}^{t}(1-x_i) \prod_{1\le i<j\le t} [(x_i - x_j)(x_i x_j - 1)]}. \end{aligned} \qquad (4.1)$$

Remark that the second equality is obtained by employing a case of the Weyl denominator formulae, whose simple inductive proof can be found in [5]. Since $Z_{t,n}(x_1, \cdots, x_t)$ is defined as a finite summation of polynomials in $x_1, \cdots, x_t$, the numerator should be divisible by the denominator. This fact is easily confirmed, since we can see that the determinant in numerator vanishes if $x_i = 1$ $(1 \le i \le t)$ or $x_i = x_j$ $(1 \le i < j \le t)$ or $x_i x_j = 1$ $(1 \le i < j \le t)$.

In the present section we show the following identities are derived from Theorem 4.2.

Corollary 4.3

$$(i) \quad Z_{t,n}(q) = \prod_{1 \le i \le j \le t} \frac{1 - q^{n+i+j-1}}{1 - q^{i+j-1}}.$$
$$(ii) \quad \lim_{q \to 1} Z_{t,n}(q) = \prod_{1 \le i \le j \le t} \frac{n+i+j-1}{i+j-1}.$$

By Corollaries 2.7 and 2.8, these identities gives the Bender-Knuth formula (Formula 1.2) and the Arrowsmith-Mason-Essam formula (Formula 1.1), respectively.

First we prove the following lemma.

Lemma 4.4 *For $t \in \mathbf{N}, n \in \mathbf{Z}^+$ and $q \in \mathbf{C}$, define a $t \times t$ matrix*

$$\begin{aligned} M(t,n;q) &= (q^{(t-i+1)(j-1)} - q^{(t-i+1)(n+2t-j)})_{1 \le i,j \le t} \\ &= \begin{pmatrix} 1 - q^{t(n+2t-1)} & q^t - q^{t(n+2t-2)} & \cdots & q^{t(t-1)} - q^{t(n+t)} \\ 1 - q^{(t-1)(n+2t-1)} & q^{t-1} - q^{(t-1)(n+2t-2)} & \cdots & q^{(t-1)(t-1)} - q^{(t-1)(n+t)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 - q^{2(n+2t-1)} & q^2 - q^{2(n+2t-2)} & \cdots & q^{2(t-1)} - q^{2(n+t)} \\ 1 - q^{n+2t-1} & q - q^{n+2t-2} & \cdots & q^{t-1} - q^{n+t} \end{pmatrix} \end{aligned}$$

and let $D(t,n;q) = \det M(t,n;q)$. Then

$$\begin{aligned} D(t,n;q) &= \prod_{i=1}^{t} (1 - q^{n+(t-i+1)}) \prod_{1 \le i < j \le t} \left[(q^{t-i+1} - q^{t-j+1})(q^{n+(t-i+1)+(t-j+1)} - 1) \right] \\ &= \prod_{1 \le i < j \le t} (q^i - q^j) \prod_{1 \le i \le j \le t} (1 - q^{n+i+j-1}). \end{aligned} \tag{4.2}$$

Proof. (i) For a fixed k $(1 \le k \le t)$, assume that $q^{n+(t-k+1)} = 1$. Then $M(t,n;q)_{i,j} = q^{(t-i+1)(j-1)} - q^{(t-i+1)(t+k-j-1)}$. Therefore $M(t,n;q)_{i,1} = 1 - q^{(t-i+1)(t+k-2)} = -M(t,n;q)_{i,t+k-1}$ for $1 \le \forall i \le t$. Then $D(t,n;q) = 0$, since the first column is equal to $(-1) \times (t+k-1)$-th column, and thus $D(t,n;q)$ should have the factors $\prod_{k=1}^{t} (1 - q^{n+(t-k+1)})$. (ii) Fix a pair (k,ℓ) $(1 \le k < \ell \le t)$ and assume that $q^{t-k+1} - q^{t-\ell+1} = 0$. Then $M(t,n;q)_{k,j} = q^{(t-k+1)(j-1)} - q^{(t-k+1)(n+2t-j)} = q^{(t-\ell+1)(j-1)} - q^{(t-\ell+1)(n+2t-j)} = M(t,n;q)_{\ell,j}$ for $1 \le \forall j \le t$. That is, the k-th row is equal to the ℓ-th row and thus $D(t,n:q) = 0$. Therefore $D(t,n;q)$ should have the factors $\prod_{1 \le k < \ell \le t} (q^{t-k+1} - q^{t-\ell+1})$. (iii) By the similar argument we can show that $D(t,n;q)$ should have the factors $\prod_{1 \le k < \ell \le t} (q^{n+(t-k+1)+(t-\ell+1)} - 1)$.

Let $\bar{D}(t,n;q)$ be the RHS of the first equality of (4.2). Then we can conclude that

$$D(t,n;q) = c(t,n;q) \bar{D}(t,n;q)$$

with a polynomial $c(t,n;q)$ of q. We can see that the degree of $\bar{D}(t,n;q)$ in q is

$$\sum_{i=1}^{t}[n+(t-i+1)]+\sum_{i=1}^{t}\sum_{j=i+1}^{t}(t-i+1)+\sum_{i=1}^{t}\sum_{j=i+1}^{t}(n+2t+2-i-j)=t(t+1)(5t+3n-2)/6,$$

and by definition of $D(t,n;q)$ the degree of $D(t,n;q)$ in q is evaluated as

$$\sum_{i=1}^{t}(t-i+1)(n+2t-i)=t(t+1)(5t+3n-2)/6.$$

Furthermore we can see that both of the coefficients of these largest powers of q in $\bar{D}(t,n;q)$ and $D(t,n;q)$ are given by $(-1)^t$. Then we can conclude that $c(t,n;q)=1$ and the first equality of (4.2) is proved. Let $i'=t-i+1$ and $j'=t-j+1$. Then

$$\begin{aligned} D(t,n;q) &= \prod_{i'}(1-q^{n+i'})\prod_{1\le j'<i'\le t}(q^{i'}-q^{j'})(q^{n+i'+j'}-1) \\ &= \prod_{i=1}^{t}(1-q^{n+i})\prod_{1\le i<j\le t}(q^{i}-q^{j})(1-q^{n+i+j}). \end{aligned}$$

Here we can see that $\prod_{1\le i<j\le t}(1-q^{n+i+j})=\prod_{1\le i\le j\le t}(1-q^{n+i+j-1})/\prod_{i=1}^{t}(1-q^{n+i})$, then the second equality of (4.2) is proved. ∎

Proof of Corollary 4.3. With definitions of $Z_{t,n}(q)$ and $D(t,n;q)$ Theorem 4.2 gives

$$Z_{t,n}(q)=\frac{D(t,n;q)}{D(t,0;q)}.$$

The equality (i) is obtained by applying the second equality of (4.2) in Lemma 4.4 to this expression. The equality (ii) is immediately obtained from (i). ∎

In the context of vicious walkers $\lim_{q\to 1}Z_{t,n}(q)$ gives the total number of *star* configurations of n vicious walkers and the different expressions from (ii) of Corollary 4.3, which are, however, equivalent to it, were conjectured by direct evaluations;

$$\begin{aligned} \prod_{1\le i\le j\le t}\frac{n+i+j-1}{i+j-1} &= \prod_{i=1}^{[(t+1)/2]}\frac{(n+2i-1)_{2t-4i+3}}{(2i-1)_{2t-4i+3}} \\ &= 2^{nt}\prod_{i=1}^{n}\frac{\left(\frac{i+1}{2}\right)_i}{(i)_i}. \end{aligned}$$

The second expression was given by Arrowsmith, Mason and Essam [1] as mentioned in Section 1. The third expression is found in Essam and Guttmann [9].

5 Recurrence Equation and Macdonald Formula as its Solution

In this section we review an "elementary proof" of the Macdonald formula (Formula 1.3 or Theorem 4.2) given by Bressoud [5]. He proved the formula by induction on t. Here we claim that this proof essentially bases on the fact that $Z_{t,n}(x_1,\cdots,x_t)$ satisfies a recurrence equation, which was not explicitly mentioned in [5].

Lemma 5.1 *For $t=2,3,\cdots,$*

$$Z_{t,n}(x_1,\cdots,x_t)=\sum_{m=0}^{n}(x_1\cdots x_t)^m\sum_{k=1}^{t}\prod_{i=1,i\neq k}^{t}\frac{x_i}{x_i-x_k}Z_{t-1,n-m}(x_1,\cdots,\hat{x}_k,\cdots,x_t), \tag{5.1}$$

where $\hat{x}_k$ means omitting x_k in $x_1,\cdots,x_t$.

Proof. By Definition 4.1, Theorem 2.5 and (2.5),

$$\begin{aligned}Z_{t,n}(x_1,\cdots,x_t)=&\frac{1}{\prod\limits_{1\le i<j\le t}(x_i-x_j)}\sum_{n(-t)\ge 0}\cdots\sum_{n(t-2)\ge 0}\mathbf{1}\left(\sum_{k=0}^{t-1}n(-t+2k)\le n\right)\\&\times\det\left(x_i^{\sum_{k=0}^{t-j}n(-t+2k)+t-j}\right)_{1\le i,j\le t}.\end{aligned}$$

Expand the determinant along the t-th column, we have

$$\begin{aligned}&\det\left(x_i^{\sum_{k=0}^{t-j}n(-t+2k)+t-j}\right)\\&=\sum_{k=1}^{t}(-1)^{t+k}x_k^{n(-t)}\det\left(x_i^{\sum_{\ell=0}^{t-j}n(-t+2\ell)+t-j}\right)_{1\le i\le t;i\neq k,1\le j\le t-1}\\&=\sum_{k=1}^{t}(-1)^{t+k}x_k^{n(-t)}\prod_{i=1,i\neq k}^{t}x_i^{n(-t)+1}\det\left(x_t^{\sum_{\ell=1}^{t-j}n(-t+2\ell)+t-j-1}\right)_{1\le i\le t;i\neq k,1\le j\le t-1}.\end{aligned}$$

Remark that

$$\begin{aligned}&\sum_{n(-t)\ge 0}\cdots\sum_{n(t-2)\ge 0}\mathbf{1}\left(0\le\sum_{k=0}^{t-1}n(-t+2k)\le n\right)(\cdots)\\&=\sum_{n(-t)=0}^{n}\sum_{n(-t+2)\ge 0}\cdots\sum_{n(t-2)\ge 0}\mathbf{1}\left(0\le\sum_{k=1}^{t-1}n(-t+2k)\le n-n(-t)\right)(\cdots)\end{aligned}$$

and that

$$\prod_{1\le i<j\le t}(x_i-x_j)=\prod_{1\le i<j\le t;j\neq k}(x_i-x_j)\times\prod_{i=1;i\neq k}^{t}(x_i-x_k)(-1)^{t+k}.$$

Then

$$
\begin{aligned}
&Z_{t,n}(x_1,\cdots,x_t)\\
&= \sum_{n(-t)=0}^{n}\sum_{k=1}^{t}\frac{x_k^{n(-t)}}{\prod_{i=1;i\neq k}^{t}(x_i-x_k)}\prod_{i=1;i\neq k}^{t}x_i^{n(-t)+1}\\
&\quad\times\frac{1}{\prod_{1\leq i<j\leq t;i,j\neq k}(x_i-x_j)}\sum_{n(-t+2)\geq 2}\cdots\sum_{n(t-2)\geq 0}\mathbf{1}\left(0\leq\sum_{k=1}^{t-1}n(-t+2k)\leq n-n(-t)\right)\\
&\qquad\times\det\left(x_i^{\sum_{\ell=1}^{t-j}n(-t+2\ell)+t-j-1}\right)_{1\leq i\leq t;i\neq k,1\leq j\leq t-1}.
\end{aligned}
$$

Let $m=n(-t)$ and $\tilde{n}(s-1)=n(s), s=-t+2,\cdots,t-2$. Then

$$
\begin{aligned}
&Z_{t,n}(x_1,\cdots,x_t)\\
&= \sum_{m=0}^{n}(x_1\cdots x_t)^m\sum_{k=1}^{t}\prod_{i=1;i\neq k}^{t}\frac{x_i}{x_i-x_k}\\
&\times\frac{1}{\prod_{1\leq i<j\leq t;i,j\neq k}(x_i-x_j)}\sum_{\tilde{n}(-(t-1))\geq 0}\cdots\sum_{\tilde{n}((t-1)-2)\geq 0}\mathbf{1}\left(0\leq\sum_{k=0}^{(t-1)-1}\tilde{n}(-(t-1)+2k)\leq n-m\right)\\
&\qquad\times\det\left(x_i^{\sum_{\ell=0}^{(t-1)-j}\tilde{n}(-(t-1)+2\ell)+(t-1)-j}\right)_{1\leq i\leq t;i\neq k,1\leq j\leq t-1}\\
&= \sum_{m=0}^{n}(x_1\cdots x_t)^m\sum_{k=1}^{t}\prod_{i=1;i\neq k}^{t}\frac{x_i}{x_i-x_k}Z_{t-1,n-m}(x_1,\cdots,\hat{x}_k,\cdots,x_t).
\end{aligned}
$$

∎

For $t=1$

$$
Z_{1,n}(x_1)=\sum_{n(-1)=0}^{n}x_1^{n(-1)}=\frac{1-x_1^{n+1}}{1-x_1}. \tag{5.2}
$$

Then the Macdonald formula is equivalent with the following statement, which was proved in [5].

Proposition 5.2 (Bressoud) *Consider (5.1) as a recurrence equation on $t=1,2,3,\cdots$. Set the initial condition as (5.2). Then the unique solution of (5.1) is given by (4.1).*

6 Okada's Pfaffian and Macdonald Formula

For $t\in\mathbf{N}, n\in\mathbf{Z}^+$ and $x_i\in\mathbf{C}, 1\leq i\leq t$, define a $t\times(n+t)$ matrix X as

$$
X = \det(x_i^{n+t-j})_{1\leq i\leq t,1\leq j\leq n+t}
$$

$$= \begin{pmatrix} x_1^{n+t-1} & x_1^{n+t-2} & \cdots & x_1 & 1 \\ x_2^{n+t-1} & x_2^{n+t-2} & \cdots & x_2 & 1 \\ & \cdots & \cdots & \cdots & \\ x_t^{n+t-1} & x_t^{n+t-2} & \cdots & x_t & 1 \end{pmatrix}.$$

Let $\{k_1, \cdots, k_t\} \subset \{1, 2, \cdots, n+t\}$ and write a $t \times t$ submatrix of X as

$$\begin{aligned} X_{k_1,\cdots,k_t} &= \det(x_i^{n+t-k_\ell})_{1\le i,\ell\le t} \\ &= \begin{pmatrix} x_1^{n+t-k_1} & x_1^{n+t-k_2} & \cdots & x_1^{n+t-k_t} \\ x_2^{n+t-k_1} & x_2^{n+t-k_2} & \cdots & x_2^{n+t-k_t} \\ & \cdots & \cdots & \cdots \\ x_t^{n+t-k_1} & x_t^{n+t-k_2} & \cdots & x_t^{n+t-k_t} \end{pmatrix}. \end{aligned}$$

Then Definition 4.1, Theorem 2.5 and (2.5) give the following lemma.

Lemma 6.1

$$Z_{t,n}(x_1, \cdots, x_t) = \frac{1}{\prod\limits_{1\le i<j\le t} (x_i - x_j)} \sum_{1\le k_1<k_2<\cdots<k_t\le n+t} \det(X_{k_1,\cdots,k_t}).$$

That is, $Z_{t,n}(x_1, \cdots, x_t)$ is given as a summation of minors of X.

Hereafter we assume that t is even positive integer in this section for simplicity of the description.

6.1 Okada's minor-summation formula

First we give the definition of Pfaffian of anti-symmetric matrix.

Definition 6.2 *Let $2m$ be an even integer and define a subset $\mathcal{F}_{2m}$ of the symmetric group $\mathcal{S}_{2m}$ by*

$$\mathcal{F}_{2m} = \{\sigma \in \mathcal{S}_{2m} : \sigma(1) < \sigma(3) < \cdots < \sigma(2m-1),\ \sigma(2i-1) < \sigma(2i)\ (1 \le \forall i \le m)\}.$$

Then the Pfaffian of a $2m \times 2m$ anti-symmetric matrix $A = (a_{ij})_{1\le i,j\le 2m}$ is defined as

$$\mathrm{Pf}(A) = \sum_{\sigma\in\mathcal{F}_{2m}} (-1)^{\mathcal{I}(\sigma)} a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2m-1)\sigma(2m)},$$

where $\mathcal{I}(\sigma)$ is the number of inversions in σ.

The following properties of Pfaffian will be used in this section.

Lemma 6.3 *Let $m \in \mathbf{N}$ and $A = (a_{ij})_{1\le i,j\le 2m}$ be a $2m \times 2m$ anti-symmetric matrix.*

$$(i) \quad \mathrm{Pf}((-a_{ij})_{1\le i,j\le 2m}) = (-1)^m \mathrm{Pf}((a_{ij})_{1\le i,j\le 2m}).$$

$$(ii) \quad \mathrm{Pf}((x_i x_j a_{ij})_{1\le i,j\le 2m}) = \prod_{i=1}^{2m} x_i \times \mathrm{Pf}((a_{ij})_{1\le i,j\le 2m}).$$

$$(iii) \quad \mathrm{Pf}\left(\left(\frac{x_i - x_j}{1 - x_i x_j}\right)_{1\le i,j\le 2m}\right) = \prod_{1\le i<j\le 2m} \frac{x_i - x_j}{1 - x_i x_j}.$$

The property (i) is immediately found by the definition of Pfaffian. Other two properties are proved in Proposition 2.3 of Stembridge [27].

Assume that n_1 and n_2 are integers with $1 \le n_1 < n_2$ and n_1 is even. Consider an arbitrary $n_1 \times n_2$ rectangular matrix $M = (m_{ij})_{1\le i\le n_1, 1\le j\le n_2}$ and its $n_1 \times n_1$ square submatrices

$$M_{k_1,k_2,\cdots,k_{n_1}} = (m_{ik_j})_{1\le i,j\le n_1},$$

where $\{k_1, k_2, \cdots, k_{n_1}\} \subset \{1, 2, \cdots, n_2\}$. Okada proved the following general identity which is called the minor-summation formula (Theorem 3 in [23]).

Theorem 6.4 (Okada) *Let*

$$y_{ij} = \sum_{1\le k_1<k_2\le n_2} \begin{vmatrix} m_{ik_1} & m_{ik_2} \\ m_{jk_1} & m_{jk_2} \end{vmatrix}$$

for $1 \le i < j \le n_1$. *Define an* $n_1 \times n_1$ *anti-symmetric matrix*

$$Y = \begin{pmatrix} 0 & y_{12} & y_{13} & \cdots & y_{1n_1} \\ -y_{12} & 0 & y_{23} & \cdots & y_{2n_1} \\ -y_{13} & -y_{23} & 0 & \cdots & y_{3n_1} \\ & \cdots & & & \\ -y_{1n_1} & -y_{2n_1} & -y_{3n_1} & \cdots & 0 \end{pmatrix}.$$

Then

$$\sum_{1\le k_1<k_2<\cdots<k_{n_1}\le n_2} \det(M_{k_1,k_2,\cdots,k_{n_1}}) = \mathrm{Pf}(Y).$$

6.2 From Pfaffian to Determinant

Define

$$y(x_i, x_j) = \sum_{1\le k_1<k_2\le n+t} \begin{vmatrix} x_i^{n+t-k_1} & x_i^{n+t-k_2} \\ x_j^{n+t-k_1} & x_j^{n+t-k_2} \end{vmatrix}$$

for $1 \le i < j \le t$. Then we have

$$y(x_i, x_j) = \frac{(x_i - x_j)\{1 - (x_i x_j)^{n+t}\} - (1 - x_i x_j)(x_i^{n+t} - x_j^{n+t})}{(1 - x_i)(1 - x_j)(1 - x_i x_j)}. \tag{6.1}$$

Combining Lemma 6.1 and Theorem 6.4 gives the following identity.

Proposition 6.5 *Define a* $t \times t$ *anti-symmetric matrix*

$$Y(x_1, \cdots, x_t) = \begin{pmatrix} 0 & y(x_1,x_2) & y(x_1,x_3) & \cdots & y(x_1,x_t) \\ -y(x_1,x_2) & 0 & y(x_2,x_3) & \cdots & y(x_2,x_t) \\ -y(x_1,x_3) & -y(x_2,x_3) & 0 & \cdots & y(x_3,x_t) \\ & & \cdots & & \\ -y(x_1,x_t) & -y(x_2,x_t) & -y(x_3,x_t) & \cdots & 0 \end{pmatrix}, \tag{6.2}$$

where $y(x_i, x_j)$ *is defined by (6.1). Then*

$$Z_{t,n}(x_1, \cdots, x_t) = \frac{1}{\prod_{1 \le i < j \le t} (x_i - x_j)} \mathrm{Pf}(Y(x_1, \cdots, x_t)).$$

Okada showed that the Macdonald formula (Theorem 4.2) is obtained from Proposition 6.5 by using the following Pfaffian identities.

Lemma 6.6 *Let* $Y(x_1, \cdots, x_t)$ *be the* $t \times t$ *anti-symmetric matrix defined by (6.2). Then*

$$\mathrm{Pf}(Y(x_1, \cdots, x_t)) = \frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1 \le i,j \le t}}{\prod_{i=1}^{t}(1 - x_i) \prod_{1 \le i < j \le t} (x_i x_j - 1)}.$$

Proof. By (i) and (ii) of Lemma 6.3

$$\mathrm{P}(Y(x_1, \cdots, x_t)) = \frac{(-1)^{t/2}}{\prod_{i=1}^{t}(1 - x_i)} \mathrm{Pf}(\bar{Y}(x_1, \cdots, x_t)),$$

where $\bar{Y}(x_1, \cdots, x_t)$ is obtained from (6.2) by replacing $y(x_i, x_j)$ by

$$\bar{y}(x_i, x_j) = -(1 - x_i)(1 - x_j) y(x_i, x_j).$$

Okada proved the identity (Corollary 4.6 in [24])

$$\mathrm{Pf}(\bar{Y}(x_1, \cdots, x_t)) = \frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1 \le i,j \le t}}{\prod_{1 \le i < j \le t} (1 - x_i x_j)}.$$

Since $\prod_{1 \le i < j \le t} (1 - x_i x_j) = (-1)^{\frac{1}{2}t(t-1)} \prod_{1 \le i < j \le t} (x_i x_j - 1)$ and $t(t-1)/2 = t/2 \mod 2$ for even t, the proof is completed. ∎

6.3 $n \to \infty$ Limit

For the usage of the later sections, here we consider the $n \to \infty$ limit. Assume that $0 < x_i < 1, 1 \le i \le t$, in this subsection. Then (6.1) defines

$$\begin{aligned} y_\infty(x_i, x_j) &\equiv \lim_{n \to \infty} y(x_i, y_j) \\ &= \frac{x_i - x_j}{(1 - x_i)(1 - x_j)(1 - x_i x_j)}. \end{aligned}$$

Define $Y_\infty(x_1, \cdots, x_t)$ be the $t \times t$ anti-symmetric matrix obtained from the matrix (6.2) by replacing $y(x_i, x_j)$ by the above $y_\infty(x_i, x_j)$. Then Okada's minor-summation formula (Theorem 6.4) gives

$$\lim_{n\to\infty} Z_{t,n}(x_1, \cdots, x_t) = \frac{1}{\prod_{1\le i<j\le t}(x_i - x_j)} \mathrm{Pf}(Y_\infty(x_1, \cdots, x_t)).$$

Here we can use (ii) and (ii) of Lemma 6.3 to calculate the Pfaffian of Y_∞ as

$$\mathrm{Pf}(Y_\infty(x_1, \cdots, x_t)) = \frac{\mathrm{Pf}\left(\left(\frac{x_i - x_j}{1 - x_i x_j}\right)_{1\le i<j\le t}\right)}{\prod_{i=1}^{t}(1 - x_i)} = \frac{\prod_{1\le i<j\le t} \frac{x_i - x_j}{1 - x_i x_j}}{\prod_{i=1}^{t}(1 - x_i)}.$$

On the other hand, since the independence of $F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)$ on $n(t)$ claimed in Theorem 2.5 implies that

$$\begin{aligned} Z_{t,n}(x_1, \cdots, x_t) &= \sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t) \\ &\quad \times \sum_{n(t)\ge 0} \mathbf{1}\left(\sum_{k=0}^{t} n(-t + 2k) = n\right) \\ &= \sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} \mathbf{1}\left(\sum_{k=0}^{t-1} n(-t + 2k) \le n\right) F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t), \end{aligned}$$

and then

$$\lim_{n\to\infty} Z_{t,n}(x_1, \cdots, x_t) = \sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t).$$

The following identities are then concluded.

Lemma 6.7 *Assume that* $0 \le x_i < 1, i = 1, 2, \cdots, t$. *Then*

$$\sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t) = \prod_{i=1}^{t}(1 - x_i)^{-1} \prod_{1\le i<j\le t}(1 - x_i x_j)^{-1}. \quad (6.3)$$

In particular, if $0 < \alpha, q < 1$,

$$\sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)|_{x_1=\cdots x_t=\alpha} = (1 - \alpha)^{-t}(1 - \alpha^2)^{-\frac{1}{2}t(t-1)}, \quad (6.4)$$

and

$$\begin{aligned} \sum_{n(-t)\ge 0} \cdots \sum_{n(t-2)\ge 0} F_t(n(-t), \cdots, n(t); q) &= \prod_{i=1}^{t}(1 - q^{t-i+1})^{-1} \prod_{1\le i<j\le t}(1 - q^{2t-i-j+2})^{-1} \\ &= \prod_{1\le i\le j\le t}(1 - q^{i+j-1})^{-1}. \quad (6.5) \end{aligned}$$

It should be noted that, when $0 < x_i < 1$ for $1 \le i \le t$ and $0 < q < 1$, (6.3) and (6.5) are obtained directly from Theorem 4.2 (Macdonald formula) and (i) of Corollary 4.3 (the Bender-Knuth formula) by taking the limit $n \to \infty$, respectively

7 Determinantal Expressions with Orthogonal Polynomials

7.1 Gaussian Ensembles of Random Matrices and Hermite Polynomials

Consider the Gaussian orthogonal ensemble of $N \times N$ random matrices ($\beta = 1$ case). The distribution function of eigenvalues $\{y_i\}_{i=1}^N$ is described by

$$P_{N1}(y_1, \cdots, y_N) = c_1 \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^N y_i^2} \prod_{1 \le i < j \le N} |y_i - y_j|, \tag{7.1}$$

where c_1 is a normalized constant to be determined so that $\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N P_{N1}(y_1, \cdots, y_N) = 1$.

A standard way in the random matrix theory found in Chapter 5 of [21] or Chapter 5 of [8] is started by giving a determinantal expression to the distribution function, whose matrix elements are given by orthogonal polynomials. The argument is the following. By the Vandermonde determinant (2.5),

$$\begin{aligned} \prod_{1 \le i < j \le N} (y_i - y_j) &= \begin{vmatrix} y_1^{N-1} & y_1^{N-2} & \cdots & 1 \\ y_2^{N-1} & y_2^{N-2} & \cdots & 1 \\ & \cdots & \cdots & \\ y_N^{N-1} & y_N^{N-2} & \cdots & 1 \end{vmatrix} \\ &= (-1)^{\frac{1}{2}N(N-1)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_N \\ & \cdots & \cdots & \\ y_1^{N-1} & y_2^{N-1} & \cdots & y_N^{N-1} \end{vmatrix}. \end{aligned} \tag{7.2}$$

Let $H_i(y)$ be the Hermite polynomial of order i defined by

$$H_i(y) = \mathrm{e}^{y^2} \left(-\frac{d}{dy} \right)^i \mathrm{e}^{-y^2} = i! \sum_{k=0}^{[i/2]} (-1)^k \frac{(2y)^{i-2k}}{k!(i-2k)!}.$$

In the last expression of (7.2) multiply the i-th row of the matrix by 2^{i-1} for $1 \le i \le t$ and then add to each row an appropriate linear combination of the other rows with lower power of y's, so that we have

$$\prod_{1 \le i < j \le N} (y_i - y_j) = \text{const.} \times \det(H_{i-1}(y_j))_{1 \le i,j \le N}.$$

We further multiply the i-th row of the matrix by a factor $(2^{i-1}(i-1)!\sqrt{\pi})^{-1/2}$, $1 \le i \le t$, and the j-th column by $\mathrm{e}^{-x_j^2/2}$, $1 \le j \le t$, and we obtain

$$\mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{N} y_i^2} \prod_{1\le i<j\le N} (y_i - y_j) = \text{const.} \times \det(\varphi_{i-1}(y_j))_{1\le i,j\le N},$$

where

$$\varphi_i(y) = \left(2^i i! \sqrt{\pi}\right)^{-1/2} \mathrm{e}^{-\frac{1}{2}y^2} H_i(y).$$

The polynomials $\varphi_i(y), i = 0, 1, 2, \cdots$, are orthogonal over $(-\infty, \infty)$ as

$$\int_{-\infty}^{\infty} \varphi_i(y)\varphi_j(y)dy = \delta_{i,j} \quad \text{for} \quad i, j = 0, 1, 2, \cdots,$$

where $\delta_{i,j}$ is the Kronecker delta. Then we have the following determinantal expression using the orthogonal polynomials for (7.1),

$$P_{N1}(y_1, \cdots, y_N) = c_2 \det(\varphi_{i-1}(y_j))_{1\le i,\le N},$$
$$\text{under the condition that } -\infty < y_1 \le y_2 \le \cdots \le y_N < \infty,$$

where c_2 is a constant to be determined so that

$$c_2 N! \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \mathbf{1}(y_1 \le y_2 \le \cdots \le y_N) \det(\varphi_{i-1}(y_j))_{1\le i,j\le N} = 1.$$

7.2 Anisotropic-Flow Polynomials and Meixner Polynomials

Let $0 < \alpha < 1$ and define the polynomials of α for each given set $\{n(-t+2k)\}_{k=0}^{t}$.

Definition 7.1

$$F_t^{\alpha}(n(-t), n(-t+2), \cdots, n(t)) = F_t(n(-t), n(-t+2), \cdots, n(t); x_1, \cdots, x_t)\big|_{x_1=\cdots=x_t=\alpha}.$$

By the definition of $F_t(n(-t), \cdots, n(t); x_1, \cdots, x_t)$, it is easy to see that the power of α of each term represents the number of arcs $\{a\} \subset A_t$, which are oriented up-left-wards and support a strictly positive flow, $\phi(a) > 0$. Then the variable α measures the left-right anisotropy of flow and we call the polynomial F_t^{α} the anisotropic-flow polynomials here See [15] and [2] for anisotropic vicious walker models.

Put $x_i = \alpha y_i, 1 \le i \le t$, in (2.6). Then we set $y_i = q^{t-i+1}$ and take the $q \to 1$ limit to have

$$F_t^{\alpha}(n(-t), \cdots, n(t)) = \frac{\alpha^{\sum_{i=1}^{t}(\lambda_i+t-i)}}{\alpha^{\frac{1}{2}t(t-1)}} \prod_{1\le i<j\le t} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Define

$$h_i = \lambda_i + t - i, \quad 1 \le i \le t, \tag{7.3}$$

and by using the Vandermonde determinant (2.5) we rewrite the equality as

$$\begin{aligned} F_t^\alpha(n(-t),\cdots,n(t)) &= \frac{\alpha^{\sum_{i=1}^t h_i}}{\alpha^{\frac{1}{2}t(t-1)}\prod_{i=1}^{t-1} i!} \times \prod_{1\le i<j\le t}(h_i-h_j) \\ &= \frac{(-1)^{\frac{1}{2}t(t-1)}}{\alpha^{\frac{1}{2}t(t-1)}\prod_{i=1}^{t-1} i!} \times \begin{vmatrix} \alpha^{h_1} & \alpha^{h_2} & \cdots & \alpha^{h_t} \\ h_1\alpha^{h_1} & h_2\alpha^{h_2} & \cdots & h_t\alpha^{h_t} \\ & \cdots & \cdots & \\ h_1^{t-1}\alpha^{h_1} & h_2^{t-1}\alpha^{h_2} & \cdots & h_t^{t-1}\alpha^{h_t} \end{vmatrix}. \end{aligned} \tag{7.4}$$

Now we define the polynomials $M_i(h;\alpha), i = 0,1,2,\cdots,$ by

$$M_i(h;\alpha) = F\left({}^{-i,-h}_{\ \ 1}|1-\frac{1}{\alpha^2}\right), \tag{7.5}$$

where $F\left({}^{a_1,a_2}_{\ \ b}|z\right)$ is the hypergeometric series defined by

$$F\left({}^{a_1,a_2}_{\ \ b}|z\right) = \sum_{k\ge 0}\frac{(a_1)_k(a_2)_k}{(b)_k}\frac{z^k}{k!}.$$

They are the special case of the Meixner polynomials $M_n(x;\beta,c)$ [19, 17]; $M_i(h;\alpha) = M_i(h;1,\alpha^2)$. We found that $M_i(h;\alpha)$ has the generating function

$$\left(1-\frac{z}{\alpha^2}\right)(1-z)^{-h-1} = \sum_{i=0}^{\infty} M_i(h;\alpha)z^i$$

and satisfy the orthogonality over the discrete variable $h = 0,1,2,\cdots$,

$$\sum_{h=0}^{\infty}\alpha^{2h}M_i(h;\alpha)M_j(h;\alpha) = \frac{1}{\alpha^{2i}(1-\alpha)}\delta_{i,j}.$$

The normalized discrete orthogonal polynomials $\{m_i(h;\alpha)\}$ are then defined as

$$m_i(h;\alpha) = \sqrt{1-\alpha^2}\ \alpha^i\ M_i(h)\alpha^h, \tag{7.6}$$

such that

$$\sqrt{1-\alpha^2}(\alpha-z)^h(1-\alpha z)^{-h-1} = \sum_{i=0}^{\infty} m_i(h;\alpha)z^i$$

and

$$\sum_{h=0}^{\infty} m_i(h;\alpha)m_j(h;\alpha) = \delta_{i,j} \quad \text{for} \quad i,j = 0,1,2,\cdots,$$

First few polynomials are given as follows,

$$m_0(h;\alpha) = \sqrt{1-\alpha^2}\alpha^h,$$

$$
\begin{aligned}
m_1(h;\alpha) &= \frac{\sqrt{1-\alpha^2}}{\alpha}\alpha^h\{(\alpha^2-1)h+\alpha^2\},\\
m_2(h;\alpha) &= \frac{\sqrt{1-\alpha^2}}{2\alpha^2}\alpha^h\{(\alpha^2-1)^2h^2+(\alpha^2-1)(3\alpha^2+1)h+2\alpha^4\},\\
m_3(h;\alpha) &= \frac{\sqrt{1-\alpha^2}}{6\alpha^3}\alpha^h\{(\alpha^2-1)^3h^3+3(\alpha^2-1)^2(2\alpha^2+1)h^2\\
&\qquad +(\alpha^2-1)(11\alpha^4+5\alpha^2+2)h+6\alpha^6\},\cdots.
\end{aligned}
$$

By definition (7.5) we find that

$$m_i(h;\alpha) = A_i(\alpha)\alpha^h\{h^i+\mathcal{O}(h^{i-1})\} \qquad \text{for} \quad i=1,2,\cdots$$

with

$$A_i(\alpha) = \frac{(-1)^i(1-\alpha^2)^{i+\frac{1}{2}}}{i! \quad \alpha^i}.$$

Then in the matrix in (7.4) we multiply the i-th row by a factor $A_{i-1}(\alpha)$ for $1 \le i \le t$ and after that we add to each row an appropriate linear combination of the other rows with lower power of h, so that

$$F_t^\alpha(n(-t),\cdots,n(t)) = \frac{(-1)^{\frac{1}{2}t(t-1)}}{\alpha^{\frac{1}{2}t(t-1)}\prod_{i=1}^{t} i!} \times \frac{\det(m_{i-1}(h_j;\alpha))_{1\le i,j\le t}}{\prod_{i=1}^{t-1} A_i(\alpha)}.$$

It is easy to see

$$\prod_{i=1}^{t-1} A_i(\alpha) = \frac{(-1)^{\frac{1}{2}t(t-1)}(1-\alpha^2)^{\frac{1}{2}t^2}}{\alpha^{\frac{1}{2}t(t-1)}\prod_{i=1}^{t-1} i!}.$$

Then we have the following equality.

Lemma 7.2 *Let $m_i(h;\alpha)$ be the discrete orthogonal polynomials defined by (7.6) using the special case of the Meixner polynomials (7.5). Then*

$$F_t^\alpha(n(-t),\cdots,n(t)) = (1-\alpha^2)^{-\frac{1}{2}t^2} \ \det(m_{i-1}(h_j;\alpha))_{1\le i,j\le t},$$

where $h_i = \sum_{k=0}^{t-i} n(-t+2k)+t-i$, $i=1,2,\cdots,t$.

Combining this relation with (6.4) in Lemma 6.7 gives the following summation formula of determinants.

Proposition 7.3 *For $t \in \mathbf{N}, 0<\alpha<1$*

$$\sum_{0\le h_t<h_{t-1}<\cdots<h_1} \det(m_{i-1}(h_j;\alpha))_{1\le i,j\le t} = \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{t}{2}}.$$

7.3 q-Flow Polynomials and Little q-Jacobi Polynomials

By using the Vandermonde determinant (2.5) the formula given by Corollary 2.7 is rewritten as

$$\begin{aligned} F_t(n(-t),\cdots,n(t);q) &= \frac{q^{\sum_{i=1}^t \lambda_i}}{\prod\limits_{1\le i<j\le t}(q^{t-i}-q^{t-j})}\prod_{1\le i<j\le t}(q^{\lambda_i+t-i}-q^{\lambda_j+t-j}) \\ &= \frac{1}{q^{\frac{1}{6}(t-1)t(t+1)}\prod\limits_{i=1}^{t-1}(1-q^i)^{t-i}} \times \begin{vmatrix} q^{h_1} & q^{h_2} & \cdots & q^{h_t} \\ q^{2h_1} & q^{2h_2} & \cdots & q^{2h_t} \\ & \cdots & \cdots & \\ q^{th_1} & q^{th_2} & \cdots & q^{th_t} \end{vmatrix}, \end{aligned} \tag{7.7}$$

where h_i's are defined as (7.3).

We consider a special case of the little q-Jacobi polynomials [19], which can be regarded as a q-analogue of the Jacobi polynomials,

$$p_i(x) = \phi\left({}^{q^{-i},q^{i+2}}_{\quad q^2}|q;qx\right), \tag{7.8}$$

where

$$\phi\left({}^{a_1,a_2}_{\ b}|q;z\right) = \sum_{k\ge 0}\frac{(a_1;q)_k(a_2;q)_k}{(b;q)_k}\frac{z^k}{(q;q)_k}$$

is the q-hypergeometric series with the q-analogue of the Pochhammer-symbol $(a;q)_k$; $(a;q)_0 = 1$ and $(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1})$, $k = 1,2,3,\cdots$. The first few polynomials are given as

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= 1-\frac{1-q^3}{1-q^2}x, \\ p_2(x) &= 1-\frac{1-q^4}{q(1-q)}x+\frac{q(1-q^4)(1-q^5)}{(1-q^2)(1-q^3)}x^2, \\ p_3(x) &= 1-\frac{(1-q^3)(1-q^5)}{q^2(1-q)(1-q^2)}x+\frac{(1-q^5)(1-q^6)}{q^3(1-q)(1-q^2)}x^2 \\ &\quad -\frac{(1-q^5)(1-q^6)(1-q^7)}{q^3(1-q^2)(1-q^3)(1-q^4)}x^3, \cdots. \end{aligned}$$

The orthogonality is

$$\sum_{h=0}^{\infty} q^{2h}p_i(q^h)p_j(q^h) = \frac{q^{2i}(1-q)^2}{(1-q^{2i+2})(1-q^{i+1})^2}\delta_{i,j} \quad \text{for} \quad i,j = 0,1,2,\cdots.$$

We define the normalized polynomials as

$$\psi_i(h;q) = \frac{\sqrt{1-q^{2i+2}}(1-q^{i+1})}{q^i(1-q)} \times q^h p_i(q^h), \tag{7.9}$$

so that

$$\sum_{h=0}^{\infty} \psi_i(h;q)\psi_j(h;q) = \delta_{i,j} \quad \text{for} \quad i,j = 0,1,2,\cdots.$$

Let

$$\begin{aligned} B_i(q) &= \frac{\sqrt{1-q^{2i+2}}(1-q^{i+1})}{q^i(1-q)} \times \frac{(q^{-i};q)_i(q^{i+2};q)_i}{(q^2;q)_i(q;q)_i} \\ &= (-1)^i(1-q^{2i+2})^{1/2} \prod_{j=1}^{i} \frac{1-q^{i+j+1}}{q^j(1-q^j)}. \end{aligned}$$

Then by definition

$$\psi_i(h;q) = B_i(q)z^{i+1} + \mathcal{O}(z^i), \quad i = 0,1,2,\cdots,$$

as a power series of $z = q^h$. The same argument on the matrix in (7.7) as in the previous subsections and the equality

$$\prod_{i=1}^{t-1} B_i(q) = \frac{(-1)^{\frac{1}{2}t(t-1)} \prod_{i=1}^{t} \left(\frac{1+q^i}{1-q^i}\right)^{1/2} \prod_{1\le i\le j\le t}(1-q^{i+j-1})}{q^{\frac{1}{6}(t-1)t(t+1)} \prod_{i=1}^{t-1}(1-q^i)^{t-i}}$$

lead to the following identity.

Lemma 7.4 *Let $\psi_i(h;q)$ be the discrete q-orthogonal polynomials defined by (7.9) using the special case of the little q-Jacobi polynomials (7.8). Then*

$$F_t(n(-t),\cdots,n(t);q) = (-1)^{\frac{1}{2}t(t-1)} \prod_{i=1}^{t} \left(\frac{1-q^i}{1+q^i}\right)^{\frac{1}{2}} \prod_{1\le i\le j\le t}(1-q^{i+j-1})^{-1} \times \det(\psi_{i-1}(h_j;q))_{1\le i,j\le t}$$

where $h_i = \sum_{k=0}^{t-i} n(-t+2k) + t - i$, $i = 1,2,\cdots,t$.

This identity and (6.5) in Lemma 6.7 give the summation formula.

Proposition 7.5 *For $t \in \mathbf{N}, 0 < q < 1$*

$$\sum_{0\le h_t<h_{t-1}<\cdots<h_1} \det(\psi_{i-1}(h_j;q))_{1\le i,j\le t} = (-1)^{\frac{1}{2}t(t-1)} \prod_{i=1}^{t} \left(\frac{1+q^i}{1-q^i}\right)^{\frac{1}{2}}.$$

8 Determinantal Probability Measures for Flow Polynomials

Here we fix t, that is, fix the directed graph $H_t = (V_t, A_t)$. For each set of strengths of sinks, $\{n(-t+2k)\}_{k=0}^{t}$, flow polynomials $F_t^{\alpha}(n(-t),\cdots,n(t))$ and $F_t(n(-t),\cdots,n(t);q)$ are uniquely determined as polynomials of α and q, respectively. Now we consider ensembles of these polynomials by allowing $n(-t+2k)$'s to take arbitrary nonnegative integers and introduce the following probability measures.

Definition 8.1 *For $t \in \mathbf{N}$ and $n \in \mathbf{Z}^+$ consider the state space $X_t = (\mathbf{Z}^+)^t$. For $0 < \alpha, q < 1$ the probability measures on X_t are defined as*

$$\mu_t^\alpha(n_1, \cdots, n_t) = \frac{F_t^\alpha(n_1, \cdots, n_{t+1})}{Z(\mu_t^\alpha)}$$

and

$$\nu_t^q(n_1, \cdots, n_t) = \frac{F_t(n_1, \cdots, n_{t+1}; q)}{Z(\nu_t^q)}$$

with

$$Z(\mu_t^\alpha) \equiv \sum_{(n_1,\cdots,n_t)\in X_t} F_t^\alpha(n_1, \cdots, n_t) = (1-\alpha)^{-t}(1-\alpha^2)^{-\frac{1}{2}t(t-1)}$$

and with

$$Z(\nu_t^q) \equiv \sum_{(n_1,\cdots,n_t)\in X_t} F_t(n_1, \cdots, n_t; q) = \prod_{1\le i\le j\le t} (1-q^{i+j-1})^{-1}.$$

We can also define the following determinantal probability measures for strictly decreasing finite series of nonnegative integers.

Definition 8.2 *For $t \in \mathbf{N}$ consider the state space $X_t^- = \{(h_1, h_2, \cdots, h_t) \in (\mathbf{Z}^+)^t : h_1 > h_2 > \cdots > h_t\}$. For $0 < \alpha, q < 1$ the probability measures on X_t^- are defined as*

$$\hat{\mu}_t^\alpha(h_1, \cdots, h_t) = \frac{\det(m_{i-1}(h_j; \alpha))_{1\le i,j\le t}}{Z(\hat{\mu}_t^\alpha)}$$

and

$$\hat{\nu}_t^q(h_1, \cdots, h_t) = \frac{\det(\psi_{i-1}(h_j; q))_{1\le i,j\le t}}{Z(\hat{\nu}_t^q)}$$

with

$$Z(\hat{\mu}_t^\alpha) \equiv \sum_{0\le h_t<h_{t-1}<\cdots<h_1} \det(m_{i-1}(h_j;\alpha))_{1\le i,j\le t} = \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{t}{2}}$$

and with

$$Z(\hat{\nu}_t^q) \equiv \sum_{0\le h_t<h_{t-1}<\cdots<h_1} \det(\psi_{i-1}(h_j;\alpha))_{1\le i,j\le t} = (-1)^{\frac{1}{2}t(t-1)} \prod_{i=1}^t \left(\frac{1+q^i}{1-q^i}\right)^{\frac{1}{2}},$$

where $m_i(h;\alpha)$ and $\psi_i(h;q)$ are the discrete orthogonal polynomials defined by (7.6) and (7.9), respectively.

The expectations and probabilities are defined as usual on the probability measures as

$$\begin{aligned} E_{\mathcal{M}}(\,\cdot\,) &= \sum_{(m_1,\cdots,m_t)\in Y} \mu_{\mathcal{M}}(m_1,\cdots,m_t)(\,\cdot\,), \\ P_{\mathcal{M}}(\,\cdot\,) &= E_{\mathcal{M}}(\mathbf{1}(\,\cdot\,)) \end{aligned}$$

for each probability measure $\mathcal{M}$ on Y. Lemmas 7.2 and 7.4 state that, if $h_i = \sum_{j=1}^{t-i+1} n_j + t - i$ with $i = 1, 2, \cdots, t$, then

$$\mu_t^\alpha(n_1,\cdots,n_t) = \hat{\mu}_t^\alpha(h_1,\cdots,h_t)$$

and

$$\nu_t^q(n_1,\cdots,n_t) = \hat{\nu}_t^q(h_1,\cdots,h_t).$$

As corollaries of the Macdonald formula (Theorem 4.2), the following probability laws on the measures are concluded.

Corollary 8.3 *Fix* $t \in \mathbf{N}$ *and assume* $0 < \alpha, q < 1$. *For* $n \in \mathbf{Z}^+$

$$\begin{aligned} P_{\mu_t^\alpha}\left(\sum_{i=1}^t n_i \le n\right) &= P_{\hat{\mu}_t^\alpha}(h_1 \le t+n-1) \\ &= (1-\alpha)^t(1-\alpha^2)^{\frac{1}{2}t(t-1)} \times \left.\frac{\det(x_i^{j-1} - x_i^{n+2t-j})_{1\le i,j\le t}}{\det(x_i^{j-1} - x_i^{2t-j})_{1\le i,j\le t}}\right|_{x_1=\cdots=x_t=\alpha} \end{aligned}$$

and

$$P_{\nu_t^q}\left(\sum_{i=1}^t n_i \le n\right) = P_{\hat{\nu}_t^q}(h_1 \le t+n-1) = \prod_{1\le i\le j\le t}(1-q^{n+i+j-1}).$$

9 Concluding Remarks

9.1 Stembridge's Pfaffian and Macdonald's Formula

The Macdonald formula is the summation formula of the Schur functions. In Section 6 we reviewed Okada's proof in which each Schur function is represented by the ratio of two $t \times t$ determinant (Jacobi-Trudi formula) and the summation is performed as a minor-summation of $t \times (n+t)$ matrix. The result is given by a $t \times t$ Pfaffian with a prefactor.

As shown in Section 3, the Gessel-Viennot theorem gives the $n \times n$ determinantal expression for the Schur functions. Stembridge gave a combinatorial proof of the fact that the summation of such $n \times n$ determinants is expressed by an $n \times n$ Pfaffian [27]. In [27] Stembridge also gave another proof of the Macdonald formula (Formula 1.3) by showing the equivalence between his Pfaffian and RHS of Formula 1.3.

9.2 Nagao-Forrester's Quaternion Determinant and Conjugate of Arrowsmith-Mason-Essam's Formula

Corollaries 2.7 and 2.8 give the product formulae to the flow polynomials in the form $\prod_{1\le i<j\le t} f_{ij}$. In Section 7 following the standard argument of the random matrix theory we rewrote these product formulae by using the $t \times t$ determinants with the elements described by discrete orthogonal polynomials. We then defined probability measures, which can be regarded as the discrete analogues of the Gaussian orthogonal ensemble of random matrices. We showed in Section 8 that the Macdonald formula proves the probability laws on these measures.

There will be two problems concerning the results. The first one is whether we can derive these probability laws (Corollary 8.3) by using the technique of the random matrix theory [21, 8]. This question is asking the possibility of constructing new proofs of the Arrowsmith-Essam-Mason, the Bender-Knuth, and the Macdonald formulae.

The second problem is about the conjugate expressions. In Section 3 we have shown that the Gessel-Viennot theorem gives the $n \times n$ determinantal expressions and then Essam-Guttmann's expression for the flow polynomial is the product form of $\prod_{1\le i<j\le n} g_{ij}$. This conjugate expression implies the following formula

$$\sum_{0\le\lambda'_n\le\lambda'_{n-1}\le\cdots\le\lambda'_1\le t} \prod_{i}^{n} \frac{(t+n-i)!}{(\lambda'_i+n-i)!(t-\lambda'_i+i-1)!} \prod_{1\le i<j\le n} (\lambda'_i-\lambda'_j+j-i)$$
$$= \prod_{1\le i\le j\le t} \frac{n+i+j-1}{i+j-1}, \tag{9.1}$$

which we can call the conjugate Arrowsmith-Essam-Mason formula. The question is whether we can prove (9.1) independently of the proof of the original Arrowsmith-Mason-Essam formula (Formula 1.1).

We find in a quite recent paper by Nagao and Forrester [22] that the second problem was solved by using the technique of random matrix theory. That means, they also gave a partial answer to our first problem mentioned above. Nagao and Forrester gave such a quaternion determinantal expression to the Essam-Guttmann formula (3.5) that the summation of LHS of (9.1) can be performed explicitly. It should be noted that as mentioned on page 127 of [21] their quaternion determinant should be equal to some $n \times n$ Pfaffian. To clarify the relationship among Okada's Pfaffian, Stembridge's Pfaffian and Nagao-Forrester's quaternion determinant will be an interesting future problem.

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References

[1] D. K. Arrowsmith, P. Mason and J. W. Essam, Vicious walkers, flows and directed percolation, *Physica* **A 177** (1991) 267-272.

[2] J. Baik, Random vicious walks and random matrices, *Commun. Pure and Appl. Math.* **53** (2000) 1385-1410.

[3] J. Baik, Lecture notes: Longest increasing subsequences and random growth models, http://www.math.princeton.edu/ jbaik/.

[4] R. Brak, J. W. Essam and A. L. Owczarek, From the Bethe ansatz to the Gessel-Viennot theorem, *Ann. Combin.* **3** (1999) 251-263.

[5] D. M. Bressoud, Elementary proof of MacMahon's conjecture, *J. Alg. Combin.* **7** (1998) 253-257.

[6] E. A. Bender and D. E. Knuth, Enumeration of plane partitions, *J. Combin. Theory* Ser. **A 13** (1972) 40-54.

[7] J. Cardy and F. Colaiori, Directed percolation and generalized friendly random walkers, *Phys. Rev. Lett.* **82** (1999) 2232-2235.

[8] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, American Math. Society, Providence, 2000.

[9] J. W. Essam and A. J. Guttmann, Vicious walkers and directed polymer networks in general dimensions, *Phys. Rev.* **E 52** (1992) 5849-5862.

[10] M. E. Fisher Walks, walls, wetting, and melting, *J. Stat. Phys.* **34** (1984) 667-729.

[11] W. Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry*, Cambridge Univ. Press, Cambridge, 1997.

[12] B. Gordon, A proof of the Bender-Knuth conjecture, *Pacific J. Math.* **108** (1983) 99-113.

[13] A. J. Guttmann, A. L. Owczarek and X. G. Viennot, Vicious walkers and Young tableaux I: Without walls, *J. Phys. A: Math. Gen.* **31** (1998) 8123-8135.

[14] I. M. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985) 300-321.

[15] A. J. Guttmann and M. Vöge, Lattice paths: vicious walkers and friendly walkers, to appear in *J. Stat. Interface and Planning.*

[16] N. Inui and M. Katori, Fermi partition functions of friendly walkers and pair connectedness of directed percolation, *J. Phys. Soc. Jpn.* **70** (2001) 1-4.

[17] K. Johansson, Shape fluctuations and random matrices, *Commun. Math. Phys.* **209** (2000) 437-476.

[18] C. Krattenthaler, A. J. Guttmann and X. G. Viennot, Vicious walkers, friendly walkers and Young tableaux: II. With a wall, *J. Phys. A: Math. Gen.* **33** (2000) 8835-8866.

[19] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, LANL e-print Archive math.CA/9602214; http://xxx.lanl.gov/.

[20] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Univ. Press, London, 1995.

[21] M. L. Mehta, *Random Matrices*, second edition, Academic Press, London, 1991.

[22] T. Nagao and P. J. Forrester, Vicious random walkers and a discretization of Gaussian random matrix ensembles, LANL e-print Archive cond-mat/0107221; http://xxx.lanl.gov/.

[23] S. Okada, On the generating functions for certain classes of plane partitions, *J. Combin. Theory* Ser. **A 51** (1989) 1-23.

[24] S. Okada, Applications of minor summation formulas to rectangular-shaped representations of classical groups, *J. Algebra* **205** (1998) 337-367.

[25] R. A. Proctor, Equivalence of the combinatorial and the classical definitions of Schur functions, *J. Combin. Theory,* Ser. **A 51** (1989) 135-137.

[26] R. P. Stanley, *Enumerative Combinatorics*, vol.2, Cambridge Univ. Press, Cambridge, 1999.

[27] J. R. Stembridge, Nonintersecting paths, pfaffians, and the plane partitions, *Adv. in Math.* **83** (1990) 96-131.

[28] T. Tsuchiya and M. Katori, Chiral Potts models, friendly walkers and directed percolation problem, *J. Phys. Soc. Jpn.* **67** (1998) 1655-1666.