Generating functions and duality for non-crossing walks on a plane graph.

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Abstract

The generating function for the number of non-crossing walk configurations of n walks between the roots of a two-rooted directed plane graph is introduced. This is shown to be a rational function and the structure and symmetry property of its numerator are discussed. The walk configurations correspond to flows and the equivalent dual generating function for potentials is investigated independently. Also equivalences are drawn with the partially order sets that can be constructed from the walk configurations. Finally, the general results developed here are applied to the directed square and honeycomb lattices.

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1 Introduction

1.1 Background and definitions.

Consider a two-rooted directed plane graph H with source u and sink v. It will be assumed that H has no directed cycles and that every arc belongs to at least one directed path between the roots: such graphs will be said to be *coverable* (by walks).

We define the generating function

$$G^{Bose}(H;z) \equiv \sum_{n=0}^{\infty} f_n^{Bose}(H) z^n \tag{1.1}$$

where $f_n^{Bose}(H)$ is the number of non-crossing configurations of n walks between the roots. It will turn out that $f_0^{Bose}(H) = 1$. The superscript *Bose* is to distinguish the configurations from Fermi configurations for which a given path may be used by at most one walk. The terminology was introduced in previous work by Inui and Katori [11].

Bose configurations may be generated by replacing each walk in a Fermi configuration by $1, 2, 3, \ldots$ walks using the same path. This is implemented by replacing z by $\frac{z}{1-z}$ in the Fermi generating function.

$$G^{Bose}(H;z) = G^{Fermi}(H; \frac{z}{1-z}) \tag{1.2}$$

Let n_{max} be the number of walks in a maximal Fermi configuration so that $G^{Fermi}(H;z)$ is a polynomial of degree n_{max} then collecting the terms in the expansion of the right hand side of (1.2) over a common denominator we see that $G^{Bose}(H;z)$ is a rational function

$$G^{Bose}(H;z) = \frac{Q^{Bose}(H;z)}{(1-z)^{n_{max}}} \tag{1.3}$$

where

$$Q^{Bose}(H;z) = \sum_{n=0}^{n_{max}} z^n (1-z)^{n_{max}-n} f_n^{Fermi}(H). \tag{1.4}$$

An example of such rationality occurred in the earlier work of Guttmann and Vöge [10].

The union of paths between two vertices of the directed square lattice with various boundary conditions leads to a number of different examples of H for which $f_n^{Bose}(H)$ is known using a bijection [5] to vicious walker configurations [7]. These results are summarised with references to the vicious walker enumerations in a previous paper [1]. Substitution in [1.3] leads to the results tabulated in Appendix A. For example if H is the union of all paths from the origin to $(3, 4)$ which we denote $W_{3,4}$

$$Q^{Bose}(W_{3,4};z) = 1 + 22z + 113z^2 + 190z^3 + 113z^4 + 22z^5 + z^6 \tag{1.5}$$

In general we denote the degree of $Q^{Bose}(H;z)$ by k_{max}. Equation (1.4) suggests that $k_{max} = n_{max}$ but for this example $n_{max} = 13$ so $k_{max} < n_{max}$. This is true of all the polynomials listed in the appendix and arises from the cancellation of the higher terms in (1.4). The polynomials are also seen to have symmetry. Here we determine for a general graph H under what conditions this symmetry occurs and obtain a formula k_{max}.

The graph in Figure 1 (a) has m parallel arcs and in a Fermi configuration each one of them may be used by at most one walk hence $f_n^{Fermi}(H_1) = \binom{m}{n}$. Carrying out the defining

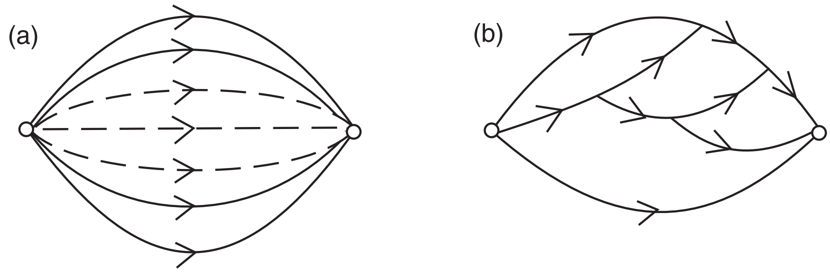

Figure 1: *Examples of coverable plane graphs (a)* H_1 *with* m *arcs and (b)* H_2*, with the same* $G^{Bose}(H_i; z)$ *and* $G^{Fermi}(H_i; z)$ *generating functions for* $i = 1, 2$ *when* $m = 5$.

sum (1.1) $G^{Fermi}(H_1; z) = (1+z)^m$ and using (1.2) $G^{Bose}(H_1; z) = (1+\frac{z}{1-z})^m = (1-z)^{-m}$. For future reference note that each arc contributes a factor $1 + z$ to $G^{Fermi}(H_1; z)$ and a factor $1/(1 - z)$ to $G^{Bose}(H_1; z)$. Clearly $n_{max} = m$ in this example, $Q^{Bose}(H_1; z)$ has degree zero and is identically 1, (cf. eqn. (1.3)) the number of 0-walk configurations is equal to 1 in agreement with our general convention. In general it is clear that for graphs in parallel the generating functions factorise whereas for graphs in series it is the number of configurations which factorises.

The graph H_2 in Figure 1 (b) has walk configurations identical to that of the parallel graph in Figure 1 (a) for the case $m = 5$. This is because a third degree vertex with in-degree one can always be replaced by two parallel paths. This will be used in relating the generating functions for the honeycomb lattice to those for the square lattice.

1.2 Main results.

Setting $z = 1$ in (1.4), $Q^{Bose}(H; 1) = f^{Fermi}_{n_{max}}(H)$, that is the sum of the numerator coefficients is equal to the number of maximal Fermi walk configurations. Proposition 1 (a) states that the coefficient of z^k in $Q^{Bose}(H; z)$ is the number of maximal Fermi walk configurations with k *flopped walks* (section 2.2).

In subsection 2.1 the construction of the directed dual graph H^* of H is described. This is used in the following section to define Young labellings of the faces of H. These labellings are in one-one correspondence with maximal Fermi walk configurations and flopped walks correspond to ascents in the labelling. Young labellings of a square lattice graph are the same as Young tableaux.

Proposition 1 (b) relates the generating function for Bose configurations to that for non-crossing configurations which cover all the arcs of H. From this relation it follows that the degree of $Q^{Bose}(H; z)$ is related to the number of walks n_{min} in a minimum cover by $k_{max} = n_{max} - n_{min}$.

In section3 use is made of the well-known flow-potential duality (recalled in section 3.1) to obtain an alternative proof of proposition 1. Young labellings are transferred from the faces of H to the vertices of H^*. Taking the labelling with no ascents as standard allows the definition of *reversals* in the case of other labellings. Potentials are enumerated by partitioning them according to the number of reversals which plays the same role as the number of flopped paths in Bose walk or flow enumeration.

In section 3.3 we note that equations (3.3) and (3.4) which arise in the alternative proof of proposition 1 are examples of theorem 4.5.14 of Stanley [13]. We also note the parallels of flows and potentials on graphs in terms of fundmental constructs in the theory of partially ordered sets [13]. The duality between flows and potentials which is well-known in physics becomes a relation between the order polynomial and the zeta function.

Finally section 4 is concerned with applications to the square lattice. We give an example of the use of ascents in shifted Young tableaux to obtain the Bose generating function for star configurations. Also we give examples of optimal covers for certain classes of subgraph for which an independent derivation of the formula for k_{max}. The value of k_{max} is given for general members of these classes.

2 Bose and Fermi generating functions for coverable plane graphs.

2.1 The directed dual of a two-rooted plane graph.

In the following sections we need the directed dual graph H^* of H.

Definition 1. *The directed anticlockwise inner dual graph $\breve{H}^*$ of the plane graph H has a vertex in each face of H and for each arc (w, x) of H, we have a dual arc (y, z) in $\breve{H}^*$ that crosses (w, x) from right to left and joins the vertices of adjacent faces of H. The dual graph H^* has two vertices in the infinite face, $u^*(v^*)$ connects to vertices of $\breve{H}^*$ in faces adjacent to the right-hand (left-hand) bounding path.*

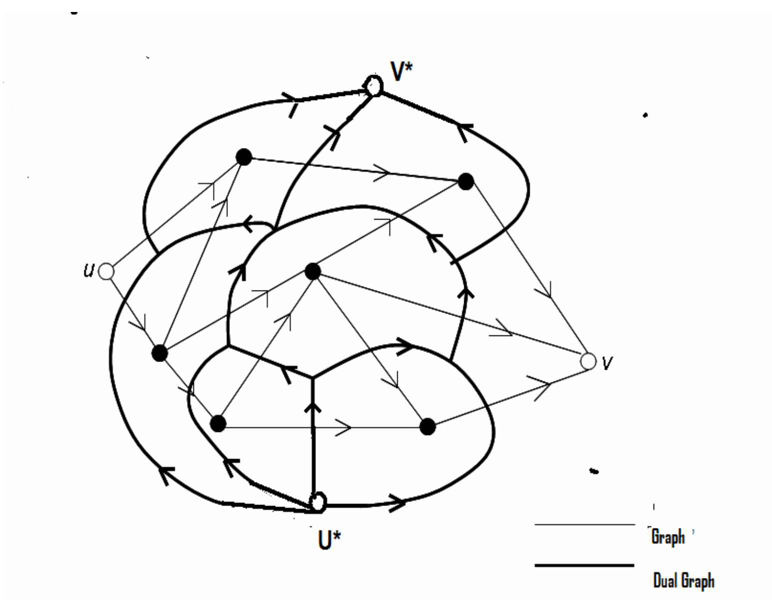

Figure 2: *A two rooted graph and its anti-clockwise dual.*

Notes:

1. Since H is coverable it is bounded by two paths directed from root u to root v. Consequently one of the two vertices of H^* not belonging to a finite face of H is a source u^* and the other v^* is a sink.

2. H^* has no other sources or sinks as these would correspond to cycles on H. H^* is coverable by paths directed from u^* to v^*.

In order to state our main proposition we need to introduce the following terms.

2.2 Young labellings, flipflops and ascents.

Maximal Fermi configurations may be constructed as follows.

(i) Label 0 the walk following the rightmost path leaving the the source.

(ii) Having contructed walks labelled $1, 2, \ldots, i-1$ choose a face bordered on the right by walk $i-1$ such that the walk may be diverted to the left of the face and that there is just one less face to the left of walk $i-1$. Label the walk i obtained by diverting walk $i-1$ in this way.

(iii) Continue until there are no more faces to the left of the last path.

This process gives the various maximal Fermi walk configurations by using the choice of sequenced faces allowed by rule (ii) above. An example of maximal Fermi walk configurations is shown in figure 3.

One way of describing the maximal walk configurations is to define the *standard configuration* as the one obtained by imposing the further condition

(ii+) the face which is chosen to be enclosed by walks $i-1$ and i is the first adjacent face available as the walk $i-1$ is traversed.

Now attach label i to the face chosen in constructing the i^{th} walk of the standard configuration. Call this the *standard face labelling*. The other maximal configurations can be described by sequences of standard labels in the order in which the faces are chosen. It will be useful later to think of the possible orderings in terms of a rooted *search tree* $T(H)$ where the nodes are given standard labels and each path from the root to a terminal node determines a maximal walk configuration. The tree branches whenever a choice has to be made between different faces as the configuration is constructed. Figure 4 shows the tree which corresponds to the graph in figure 3.

An alternative description is in terms of the walk labels. If the face used in constructing the i^{th} walk is also labelled i the final configuration will correspond to a labelling of the faces with the numbers $1, 2, \ldots, c(H)$ in such a way that they increase in the direction of the arcs of the anti-clockwise dual H^*. This assignment of numbers to the faces will be called a *Young labelling* since if H is a rectangular grid with source at the top right corner and sink at the bottom left corner it will be a standard Young tableau. Denote the set of all Young labellings by $Y(H)$.

Given a Young labelling, the walk configuration may be recovered starting with the left most path (labelled $c(H)$) and iteratively constucting path $i-1$ from path i by diverting it from the left to the right of face i. The construction therefore gives rise to the following bijection.

Bijection 1. *The Young labellings $Y(H)$ are in one-to-one correspondence with the maximal Fermi walk configurations on H.*

An immediate consequence of the bijection is that $n_{max} = c(H) + 1$, and

$$f_{nmax}^{Fermi}(H) = |Y(H)|. \tag{2.1}$$

Bijection 1 was considered in detail for the square lattice in [2] and used to enumerate maximal Fermi walk configurations using a hook length formula 8, 9 . Examples of the bijection are shown in figures 5 and 3. For the graph in figure 5(a), $n_{max} = 5$ and (2.1) gives $f_{nmax}^{Fermi} = 2$.

Given three adjacent walks $i-1, i, i+1$ in a maximal Fermi walk configuration there will be exactly two associated faces, i and $i+1$, enclosed by the first and last of these

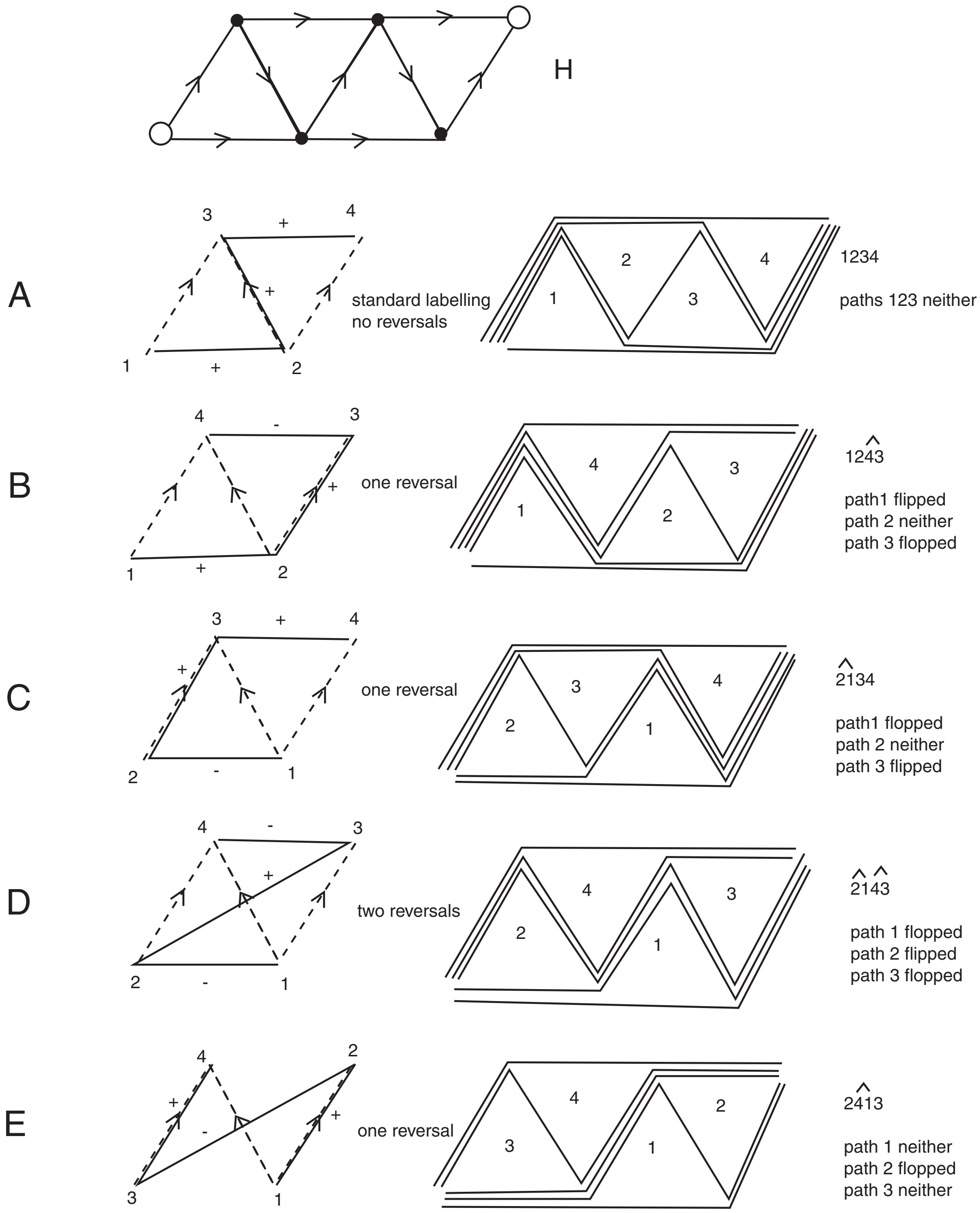

Figure 3: *The figures on the right show the* 5 *maximal Fermi walk configurations of* H *with their Young labellings. The corresponding total orderings of the inner dual (shown dotted) on the left are indicated by the solid line walks. Negative signs on the arcs indicate reversals.*

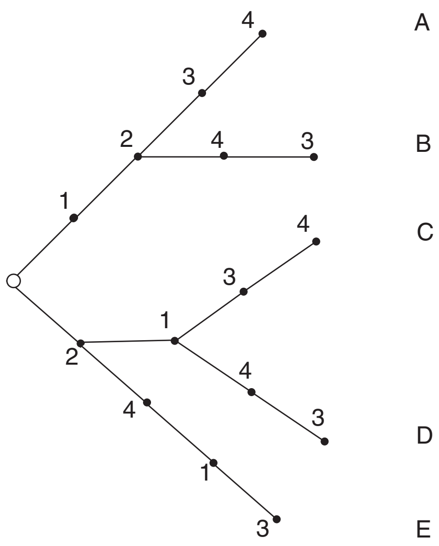

Figure 4: *The search tree $T(H)$ for the graph in figure 3.*

walks. The two faces may (a) share arcs belonging to walk i or (b) if not this walk will pass to the right of one of the faces and then to the left of the other. In case (a) the i^{th} walk will always pass to the left of the first face encountered and to the right of the second. In the case (b) the triple of paths will be called a *flipflop*. If the central walk of a flipflop passes to the right of the first face encountered it will be called the *flopped walk* and if it passes to the left it will be called the *flipped walk*. In case (a) and the flipped walk of case (b) the Young labels of the faces encountered will be in the order $i \to i+1$ but for a flopped walk they will be in the order $i+1 \to i$. We say that this reversed order is an *ascent* of the Young labelling so ascents biject to flopped walks. Note that the flopped and flipped walks cross and therefore cannot be used in the same walk configuration.

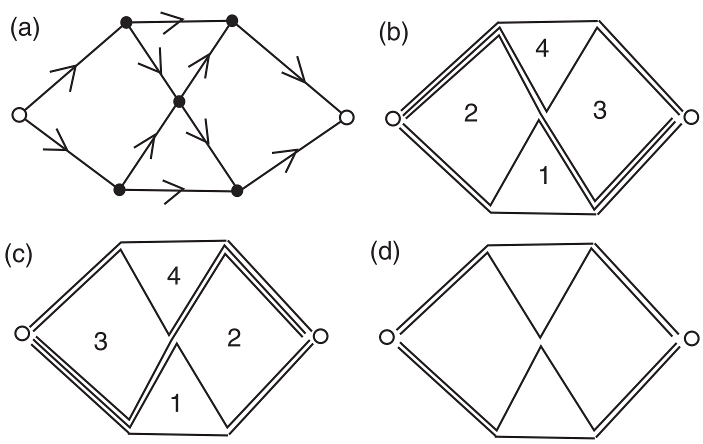

Figure 5: *A directed plane graph (a) with two maximal Fermi configurations (b) and (c) each consisting of 5 walks with their Young labellings identified. The walk configuration (d) is the single optimal configuration consisting of 4 walks.*

By definition the standard configuration has no flopped walks. In terms of the standard labelling a flopped walk is recognised by a *descent* in the labellings of the node sequence on H^* defining the configuration in the search tree $T(H)$. To see this, notice that the Young label of a node is the number of steps down the node sequence. Thus let s_i be the standard label of the i^{th} node in the sequence so $s_i > s_{i+1}$ is a descent. In this case, the face with Young label i will be further downstream than the face with Young label $i+1$ which gives an ascent of the Young labelling and hence corresponds to a flopped walk. A node sequence on the search tree other than the one corresponding to the standard configuration will have at least one descent in the standard label sequence and hence the configuration must have at least one flopped walk. Thus the standard configuration is the only one with no flopped walks.

The graph in Fig. 5 (a) has 2 maximal Fermi walk configurations (see (b) and (c)) with 5 walks each. Walks $1, 2$ and 3 form a flipflop, configuration (b) has a flipped walk and (c) has a flopped walk. The Young labellings of the faces are shown, note how the labels are reversed on flopping the flipped path. The flipped walk configuration has no ascents and the flopped walk configuration has just one ascent.

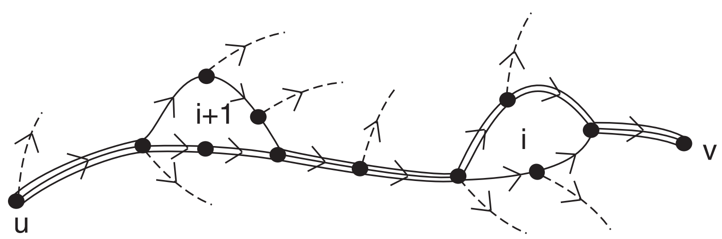

Figure 6: *A flipflop corresponding to ascent i.*

2.3 Determination of $Q^{Bose}(H; z)$ in terms of maximal Fermi walk configurations.

In order to derive the symmetry of $Q^{Bose}(H; z)$ we will also need the generating function $\hat{G}^{Bose}(H; z)$ for Bose walk configurations which cover the graph. This is also a rational function similar to (1.3) only with a different numerator $\hat{Q}^{Bose}(H; z)$.

Proposition 1. *Let $q_k(H)$ be the number of maximal Fermi walk configurations having k flopped walks or ascents and let n_{min} be the number of walks in a minimal cover of H then*

(a) $Q^{Bose}(H; z) = \sum_{k=0}^{k_{max}} q_k(H) z^k$

(b) $\hat{Q}^{Bose}(H; z) = \sum_{k=0}^{k_{max}} q_k(H) z^{n_{max}-k} = z^{n_{max}} Q^{Bose}(H; 1/z)$

(c) $k_{max} = n_{max} - n_{min}$.

(d) $q_0(H) = 1$.

The formula (1.3) for Bose walk configurations was derived by considering all Fermi walk configurations and replacing each walk in a given configuration by one or more walks. Instead we now consider only maximal Fermi walk configurations and replace each walk in a given configuration by zero or more walks. This leads to an overcounting problem which is removed by replacing a flopped walk by at least one walk.

Proof. (a) Let $F_{max}(H)$ be the set of maximal Fermi configurations and $\delta(w) = 1$ if the walk w is a flopped walk but otherwise 0. Further let $flop(\omega)$ be the number of flopped walks in the configuration ω then

$$G^{Bose}(H;z) = \sum_{\omega \in F_{max}(H)} \prod_{w \in \omega} \frac{z^{\delta(w)}}{1-z} = \frac{\sum_{\omega \in F_{max}(H)} z^{flop(\omega)}}{(1-z)^{n_{max}}} = \frac{\sum_{k=0}^{k_{max}} q_k(H) z^k}{(1-z)^{n_{max}}} \tag{2.2}$$

(b) $\hat{G}^{Bose}(H;z)$ may also be expressed as a sum over maximal Fermi walk configurations as in the proof of (a). Now non-flopped walks must be replaced by at least one walk and flopped walks are to be replaced by zero or more walks . This is because the arcs of a flopped walk are already covered by the adjacent walks. Thus $\delta(w)$ is replaced by $1 - \delta(w)$ in (2.2) and z^k is therefore replaced by $z^{n_{max}-k}$.

(c) In the sum in (b) $k = k_{max}$ gives the lowest power of z so $n_{min} = n_{max} - k_{max}$.

(d) This follows since we have shown in section 2.2 that the no flopped walk configuration is unique.

□

2.4 The symmetry property of $Q^{Bose}(H;z)$ for graphs having an optimal cover.

Definition 2. *H will be said to have an* **optimal cover** *if there is a unique minimal set of non-crossing paths which cover all of its arcs.*

The following bijection [5] is required in the proof of the symmetry of $Q^{Bose}(H;z)$ (proposition 2).

Bijection 2. *Bose walk configurations on a directed plane graph biject to integer flows where the flow on any arc is given by the number of walks traversing that arc.*

Proposition 2. *If H has an optimal cover then*

(a)

$$Q^{Bose}(H;1/z) = z^{-k_{max}} Q^{Bose}(H;z)$$

and, *(i) if k_{max} is odd, $Q^{Bose}(H;-1) = 0$*

(ii) using proposition 3(a), $G^{Fermi}(H, -\frac{1}{2}) = 0$.

(b) The coefficients of the polynomial $Q^{Bose}(H;z)$ have the symmetry property

$$q_k(H) = q_{k_{max}-k}(H).$$

(c)

$$q_{k_{max}}(H) = 1$$

Proof.

(a) Assuming that H has an optimal cover with n_{min} walks, the proper $(n_{min}+n)-$flows may be obtained by taking the unique $n_{min}-$proper flow corresponding to the optimal cover and superimposing the flows arising from the $f_n^{Bose}(H)$ Bose configurations. This construction is reversible so $\hat{f}_{n+n_{min}}(H) = f_n^{Bose}(H)$ and hence

$$\hat{Q}^{Bose}(H;z) = z^{n_{min}} Q^{Bose}(H;z). \tag{2.3}$$

Combining proposition 1 (b) and equation (2.3)

$$Q^{Bose}(H;z) = z^{n_{max}-n_{min}} Q^{Bose}(H;1/z)$$

which proves proposition 2(a).
Note: Superimposing flows is not the same as overlaying the corresponding walk configurations but by bijection 2 the superimposed flows are in $1-1$ correspondence with Bose walk configurations in which each arc of H is traversed by at least one walker.

(b) follows immediately by comparing coefficients in (a).

(c) follows from proposition 1 and (b). □

The proposition is illustrated in figure 7 which shows minimal covers for graphs in the shape of two Young diagrams. The graph H_1 of shape $\{3,2,1\}/\{1,0,0\}$ has an optimal cover but H_2 of shape $\{3,1\}$ has more than one minimal cover. $Q(H_1;z) = 1+7z+7z^2+z^3$ is symmetric but $Q^{Bose}(H_2;z) = 1+2z$ is not. The relation $k_{max} = n_{max}-n_{min}$ is satisfied for both graphs.

Figure 12 shows an optimal cover with the unique Young labelling (tableau) having the maximum number of ascents.

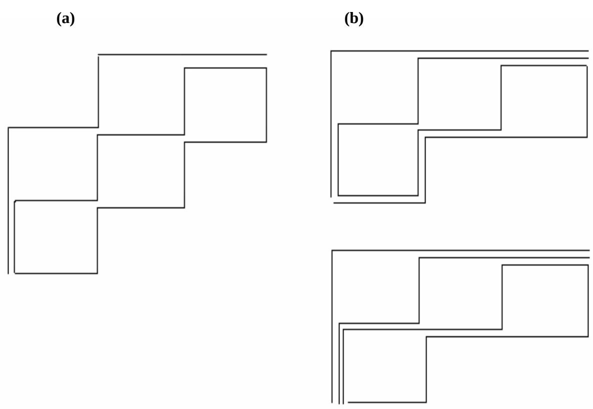

Figure 7: *(a) The graph H_1 illustrating an optimal cover. (b) The graph H_2 illustrating two minimal covers.*

2.5 The Fermi generating function

Proposition 3.

(a)

$$G^{Fermi}(H;z) = (1+z)^{n_{max}-k_{max}} Q^{Fermi}(H;z)$$

where

$$Q^{Fermi}(H;z) = (1+z)^{k_{max}} Q^{Bose}(H;\frac{z}{1+z})$$

is a polynomial of degree k_{max}.

(b) If H has an optimal cover then

$$Q^{Fermi}(H; z) = z^{k_{max}} Q^{Bose}(H; \frac{1+z}{z})$$

and

$$Q^{Fermi}(H; -z) = (-1)^{k_{max}} Q^{Fermi}(H; z-1)$$

Proof.

(a) Inverting equation (1.2) and using (1.3)

$$G^{Fermi}(H; z) = (1+z)^{n_{max}} Q^{Bose}(H; \frac{z}{1+z}) \tag{2.4}$$

and the result follows with

$$Q^{Fermi}(H; z) = \sum_{k=0}^{k_{max}} z^k (1+z)^{k_{max}-k} q_k(H) \tag{2.5}$$

$$= (1+z)^{k_{max}} Q^{Bose}(H; \frac{z}{1+z}). \tag{2.6}$$

(b) The first equality results from combining Proposition 2(a) with (a). The second equality is derived from the first and part (a). □

Figure 3 shows the 5 maximal Fermi configurations for the graph H and which of the walks are flopped. The first configuration has no flopped walks, the fourth has two flopped walks and the others have one flopped walk. By proposition 1 $Q^{Bose}(H; z) = 1 + 3z + z^2$. There are three walks in the optimal cover so $n_{min} = 3$, also $n_{max} = 5$ so by proposition 1(c) $k_{max} = 2$ in agreement with the polynomial found.

3 Duality and generating functions for potentials and order polynomials.

3.1 Potential-flow duality

Definition 3. *An $n-$***potential** *is a function $p : V(H^*) \rightarrow \{0, 1, 2, \ldots n\}$ such that $p(u^*) = 0$, $p(v^*) = n$ and for all arcs $(v, w) \in A(H^*)$, $p(w) \geq p(v)$. The associated $n-$***potential difference** *is the function $\delta p : A(H^*) \rightarrow \{0, 1, 2, \ldots n\}$ defined by $\delta p((v, w)) = p(w) - p(v)$. n-potential differences which are strictly positive on all arcs are called* **proper***.*

Notes:

1. For all arcs $a^* \in A(H^*), \delta p(a^*) \geq 0$.
2. The signed sum of potential differences round any closed path is zero where the sign of a given term is positive if the arc is parallel to the path but negative if it is anti-parallel.

Proposition 4 (Duality)**.** *There is a bijection between $n-$flows on H and $n-$potential differences on H^*.*

This bijection is illustrated in figure 8.

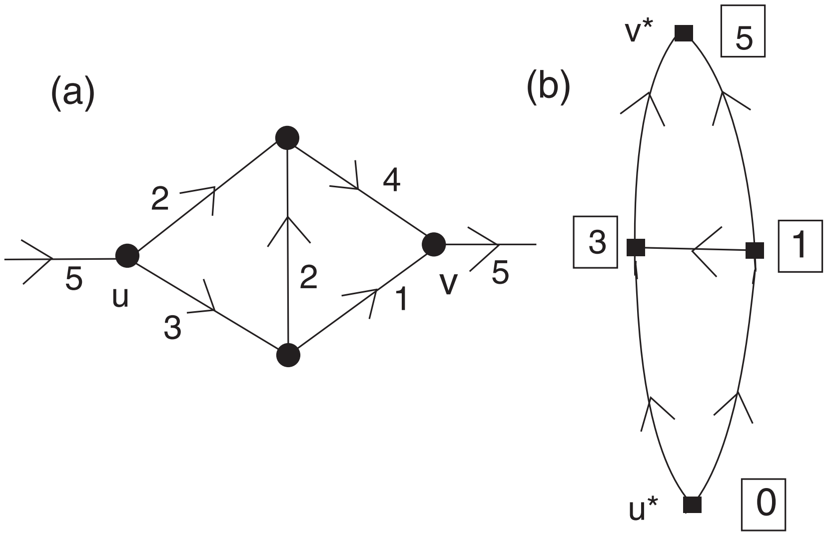

Figure 8: *An example of flow-potential duality. (a) The graph H with a root flow of 5; (b) The dual graph H^* with the corresponding potential (squared numbers) with $n = 5$. Notice that the potential differences correspond to flows on the dual arcs.*

Proof. Define $f : A(H) \to \{0, 1, 2, \ldots n\}$ by $f(a) = \delta p(a^*)$ where a^* is the arc of the dual which intersects a. The signed sum of flows into any intermediate vertex v of H, where in-flows are positive and out-flows are negative, is equal to the signed sum of potential differences round the closed path on H^* surrounding v. Since this is zero the in-flow for v is equal to its out-flow. The sum of the potential differences on the arcs of H^* which intersect the arcs of H leaving (entering) the source (sink) is equal to n. Hence f satisfies the flow condition.

The argument may be reversed which establishes the bijection. □

Let $D_n(H^*)$ be the number of $n-$potentials (or potential differences) on H^* then using bijection 2 between flows and Bose walk configurations this proposition may be summarised by

$$G^{Bose}(H; z) = \sum_{n=0}^{\infty} D_n(H^*) z^n. \tag{3.1}$$

Bose configurations which cover all of the arcs of H correspond to strictly positive potential differences on H^* enumerated by $\hat{D}_n(H^*)$ so

$$\hat{G}^{Bose}(H; z) = \sum_{n=0}^{\infty} \hat{D}_n(H^*) z^n. \tag{3.2}$$

3.2 Alternative derivation of proposition 1

The vertices of H^* may be considered as a partially ordered set where the ordering is determined by the arcs. Given a Young labelling $\eta \in Y(H)$ of the faces of H assign labels to the vertices of the inner dual so that each vertex has the same label as the face it belongs to. Now extend the labelling to H^* by assigning label 0 to u^* and label $c(H) + 1$ to v^* so that we have a total ordering of the vertices. By definition of Young labelling the vertex labels will be increasing in the direction the arcs of H^*. The ordering corresponding to the labelling η may be represented by a walk $W(\eta)$ on the vertices of H^* from $u*$ to $v*$ so that the labels increase along the walk. Not all of the arcs of $W(\eta)$ will necessarily belong to H^* (see for example figure 3). For each η the potentials of H^* consistent with the ordering of $W(\eta)$ are easily enumerated.

In the case of proper potentials, the potential differences must be strictly positive on the arcs of $W(\eta)$ belonging to H^* but may be zero on the remaining arcs. Unfortunately enumeration of the potentials for each walk and summing over walks will over count the potentials on H^* since a potential with zero differences can occur on more than one walk. To avoid this we define the *standard vertex labelling* of H^* to be the one corresponding to the standard face labelling of section 2.2 and then only allow zero potential difference on an arc of $W(\eta)$ if its direction is reversed relative to the direction implied by the standard labelling. We call such an arc a *reversal*.

The generating function for proper potentials on $W(\eta)$ will therefore have a factor $1/(1-z)$ for each reversal and a factor $z/(1-z)$ for each other arc, hence

$$(1-z)^{c(H)+1} \sum_{n=0}^{\infty} \hat{D}_n(H^*) z^n = \sum_{\eta \in Y(H)} z^{c(H)+1-r(\eta)} \tag{3.3}$$

where $r(\eta)$ is the number of reversals for the labelling η.

For ordinary potentials arcs which are not reversals are allowed to have zero potential difference and, to ovoid overcounting, reversals must be assigned strictly positive potential differences.

$$(1-z)^{c(H)+1} \sum_{n=0}^{\infty} D_n(H^*) z^n = \sum_{\eta \in Y(H)} z^{r(\eta)}. \tag{3.4}$$

Reversals clearly biject to ascents of η which in turn biject to flopped walks of the corresponding maximal walk configuration ω. Equations (3.1) and (3.2) together with (3.3) and (3.4) provide an independent derivation of proposition 1(a).

3.3 Other related partially ordered sets

Denote the partially ordered set of vertices of the dual graph H^* introduced in the previous section by $P^*(H)$. The *order polynomial* $\Omega(P, m)$ of a finite partially ordered set P is the number of order-preserving maps from P to the integers $\{1, 2, \ldots, m\}$, [13],[12]. The *strict order polynomial* $\hat{\Omega}(P, m)$ is similarly defined in terms of strict order preserving maps.

A simple comparison with definition 3 shows that

$$D_n(H^*) = \Omega(P^*(H), n+1). \tag{3.5}$$

Combining this with the duality relation (3.1) we obtain

$$f_n^{Bose}(H) = \Omega(P^*(H), n+1) \tag{3.6}$$

Theorem 4.5.14 of [13] expresses the generating functions for order polynomials of P in terms of the set $\mathcal{L}(P)$ of Jordan-Hölder permutations.

$$\sum_{m \geq 0} \Omega(P, m) z^m = \left(\sum_{\pi \in \mathcal{L}(P)} z^{d(\pi)+1} \right) (1-z)^{-p-1} \tag{3.7}$$

where $p = |P|$ and $d(\pi)$ is the number of descents of π. Equation (3.4) for the generating function of $D_n(H)$ can be seen as an example of (3.7) by taking P to be the vertex set of H^* so that $p = c(H)$. $\pi = s_1 s_2 \ldots s_{c(H)}$ where s_i is the standard label of vertex i in a Young labelling $\eta \in Y(H)$. In section 2.2 it was shown that $d(\pi)$ is the number of ascents in η. The Jordan-Hölder permutations for the graph of figure 3 are given in the figure alongside the Young labellings. The proof of (3.7) in [13] uses the more general theory of $P-$partitions developed earlier in the book. Our derivation of equation (3.4) is closer to the context of this paper.

The proper potential condition implies that proper $n-$potentials take on the values in the range 1 to $n-1$ since they must differ from the root potentials of 0 and n. Thus

$$\hat{f}_n^{Bose}(H) = \hat{D}_n(H^*) = \hat{\Omega}(P^*(H), n-1) \tag{3.8}$$

and from [13]

$$\sum_{m \geq 1} \hat{\Omega}(P,m) z^m = \left(\sum_{\pi \in \mathcal{L}(P)} z^{p-d(\pi)} \right) (1-z)^{-p-1} \tag{3.9}$$

and so equation (3.3) for $\hat{D}_n(H)$ also follows from 4.5.14 of [13].

The optimal cover condition is the equivalent to $P^*(H)$ being graded with all maximal chains of length n_{min} so proposition 2(e) is a special case of corollary 4.5.17 $b(iii)$ of [13].

The paths on H form a second partially ordered set to be denoted by $P(H)$. For paths $p_1, p_2 \in P(H)$, $p_1 > p_2$ if p_1 and p_2 are non-crossing and p_1 lies closer to v^* than p_2. Crossing paths are not comparable.

Moreover, there is an obvious bijection between non-crossing $n-$walk configurations on the directed graph H and chains/multi-chains of n elements on the partially ordered set $P(H)$. Fermi n-walk configurations biject to chains $p_1 < p_2 \cdots < p_n$ and Bose n-walk configurations biject to multi-chains $p_1 \leq p_2 \leq \cdots \leq p_n$. Maximal Fermi walk configurations biject to saturated (maximal) chains.

For $m \geq 2$ the *zeta polynomial* $Z(P,m)$ of the finite partially ordered set P is the number of multichains in P having $m-1$ elements.

The above walk-POS bijection then gives

$$f_n^{Bose}(H) = Z(P(H), n+1). \tag{3.10}$$

Combining this with the duality relation (3.6)

$$\Omega(P^*(H), n+1) = Z(P(H), n+1) \tag{3.11}$$

Further to this, the *reversals* of the walk $W(\eta)$ of the Young labelling $\eta \in Y(H)$ relative to the standard labelling of H can be identified with the descents which arise in the permutations of the Jordan-Holder set of permutations on H^* (cf. Figure 4). In this figure, the Young labellings on H^*, when seen as permutations of the Young labelling with no ascents - i.e. the standard labelling, form the elements of the Jordan-Hölder set $\mathcal{L}(P^*(H))$.

A summary of the associations between POS structures and Bose and Fermi configurations is given in tables 1 and 2 respectively.

4 Application to the directed square and honeycomb lattices.

The above results for the general graph H may be applied to t-step walk configurations on the square lattice. Each walk starts at the origin and for accordance with the usual

	dual	
n-flows on H	$\longleftrightarrow$	n-potential differences on H^*
$\updownarrow$		
n-walk Bose configurations on H		$\updownarrow$
$\updownarrow$		
n-element multi-chains on $P(H)$	$\longleftrightarrow$	order preserving maps from $P^*(H)$ to $\{0,1,\ldots,n\}$
$f_n^{Bose}(H) \equiv Z(P(H), n+1)$	$=$	$\Omega(P^*(H), n+1) \equiv D_n(H^*)$

Table 1: Structures which biject to Bose walk configurations. $P(H)$ is the partially ordered set of paths on H and $P^*(H)$ is the partially ordered set of vertices of the dual H^*.

	dual	
Maximal Fermi configurations on H	$\longleftrightarrow$	Young labellings Y(H)
$\updownarrow$		$\updownarrow$
Saturated chains on $P(H)$	$\longleftrightarrow$	Jordan-Hölder set $\mathcal{L}(P^*(H))$
$f_{n_{max}}^{Fermi}(H)$	$=$	$\lvert Y(H)\rvert = \lvert\mathcal{L}(P^*(H))\rvert$
Maximal Fermi configurations on H with k flopped paths	$\longleftrightarrow$	$\pi \in \mathcal{L}(P^*(H)) : \pi$ has k descents
$\updownarrow$		$\updownarrow$
Standard Young labelling of proper potentials on H^* with k descents	$\longleftrightarrow$	Young labellings of $Y(H)$ with k reversals

Table 2: Structures which biject to Fermi walk configurations. $P(H)$ is the partially ordered set of paths on H and $P^*(H)$ is the partially ordered set of vertices of the dual H^*.

convention for Young tableau each step is either down or to the left. If each walk makes ℓ left steps and w down steps then H, the union of all possible paths, is a rectangular grid denoted by $W_{\ell,w}$. Let H^* be its anti-clockwise dual then Young labellings of $W_{\ell,w}$ are standard Young Tableaux. The union of paths restricted to the region $y \geq x$ produces a two-rooted graph $H = \overline{W}_{\ell,w}$. These configurations are called *watermelons* [7]. If there is no restriction on the number of left and down steps other than $\ell + w = t$ the configurations are called *stars* [7] and the union of paths produces a graph S_t. The walks may terminate at any one of $t+1$ vertices. This may be reduced to a two-rooted graph $H(S_t)$ having the same generating functions by connecting each terminal vertex to a single root (see for example figure 11 (b)). Again restriction to $y \geq x$ will be indicated by an overline. The tableaux corresponding to star configurations are shifted.

Figure 9 shows a honeycomb lattice, assumed to be directed right to left, which is the union of all paths starting and ending at vertices of out-degree one. Bypassing each vertex of in-degree one by a pair of paths (as illustrated) a square lattice graph is obtained which in the example is $W_{2,3}$. The number of non-crossing configurations for the two lattices is the same. In terms of the square lattice parameters the honeycomb paths have length $2t$ and there are ℓ pairs of steps which constitute a right-step on the square lattice. By duality the number of $n-$potentials for the dual triangular lattice is also determined.

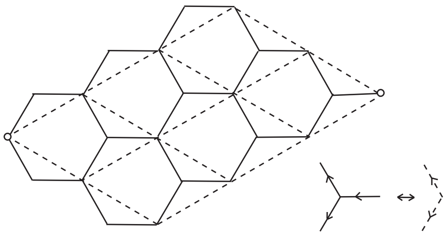

Figure 9: *An example of the honeycomb-square lattice correspondence. The generating functions are the same for both lattices.*

The ascents in the case of the square lattice occur when $i+1$ is on a row of the tableau above i (see flipflop figure 10.)

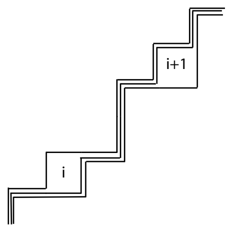

Figure 10: *A flipflop on the square lattice (cf. Figure 6).*

Table 3 shows the twelve shifted Young tableaux for S_4, ascents are indicated by the bold numbers in the figure. Using Proposition 1 yields the polynomial

$$Q^{Bose}(S_4; z) = 1 + 5z + 5z^2 + z^3.$$

1 2 3 4 5 6 7 8 9 10	1 2 3 4 5 6 8 **7** 9 10	1 2 3 5 **4** 6 7 8 9 10	1 2 3 5 **4** 6 8 **7** 9 10	1 2 3 6 4 **5** 7 8 9 10	1 2 3 6 4 **5** 8 **7** 9 10
1 2 4 5 **3** 6 7 8 9 10	1 2 4 5 **3** 6 8 **7** 9 10	1 2 3 7 4 5 8 **6** 9 10	1 2 4 6 **3** **5** 7 8 9 10	1 2 4 6 **3** **5** 8 **7** 9 10	1 2 4 7 **3** 5 8 **6** 9 10

Table 3: The twelve shifted Young tableaux corresponding to maximal Fermi walk configurations on S_4. The ascents are shown in boldface.

4.1 Optimal covers

The graphs defined above all have optimal covers two examples of which are shown in figure 11. Consequently proposition 2(a) predicts the symmetry of the $Q^{bose}(z)$ polynomials in agreement with the examples in Appendix A. The sizes n_{min} of the minimum covers are listed in table 4 along with the resulting degrees $k_{max} = n_{max} - n_{min}$ of the $Q^{Bose}(H; z)$ These also agree with the data in Appendix A.

	n_{max}	n_{min}	k_{max}
$W_{\ell,w}$	$\ell w + 1$	$\ell + w = t$	$(\ell - 1)(w - 1)$
S_t	$\frac{1}{2}t(t+1)+1$	$2t$	$\frac{1}{2}(t-1)(t-2)$
$\overline{W}_{\ell,w}$	$wd + \frac{1}{2}w(w-1)+1$	$w + d = \ell$	$(w-1)d + \frac{1}{2}(w-1)(w-2)$
$\overline{S}_{t,w}$	$(t-w)w+1$	t	$(t-w-1)(w-1)^2$

Table 4: *Number n_{min} of paths in the minimum cover and the degree k_{max} of $Q^{Bose}(H; z)$*

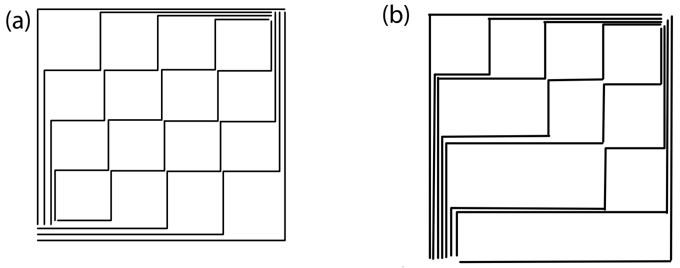

Figure 11: *The optimal covers for (a) $W_{4,4}$, the 4×4 rectangular lattice, and (b) S_4, the star configuration of length 4 (cf. [1]).*

For the graphs of figure 11, and others, the degree k_{max} of $Q^{Bose}(H; z)$ may be obtained by a simple independent argument as follows. Figure 12 shows the shifted tableau for $\overline{S}_8$ which has the maximum number of ascents. From this it is clear how the tableau with the maximum number of ascents is obtained from the optimal cover. The walks in the optimal cover partition the faces into $n_{min} - 1$ strips the faces of which meet corner to corner. The faces are to be numbered so that the numbers increase from left to right and increase sequentially up each strip. In this way a strip j with f_j faces will have $f_j - 1$ ascents and ascents are confined to these strips. The maximum number of ascents is therefore given by

$$k_{max} = \sum_{j=1}^{n_{min}-1} (f_j - 1) = c(H) - (n_{min} - 1) = n_{max} - n_{min} \tag{4.1}$$

5 Summary and concluding remarks.

We have discussed two enumeration problems on a two-rooted coverable plane graph H. The non-crossing walk problem of n walks is simply related to counting integer flows where the flow on a given arc is the number of walks which traverse it and the in and out flow is n. The second problem of enumerating potentials is related to the walk problem by the well known duality which relates $n-$flows on H to potentials on the dual graph H^* (also two-rooted and coverable) when the roots are assigned potentials zero and n. This implies equality of the generating functions.

The generating functions are rational functions and the coefficient $f_k(H)$ of z^k in the numerator is related to properties of the configurations in various ways. In the context of

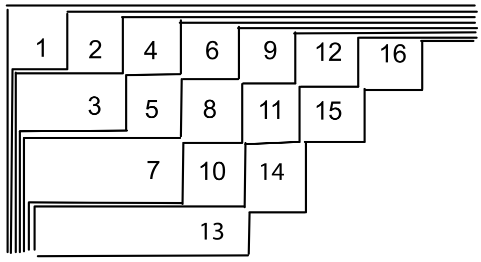

Figure 12: *The shifted skew tableau for $\overline{S}_8$, the 8-step star configuration with a wall, having the maximal number of ascents.*

the walk problem it is the number of maximal Fermi walk configurations having *k flopped walks*. An algorithm which generates maximal Fermi walk configurations finds the next walk by diverting the previous walk from right to left round a face of H. By labelling the walks as they are generated by integers from 1 to $c(H)$ this gives a total ordering to the faces of H known as a Young labelling $\eta \in Y(H)$. Incidentally this shows that a maximal Fermi walk configuration has $n_{max} = c(H) + 1$ walks. The number of flopped walks in a configuration ω is equal to the number of *ascents* in the corresponding Young labelling η.

Because the vertices of H^* correspond to faces of H the Young labellings define total orderings of these vertices. The orderings are consistent with the partial ordering defined by the arcs of H^* since, by virtue of the algorithm, the labels on an arc of H^* connecting adjacent faces of H increase by the number of walks passing between the faces. By definition the potentials are labellings of the vertices of H^* which increase along the arcs and we enumerate these by mapping them to the Young labellings. The coefficient of z^k in the generating function numerator then becomes the number of *reversals* in the labelling of H^*.

A generating function for the number of ways of extending a partially ordered set to total order is given in a theorem of Stanley [13]. The result is expressed in terms of Jordan-Hölder permutations and the coefficient of z^k in the numerator of the generating function is the number permutations having $k-$descents. We could have made use of this theorem in the potential problem by applying it to the partially ordered set of vertices of H^* but we hope that describing it in the context of walks and potentials will make it more accessible to the physics community.

As well as providing physical interpretations of the numerator coefficients we have shown that the its degree $k_{max} = n_{max} - n_{min}$ where n_{min} is the number of walks in a minimal cover of H. Also if the cover is unique then polynomial will be symmetric.

In our earlier work [4] we considered mod λ flows which satisfy the condition that the difference of the in-flow and out-flow at any internal vertex is zero mod λ. These flows are independent of the directing of the graph. However, if we assume λ is odd and restrict the flow on each arc to be even then number of flows becomes directing dependent. It was shown in [4] that if the in and out degrees of the vertices of H are at most two then these restricted flows are the integer flows considered here. These have been shown to biject to non-crossing walk configurations. The odd λ even flow configurations also arose in a $\lambda-$state chiral Potts model [3] a limiting case of which gave directed percolation. Application to directed percolation is discussed in [2].

A Numerators for Bose generating functions on the square lattice

Substituting the formulae of table1 of 1 into (1.1) and using equation 1.3 produces the Q^{Bose} polynomials listed in tables 5, 6 and 7.

t	$Q^{Bose}(S_t; z)$	$Q^{Bose}(S_t; 1)$
1	1	1
2	1	1
3	$1 + z$	2
4	$1 + 5z + 5z^2 + z^3$	12
5	$1 + 16z + 70z^2 + 112z^3 + 70z^4 + 16z^5 + z^6$	286
6	$1 + 42z + 539z^2 + 2948z^3 + 7854z^4 + 10824z^5 + 7854z^6 + 2948z^7 + 539z^8$	
	$+42z^9 + z^{10}$	33592
7	$1 + 99z + 3129z^2 + 44739z^3 + 336819z^4 + 1450761z^5 + 3753841z^6$	
	$+5999851z^7 + 5999851z^8 + 3753841z^9 + 1450761z^{10} + 336819z^{11}$	
	$+44739z^{12} + 3129z^{13} + 99z^{14} + z^{15}$	23178480

Table 5: The Bose numerator polynomials $Q^{Bose}(S_t; z)$

w	1	2	3	4
ℓ				
1	1	1	1	1
2	1	$1 + z$	$1 + 3z + z^2$	$1 + 6z + 6z^2 + z^3$
3	1	$1 + 3z + z^2$	$1 + 10z + 20z^2 + 10z^3 + z^4$	$1 + 22z + 113z^2 + 190z^3 + 113z^4 + 22z^5 + z^6$
4	1	$1 + 6z + 6z^2 + z^3$	$1 + 22z + 113z^2 + 190z^3 + 113z^4 + 22z^5 + z^6$	$1 + 53z + 710z^2 + 3548z^3 + 7700x^4 + \dots$

Table 6: The Bose numerator polynomials $Q^{Bose}(W_{\ell,w}; z)$ without a wall.

w	1	2	3	4
ℓ				
2	1	1		
3	1	$1 + z$	$1 + z$	
4	1	$1 + 3z + z^2$	$1 + 7z + 7z^2 + z^3$	$1 + 7z + 7z^2 + z^3$
5	1	$1 + 6z + 6z^2 + z^3$	$1 + 18z + 65z^2 + 65z^3 + 18z^4 + z^5$	$1 + 31z + 187z^2 + 330z^3$ $+187z^4 + 31z^5 + z^6$
6	1	$1 + 10z + 20z^2 + 10z^3 + z^4$	$1 + 35z + 279z^2 + 741z^3 + 741z^4 + 279z^5 + 35z^6 + z^7$	$1 + 75z + 1230z^2 + 6905z^3$ $+15813z^4 + 15813z^5 + \dots$

Table 7: The Bose numerator polynomials $Q^{Bose}(\overline{W}_{\ell,w}; z)$ with a wall.

B Fermi generating functions

Examples of $G^{Fermi}_{\ell,w}(z) \equiv G^{Fermi}(W_{\ell,w}; z)$ are

$$
\begin{aligned}
G^{Fermi}_{2,2}(z) &= (1 + z)^4(1 + 2z) \\
G^{Fermi}_{3,2}(z) &= (1 + z)^5(1 + 5z + 5z^2) \\
G^{Fermi}_{2,4}(z) &= (1 + z)^6(1 + 2z)(1 + 7z + 7z^2) \\
G^{Fermi}_{3,4}(z) &= (1 + z)^7(1 + 28z + 238z^2 + 882z^3 + 1596z^4 + 1386z^5 + 462z^6) \\
G^{Fermi}_{4,4}(z) &= (1 + z)^8(1 + 2z)(1 + 60z + 1050z^2 + 7986z^3 + 31020z^4 + 66066z^5 \\
&\qquad + 78078z^6 + 48048z^7 + 12012z^8)
\end{aligned}
$$

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