Maximal Fermi walk configurations on the directed square lattice and standard Young Tableaux

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Abstract

We consider configurations of n walkers each of which starts at the origin of a directed square lattice and makes the same number t of steps from node to node along edges of the lattice. Bose walkers are not allowed to cross, but can share edges. Fermi walk configurations must satisfy the additional constraint that no two walkers traverse the same path. Since, for given t, there are only a finite number of $t-$step paths there is a limit n_{max} on the number of walkers allowed by the Fermi condition. The value of n_{max} is determined for six types of boundary condition. The number of Fermi configurations of n_{max} walkers is also determined using a bijection to standard Young tableaux. In four cases there is no constraint on the endpoints of the walks and the relevant tableaux are shifted.

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1 Introduction and Definitions.

Various types of interacting walker problems are considered in the literature [7] and they model key physical entities such as gas molecules and polymers. For example, in [5] and [6], collections of "vicious" walkers are used to model directed polymer networks. Walker configurations can also be used to elicit information on other structures such as distributions of eigenvalues of real symmetric random matrices, see [15]. The combinatorics of the various walker configurations on regular lattices with different types of restriction form a rich collection of counting problems, which are not only of mathematical interest in themselves, but also relate to the combinatorics of other structures such as the Young tableaux used in this paper.

The origin of the description *Fermi* walks comes from the nomenclature for elementary particles which can be either of *Fermi* or *Bose* type. Such particles have a discrete set of energy levels and two bosons can occupy the same energy level, while for fermions only single occupation is allowed. These different properties of Fermi and Bose particles are reflected in the definitions, below, of Fermi and Bose walks.

We consider n walks in the plane each of which starts at the origin and makes t steps where the step vectors are either $(-1, 0)$ or $(0, -1)$. Negative steps are chosen to agree with the usual convention for Young tableaux introduced later. The walks are allowed to intersect but may not cross one another. Configurations which satisfy this condition are called Bose and configurations which satisfy the further condition that no two walkers traverse the same path are called Fermi. Bose and Fermi walk configurations were introduced by Inui and Katori [12] in the context of directed percolation theory. Here we will be interested only in the Fermi case.

Configurations where all walkers terminate at the same point $(-\ell, -w)$, so that $t = \ell + w$, will be called *watermelons* but configurations where the endpoints may vary are called *stars*. This terminology was introduced by Fisher [7] in the context of vicious walkers which biject [2] to Bose walkers. The union of all configurations defines a graph called $W_{\ell,w}$ in the case of watermelons (ℓ for length and w for width) and S_t for stars. See figure 1(a) for $W_{7,4}$ and figure 1(c) for S_4.

Further graphs $\bar{W}_{\ell,w}$ and $\bar{S}_t$ will arise when we introduce a wall $y = x$ which the walks are not allowed to pass below (see figures 1(b) and (d) for examples). Finally the imposition of a limit w on the number of y steps made by walks in a star configuration creates the graphs $S_{t,w}$ and $\bar{S}_{t,w}$ exemplified

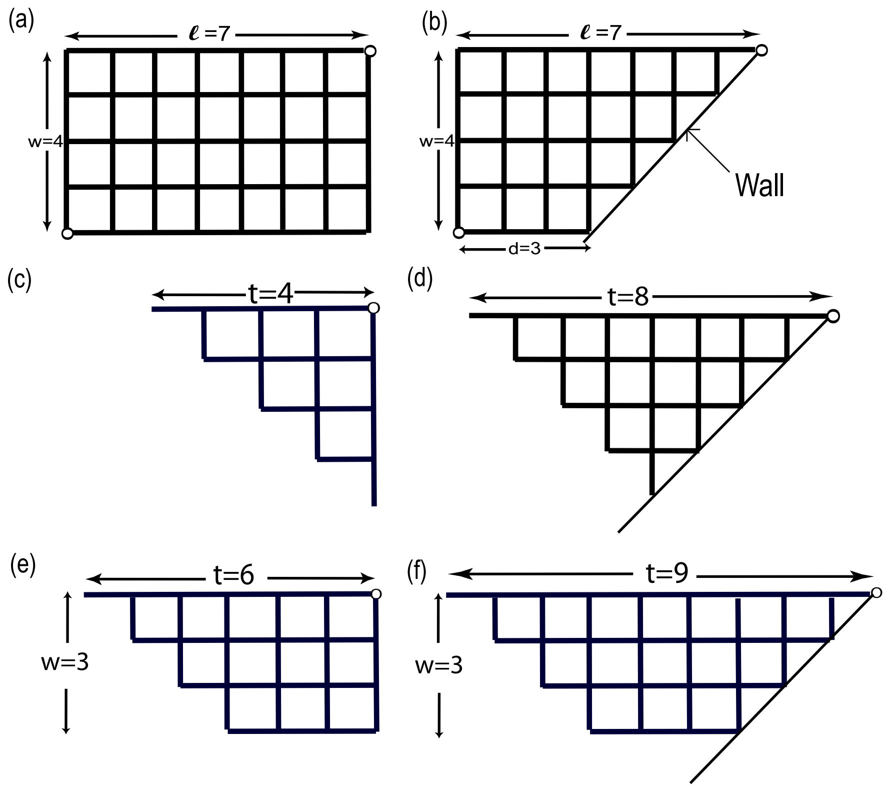

Figure 1: *The graphs(a)* $W_{7,4}$ *(b)* $\overline{W}_{7,4}$ *(c)* S_4 *(d)* $\overline{S}_8$ *(e)* $S_{6,3}$ *(f)* $\overline{S}_{9,3}$*. The graphs are considered to be directed to the left and down.*

in figures 1(e) and (f). When referring to a general graph the symbol H will be used. The number of $n-$walk Fermi configurations on the graph H will be denoted by $f_n^{Fermi}(H)$.

The cells of the grids defined by $W_{\ell,w}$ and $\bar{W}_{\ell,w}$ form *normal* Young diagrams in that the rows of *cells* are left justified and of decreasing length. However this is not so for the star graphs. In the subsequent discussion of the correspondence between Young tableaux and Fermi walk configurations (section 3) the diagrams for stars will be obtained from the graphs by adding an anti-diagonal staircase of edges connecting the possible endpoints of the $t-$step walks. This results in the inclusion of an additional cell at the beginning of each row of the diagram and a possible single cell row. In the case when a wall is present the initial step common to all walks is removed in constructing the Young diagram. This clearly has no effect on the number of configurations but the length of the top edge of the diagram is $t-1$. The resulting *shifted* Young diagrams characterised by the left-hand staircase will be denoted by Y_t, $\bar{Y}_t$, $Y_{t,w}$ and $\bar{Y}_{t,w}$. For watermelons the same notation will be used for the graph and diagram. Examples of shifted Young diagrams are shown in figures 3(b), (d) and (e). When referring to Young diagrams in general the symbol Y will be used.

Definition 1. *A standard (shifted) Young tableau is an assignment of the numbers $1, 2, \ldots, c(Y)$ to a Young diagram Y having $c(Y)$ cells such that each number is used and the entries are increasing to the right along the rows and downwards along the columns.*

The results of our walk enumerations are summarised in the next section. The bijection between maximal Fermi walk configurations and standard Young tableaux which makes these enumerations possible is obtained in section 3 and the details are given section 4. Finally we discuss the connection between Fermi walks and directed percolation in section 5.

2 Results

The Fermi condition implies that there will be a limit to the number of walkers, depending on the graph H, above which no configurations are possible. A *maximal Fermi walk configuration* is one to which no walk may be added without violating the Fermi condition. Typical maximal watermelon and star configurations are shown in figure 2.

Proposition 1. *Every maximal Fermi walk configuration on the graph H has $n_{max} = c(Y) + 1$ walks, where $c(Y)$ is the number of cells in the Young diagram corresponding to H.*

The number $f^{Fermi}_{n_{max}}(H)$ of maximal Fermi walk configurations is the main topic of this paper. In section 3 a bijection will be established between maximal Fermi walk configurations on H and standard Young tableaux on the corresponding Young diagram Y. This proves proposition 1 and enables the maximal Fermi walk configurations on H to be counted using Hook formulae [8],[16] for the numbers of Young tableaux.

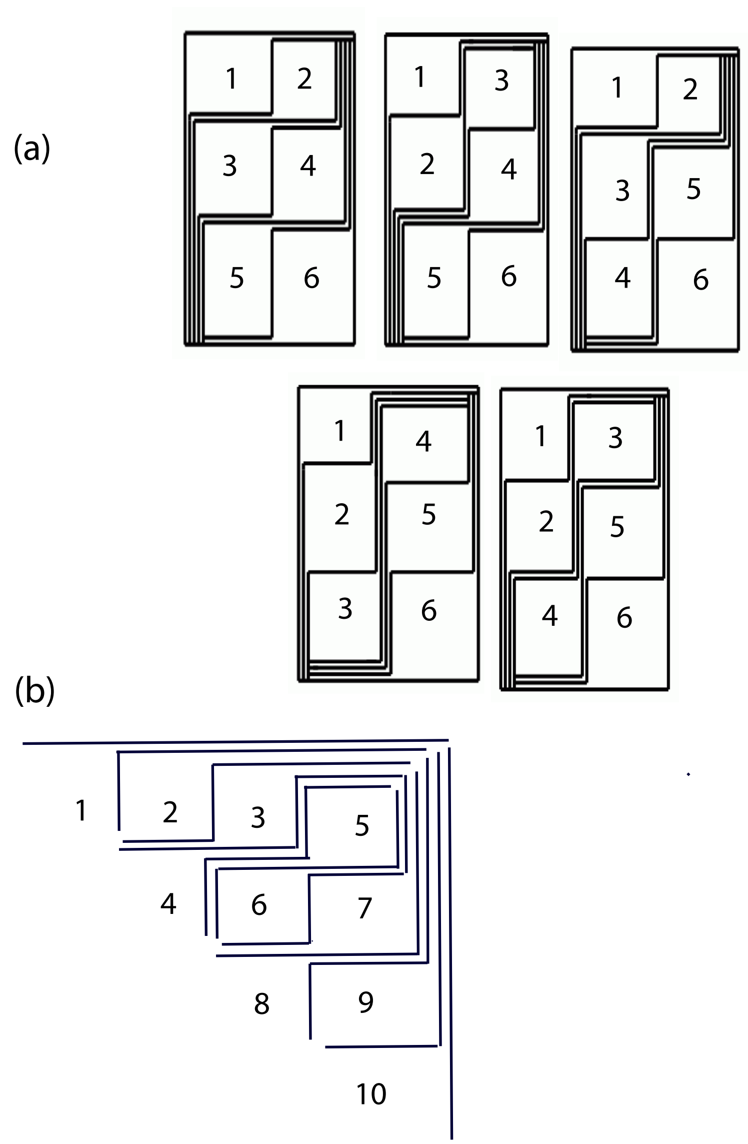

Figure 2: *Maximal Fermi walk configurations and the corresponding Young tableaux. (a) The five maximal configurations on* $W_{2,3}$*.(b) A typical maximal configuration on* S_4*.*

In the following propositions the Pochhammer symbol $(a)_b = a(a+1)\dots(a+b-1)$ is used. Also, using proposition 1, $n_{max} = c(Y) + 1$ where in the case of watermelon configurations Y is either $W_{\ell,w}$ or $\overline{W}_{\ell,w}$.

Proposition 2. *Maximal Fermi walk configurations without a wall.*

(a) For watermelon configurations without a wall $c(W_{\ell,w}) = \ell w$ *and*

$$f^{Fermi}_{n_{max}}(W_{\ell,w}) = C_{w,\ell} = \frac{(\ell w)!}{\prod_{k=1}^{w}(k)_\ell} = C_{\ell,w} \tag{2.1}$$

These are multi-dimensional Catalan numbers, see [18] - sequence A005789.

(b) For star configurations without a wall $c(Y_t) = \frac{1}{2}t(t+1)$ *and*

$$f^{Fermi}_{n_{max}}(S_t) = \frac{(\frac{1}{2}t(t+1))!}{\prod_{k=1}^{t}(k)_k} \tag{2.2}$$

These are Strict Sense Ballot numbers, see [18] - sequence A003121.

Note: Sulanke [19] attributes the formula for $C_{w,\ell}$ to MacMahon (see [17], art.93-103). Note that $C_{2,\ell}$ is a Catalan number, which is well known to enumerate Dyck paths, and $C_{w,\ell}$ has been called a $w-$dimensional Catalan number [19].

Proposition 3. *Maximal Fermi walk configurations above the wall* $y = x$.

(a) For watermelon configurations above the wall $y = x$*, introducing* $d = \ell - w \geq 0$, $c(\overline{W}_{\ell,w}) = wd + \frac{1}{2}w(w-1)$ *and*

$$f^{Fermi}_{n_{max}}(\overline{W}_{\ell,w}) = \frac{2^{\frac{1}{2}w(w-1)}(c(\overline{W}_{\ell,w}))!}{\prod_{k=1}^{w}(k)_d(d+k)_{k-1}} = \frac{2^{\frac{1}{2}w(w-1)}(c(\overline{W}_{\ell,w})!}{\prod_{k=1}^{w}(k)_{d+k-1}} \tag{2.3}$$

Note: The sequence $d = 0$, $w = \ell = 0, 1, 2, \dots$ *is the sequence of numbers of triangular standard Young tableaux, sequence A005118 of [18]. When* $\ell = 2$*, increasing* d *gives a sequence of Catalan numbers and* $\ell = 3$ *gives sequence A1123555 of [18].*

(b) For star configurations above the wall $y = x$

(i) with $t = 2w$, $c(\overline{Y}_{2w}) = w^2$ *and*

$$f^{Fermi}_{n_{max}}(\overline{S}_{2w}) = C_{w,w} = \frac{(w^2)!}{\prod_{k=1}^{w}(k)_w} = f^{Fermi}_{n_{max}}(W_{w,w}) \tag{2.4}$$

(ii) with $t = 2w+1$, $c(\overline{Y}_{2w+1}) = w(w+1)$ *and*

$$f^{Fermi}_{n_{max}}(\overline{S}_{2w+1}) = C_{w+1,w} = \frac{(w(w+1))!}{\prod_{k=1}^{w}(k)_{w+1}} = f^{Fermi}_{n_{max}}(W_{w+1,w}) \tag{2.5}$$

Proposition 4. *Maximal Fermi star configurations with* $\leq w$ $y-$*steps.*

(a) *For star configurations with* $<= w$ *down steps* $c(Y_{t,w}) = tw - \frac{1}{2}w(w-1)$ *and*

$$f^{Fermi}_{n_{max}}(S_{t,w}) = \frac{(tw - \frac{1}{2}w(w-1))!}{\prod_{k=1}^{w}(k)_{t-w+1}(k+2(t-w)+1)_{k-1}} \tag{2.6}$$

(b) *For star configurations with* $<= w$ *down steps and above the wall* $y = x$ $c(\overline{Y}_{t,w}) = (t-w)w$ *and*

$$f^{Fermi}_{n_{max}}(\overline{S}_{t,w}) = \frac{((t-w)w)!}{\prod_{k=1}^{w}(k)_{t-w}} = f^{Fermi}_{n_{max}}(W_{t-w,w}) \tag{2.7}$$

Notes:

(i) The star configurations above the wall $y = x$ with no width constraint are the special cases $t = 2w$ and $t = 2w+1$ of part (b) of this proposition.

(ii) The equality of $f^{Fermi}_{n_{max}}(\overline{S}_{t,w})$ and $f^{Fermi}_{n_{max}}(W_{t-w,w})$ will be shown to follow from a bijection ([11] Proposition 8.11) between the corresponding standard Young tableaux.

The above propositions extend earlier work [3], [4] where H was the graph $W_{\ell,w}$. In [3] these results were used to the prove and generalise certain results of Guttmann and Vöge [10] concerning the rationality of an anisotropic generating function for vicious walk configurations.

3 Maximal Fermi walk configurations and Young tableaux.

In this section a bijection between maximal Fermi walk configurations and standard Young tableaux is established. For a general reference on Young tableaux see Fulton's book [9].

The graphs $W_{\ell,w}$ and $\bar{W}_{\ell,w}$ may be considered to be cases of the general Young diagram denoted by Y.

Bijection 1. *There is a bijection between the maximal Fermi walk configurations, on the graph H and the standard Young tableaux on the corresponding Young diagram Y such that each walk, except the topmost, corresponds to an entry of the tableau.*

Fig. 2 (a) shows the bijection between the five maximal walk configurations on $W_{2,3}$ and the five standard tableaux. Note that in any one of these configurations each walk (except the topmost) differs by just two steps from the walk directly above it. The two steps are the sides of a unique cell and each cell is associated in this way with just one walk. Numbering the walks $0, 1, 2, \ldots 6$ from the top determines, by association, a number for each of the 6 cells of the Young diagram. By construction these numbers are increasing along the rows and columns to form a standard tableau. The generality of this construction is proved below. Fig. 2 (b) shows the correspondence between a maximal Fermi walk configuration on S_4 and a Young tableau on the diagram Y_4 (imagined). Notice that the fourth walk is obtained from the third by replacing its final left step by a down step and its number is placed to the left of the final downstep which, by construction of Y_4, is in the first cell of the second row of Y_4.

The proof of the bijection is constructive and requires the following two lemmas.

Lemma 1. *Any maximal Fermi walk configuration on the graph H covers all of its edges.*

Proof. Suppose there exists a maximal Fermi set of walks F which does not cover all the edges of H. An uncovered edge $[u, v]$ can be chosen such that the vertex u is on a walk $f \in F$. Now concatenate:

(i) a segment of f from the its initial vertex to u,

(ii) the edge $[u, v]$, and either,

(iii) a segment of a another walk of F from v to its final vertex, or,

(iii') a walk vertically down from v until it reaches a walk of F and then follows the segment of this walk to its final vertex.

This procedure gives a new Fermi set larger than F and therefore contradicts its maximality. □

Definition 2. *The operation whereby a walk is converted into a lower walk by replacing an adjacent left-down pair of steps by a down-left pair will be called a* **flop**. *In the case of a star configuration replacing a final left step of a walk by a down step will also be called a flop.*

Lemma 2. *Let $F = \{f_0, f_1, \dots\}$ be a maximal Fermi walk configuration with the walks listed in descending order of height then for $k \geq 1$, f_k differs from the walk f_{k-1} by just a single flop.*

Proof.
By Lemma 1 all of the edges are covered so f_k can only differ from f_{k-1} by a number of flops. The Fermi condition implies that f_k differs from the walk f_{k-1} by at least one flop. Suppose that f_k differs from f_{k-1} by more than one flop. A further walk may constructed from f_{k-1} by executing one of these flops. This walk uses a path distinct from that of f_{k-1} and lies above f_k which contradicts the maximality assumption. □

Proof of bijection 1.
→ Given a maximal Fermi walk configuration $F = \{f_0, f_1, \dots\}$ on the graph H, where the walks are listed in order of descending height, construct a labeling of the corresponding Young diagram Y as follows. By Lemma 2, for $k \geq 1$ the walk f_k contains a unique pair of steps obtained from f_{k-1} by a single flop. Label the cell of Y immediately to the left of the down step created by the flop by k. The last label will be $c(Y)$ since when all the cells have been labeled no more flops are possible. The resulting labelled diagram is a standard tableau since by construction all of the numbers $1, 2, \dots, c(Y)$ are used and the entries are increasing to the right and downwards by the ordering of the walks.

← Given a standard tableau on Y the corresponding maximal Fermi walk configuration F on the corresponding graph H is constructed as follows. f_0

is the uppermost walk. For $k = 1, 2, 3, \ldots, c(Y)$ construct f_k by flopping the steps (or step in case of a final step) of f_{k-1} which border the cell with entry k. By the increasing condition this will add a walk which uses a new path and avoids crossing previous walks. The resulting set therefore satisfies the Fermi condition and contains $c(Y) + 1$ walks. □

Proof of proposition 1. This proposition follows immediately from bijection 1. □

4 The enumeration of maximal Fermi walk configurations.

Propositions 2, 3 and 4 will now be established using bijection 1. That is the number of maximal Fermi walk configurations on H is determined by counting standard Young tableau on the corresponding diagram Y. This is achieved by using hook length formulae [8],[16]. The values of n_{max} follow immediately from proposition 1 by counting the number of cells in Y. In this section $k \in \{1, 2, \ldots w\}$ labels the rows of the Young tableaux contrary to its use as a cell entry in the previous section.

4.1 Watermelons

The following theorem is proved in [8].

Theorem 1. *The number of standard tableau on the normal Young diagram Y is given by the hook length formula $c(Y)!/\prod_{\sigma\in Y} h_\sigma$ where $c(Y)$ is the number of cells of Y and the hook length h_σ of cell $\sigma \in Y$ is 1 plus the number of cells to the right in the same row or below in the same column as σ.*

(i) Watermelons without a wall.

The hook lengths for $W_{6,4}$ are given in figure 3(a) from which the general case is clear. $c(W_{\ell,w}) = \ell w$ and the k^{th} row has ℓ terms which start at k and increase in unit steps. Thus

$$\prod_{\sigma\in W_{\ell,w}} h_\sigma = \prod_{k=1}^{w} (k)_\ell.$$

Theorem 1 together with bijection 1 derives proposition 2(a).

(ii) Watermelons above the wall $y = x$.

The hook lengths for $\overline{W}_{7,4}$ are given in figure 3(b). The product of hook lengths in the k^{th} row of the tableau $\overline{W}_{\ell,w}$ may be obtained by inserting the numbers $2, 4, \ldots, 2(k-1)$ into the row, the product of which is $2^{k-1}(k-1)!$. The product of factors in the row after insertion becomes $(2k+d-2)!$ and hence

$$\prod_{\sigma \in \overline{W}_{\ell,w}} h_\sigma = \prod_{k=1}^{w} \frac{(2k+d-2)!}{2^{k-1}(k-1)!} = \prod_{k=1}^{w} \frac{(k)_{k+d-1}}{2^{k-1}} \tag{4.1}$$

which together with Theorem 1 gives proposition 3(a).

4.2 Stars

The construction of hooks for shifted Young diagrams is described in [16].

Definition 3. *The hook of cell σ of a shifted Young diagram includes all cells in the same row and to the right of σ, or in the same column and below σ, σ included, but if this set contains a cell on the main diagonal, cell(k,k) say, then all the cells in the $(k+1)$-th row belong to the hook.*

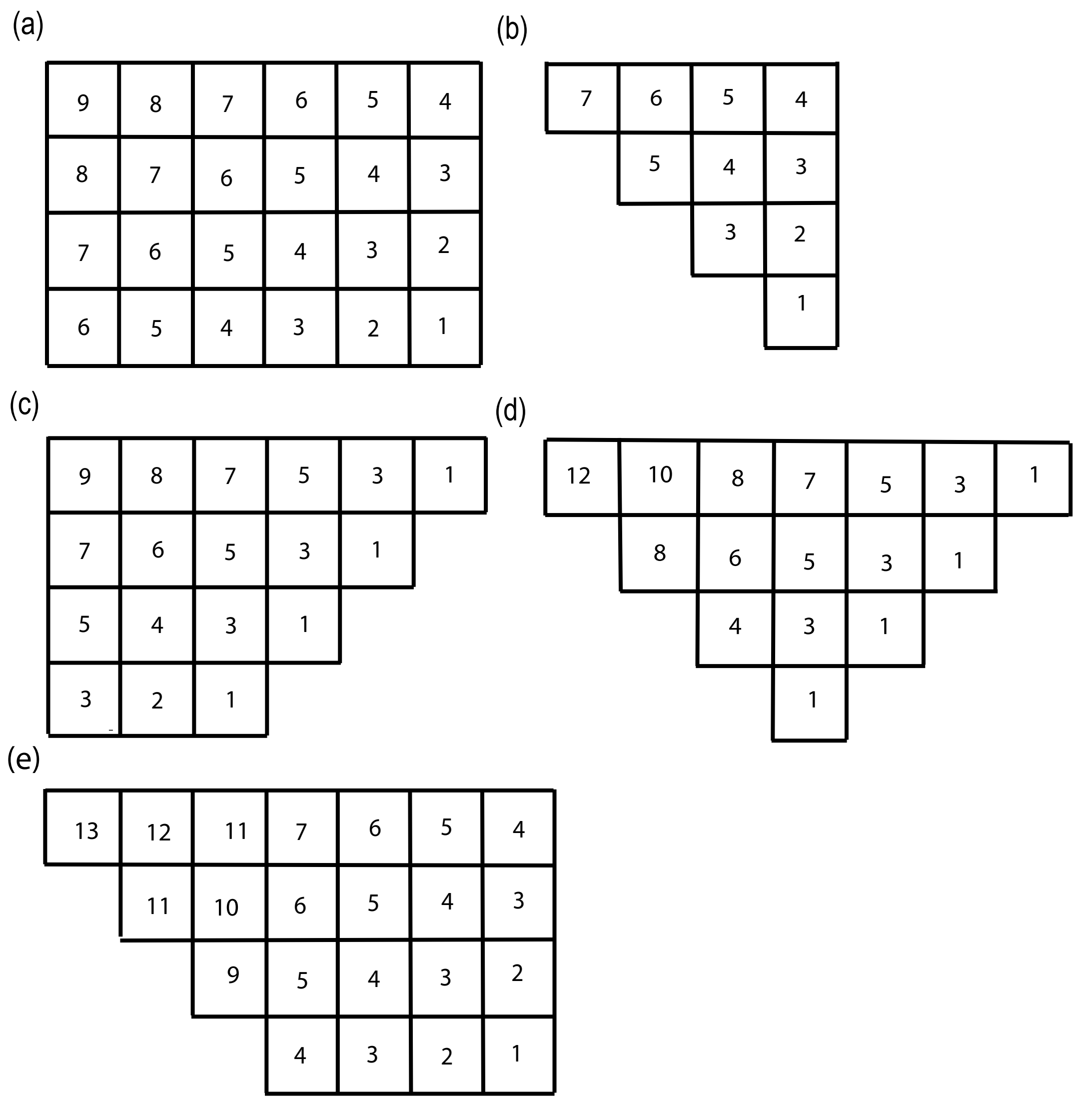

Figure 3: *Hook length tables for (a)* $W_{6,4}$ *(b)* Y_4 *(c)* $\overline{W}_{7,4}$ *(d)* $\overline{Y}_8$ *(e)* $Y_{7,4}$

(i) Stars without a wall.

A typical hook on the Young diagram for five step star walks in the absence of a surface is shown in figure 4(a). This may be converted into an inverted L shape, as shown in figure 4(b), having the same number of cells. This simplifies the hook length calculation and the hook lengths for S_4 are given in figure 3(c). The product of hook lengths arising from row k of Y_t is $(k)_k$. Substitution in the formula of Theorem 1 gives $f^{Fermi}_{n_{max}}(S_t)$ in proposition 2(b). Note that the hook lengths are a subset of those for watermelons obtained by omitting hook lengths in cells below the diagonal.

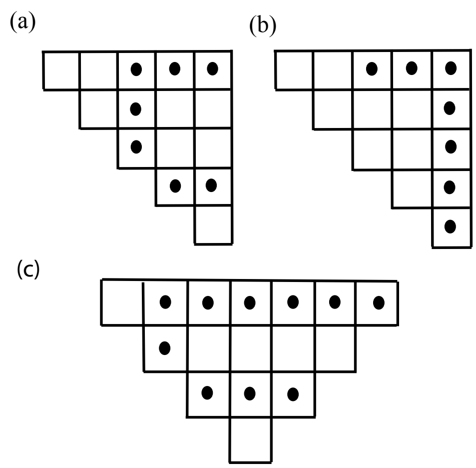

Figure 4: *Young diagrams for star configurations. (a) For five step walks showing the hook of cell (1,3), (b) the modified hook of the same length. (c) For eight step walks above a surface.*

(ii) Stars above the wall $y = x$.

In the presence of a wall it is necessary to distinguish configurations with even and odd numbers of steps.

First consider the even case $t = 2w$. Figure 4(c) shows a typical hook for eight step star walks above a wall. The general Young diagram $\overline{Y}_{2w}$ has trapezoidal shape with top row of length $2w-1$ and w rows. The rows reduce in length by steps of 2 until the last row has a single cell.

The hook lengths for $\bar{Y}_8$ are shown in figure 3(d). From figure 3 it is clear that the odd hook lengths on the right hand side and central column are the same as on the right hand side of the tableau for $\overline{W}_{7,4}$. In general the product of these lengths is therefore obtained by setting $d=1$ in (4.1) which gives $\prod_{k=1}^{w}(k)_k/2^{k-1}$.The lengths on the left side are even and the product may be written as $\prod_{k=1}^{w} 2^{k-1}(2k)_{w-k}$.

$$\prod_{\sigma\in\overline{Y}_{2w}} h_\sigma = \prod_{k=1}^{w}(2k)_{w-k}(k)_k = \prod_{k=1}^{w}(k)_w. \tag{4.2}$$

Using theorem 1 this reproduces (2.4).

Now consider the odd case $t=2w+1$.The general Young diagram $\overline{Y}_{2w+1}$ again has trapezoidal shape but with top row of length $2w$ and w rows. The rows reduce in length by steps of 2 until the last row has two cells. This time the last $w+1$ columns of $\overline{Y}_{2w+1}$ are the same as for $\overline{W}_{w+2,w}$ for which $d=2$ and from (4.1)the product of hook lengths from these columns is $\prod_{k=1}^{w}(k)_{k+1}/2^{k-1}$. The remaining $w-1$ columns have even hook lengths and removing factors of 2 leads to

$$\prod_{k=1}^{w-1}(2k+1)_{w-k}\prod_{k=2}^{w}2^{k-1} = \prod_{k=1}^{w}(2k+1)_{w-k}2^{k-1}$$

and hence the hook length product for $\bar{Y}_{2w+1}$ is

$$\prod_{\sigma\in\overline{Y}_{2w+1}} h_\sigma = \prod_{k=1}^{w}(k)_{k+1}(2k+1)_{w-k} = \prod_{k=1}^{w}(k)_{w+1}. \tag{4.3}$$

Using theorem 1 this reproduces (2.5).

The results for stars above the wall $y=x$ also follow from the following bijection which is part of proposition 8.11 of Haiman [11].

Proposition 5. *There is a bijection between standard tableaux of rectangular shape having w rows and ℓ columns with $w\le\ell$ and standard shifted tableaux of trapezoidal shape with top row of length $\ell+w-1$ and w rows.*

For the trapezoidal tableaux corresponding to $\bar{S}_{2w}$, $\ell = w$ and from the proposition these biject to those for $W_{w,w}$. For the trapezoidal tableaux corresponding to $\bar{S}_{2w+1}$, $\ell = w+1$ and from the proposition they biject to those for $W_{w+1,w}$. The equality of $f^{Fermi}_{n_{max}}(\overline{S}_{2w+m})$ and $f^{Fermi}_{n_{max}}(W_{w+m,w})$ for $m = 0, 1$ therefore follow from bijection 1 and proposition 5.

(iii) Stars with <= w down steps.

The product of hook lengths has two components. The first from the $t-w+1$ by w rectangle is the same as for $W_{t-w+1,w}$ which leaves an upper triangular section of width $w-1$. Numbering the rows of $Y_{t,w}$ from 1 to w from the bottom, row k of the triangular section has $k-1$ cells with contents increasing from $k+2(t-w)+1$.

$$\prod_{\sigma \in Y_{t,w}} h_\sigma = \prod_{k=1}^{w} (k)_{t-w+1}(k+2(t-w)+1)_{k-1} \tag{4.4}$$

using theorem 1 derives 4(a).

(iv) Stars with <= w down steps and above the wall $y = x$.

The Young diagram in this case has trapezoidal shape with w rows and $t-1$ cells in the top row. Bijection 1 together proposition 5 gives

$$f^{Fermi}_{n_{max}}(\overline{S}_{t,w}) = f^{Fermi}_{n_{max}}(W_{t-w,w}) \tag{4.5}$$

which is proposition 4(b)

5 Directed percolation and Fermi walk configurations

Let H be one of the graphs defined in the introduction, examples of which are shown in figure 1. Recall that the horizontal edges are considered to be directed to the left and the vertical edges are directed downwards. In directed bond percolation the edge e of H is open with probabilty p_e independently of the state of other edges. In site percolation the vertices are randomly deleted and p_v is the probability that vertex v is not deleted. More generally both vertices and edges may be deleted at random and we denote the vector of probabilities by $\mathbf{p}$.

Definition 4. *The connectedness function $C(H;\mathbf{p})$ for the graph H is the probability that at least one of the terminal vertices of H is connected by one or more open paths from the origin.*

Note: The functions $C(W_{\ell,w};\mathbf{p})$ and $C(\bar{W}_{\ell,w};\mathbf{p})$ are the pair connectedness functions with and without a wall. These are also the pair connectedness functions for the site $(-\ell,-w)$ of the infinite directed square lattice since there are no directed paths on the lattice which leave $W_{\ell,w}(\bar{W}_{\ell,w})$.

Inui and Katori [12] considered the walkers to be interacting polymer chains and defined a canonical partition function as a sum of Boltzmann factors over all configurations of n walks. The Boltzmann weight included a factor $\exp(-1/(k_BT))$ for each edge used by at least one of the walkers. Let $F_n(H)$ be the set of all Fermi configurations of n walks and for $\eta \in F_n(H)$ let $G(\eta)$ be the subgraph formed by the union of all walks in η. We then define a canonical partition function by

$$Z_n^{Fermi}(H;\mathbf{p}) \equiv \sum_{\eta \in F_n(H)} \prod_{e \in E(\eta)} p_e \prod_{v \in V(\eta)} p_v \tag{5.1}$$

where $E(\eta)$ and $V(\eta)$ are respectively the vertex and edge sets of $G(\eta)$. The partition function of Inui and Katori is obtained by setting $p_e = p = \exp(-1/(k_BT))$ and $p_v = 1$. Thus as the absolute temperature T varies from zero to infinity p increases from zero to one.

The following theorem is a generalisation of that proved by Inui and Katori [12] who considered the function $C(W_{\frac{1}{2}t,\frac{1}{2}t},\mathbf{p})$ with $p_e = p$ and $p_v = 1$ (see below).

Theorem 2. *The connectedness function of the graph H is related to the Fermi partition function by*

$$C(H;\mathbf{p}) = \sum_{n=1}^{n_{max}} (-1)^{n+1} Z_n^{Fermi}(H;\mathbf{p}). \tag{5.2}$$

Proof. Let $\mathcal{P}(H)$ be the set of all directed paths from the origin to some terminal vertex of H. Then, by inclusion and exclusion,

$$\begin{aligned} C(H;\mathbf{p}) &= Pr(\text{at least one path of } \mathcal{P}(H) \text{ is open }) \\ &= \sum_{\phi\subset\eta\subseteq\mathcal{P}(H)} (-1)^{n(\eta)+1} \prod_{e\in E(\eta)} p_e \prod_{v\in V(\eta)} p_v. \end{aligned} \tag{5.3}$$

where $n(\eta)$ is the number of paths in the subset η.

Notice that Fermi walk configurations are the non-crossing subsets of $\mathcal{P}(H)$ so the theorem is proven provided that we can show that subsets which contain crossing paths can be omitted from the sum. To do this we split the crossing subsets into pairs of equal weight one with an odd number of paths and one with an even number.

Let ξ be a subset with crossing paths and let u be a crossing point of ξ which is the least number of steps away from the origin. In the case of more than one such crossing point choose u to be the one which is the least number of vertical steps away. Let ξ_u be the subset of (two or more) paths of ξ which cross at u. Let G_u be the subgraph of H which is the union of paths in ξ_u. The subset ξ may or may not contain the uppermost path π on G_u. If it does then pair ξ with the configuration obtained by deleting π. If not then pair ξ with the subset obtained by adding π. This pairing has the required properties. We may therefore replace $\mathcal{P}(H)$ in (5.3) by $F(H)$, the set of all Fermi path configurations on H. Partitioning $F(H)$ by the number of walks establishes the theorem. □

Note: Since maximal Fermi walk configurations cover the whole of the graph H the last term in the sum (5.2) is

$$Z^{Fermi}_{nmax}(H,\mathbf{p}) = f^{Fermi}_{nmax}(H) \prod_{e\in E(H)} p_e \prod_{v\in V(H)} p_v \tag{5.4}$$

The proof of (5.2) by Inui and Katori [12] was for bond percolation on $H = W_{\frac{1}{2}t,\frac{1}{2}t}$ with $p_e = p$. They showed that if F_t is the set of all Fermi walk configurations on $W_{\frac{1}{2}t,\frac{1}{2}t}$ then

$$\sum_{n=1}^{n_{max}} (-1)^{n+1} Z^{Fermi}_n(W_{\frac{1}{2}t,\frac{1}{2}t};p) = \sum_{\eta\in F_t} (-1)^{c(\eta)} p^{e(\eta)} \tag{5.5}$$

where $e(\eta)$ and $c(\eta)$ are the numbers of edges and independent cycles in $G(\eta)$ and $n_{max} = 1 + (\frac{1}{2}t)^2$. They then noted that this was the formula of

Arrowsmith and Essam [1] for the pair connectedness of $W_{\frac{1}{2}t,\frac{1}{2}t}$. Combining these two proofs gives an alternative proof of the AE formula (5.5).

The percolation probability $P(\mathbf{p})$ of the square lattice is the probability that the origin belongs to an infinite cluster thus

$$P(\mathbf{p}) = \lim_{t\to\infty} C(S_t;\mathbf{p}) \tag{5.6}$$

Inui et all [13] conjectured the following surprising result for bond percolation ($p_e = p, p_v = 1$) and site percolation ($p_e = 1, p_v = p$) by comparison of series expansions.

$$\lim_{t\to\infty} C(W_{\frac{1}{2}t,\frac{1}{2}t},\mathbf{p}) = P(\mathbf{p})^2 \tag{5.7}$$

This result was subsequently proved by Katori et al [14]. To validate the result for both bond and site problems the site at the origin must always be present or a compensating factor of p must be included for the site problem.

6 Conclusion

We have developed the combinatorics of maximal Fermi walk configurations on the directed square lattice and exhibited their equivalence with standard Young tableaux. This equivalence is our main result since all our other results depend on it. Enumerations for watermelon and star configurations have been carried out for various boundary conditions by use of hook length formulae for the numbers of tableaux.

The unexpected equality of proposition 3(b) was seen to be a result of a bijection, due to Haiman [11], between the corresponding tableaux. A similar result for Bose configurations exists but in order to obtain a bijective proof it would be necessary to find a bijection which preserves ascents in the tableaux. Equation 5.7 for directed percolation is reminiscent of proposition 3(b).

Inui and Katori [12] constructed partition functions for configurations of n Fermi walks considered as interacting polymer chains and showed that these could be used to evaluate the pair connectedness for the directed bond percolation. The probability p for an open bond in percolation corresponded to the Boltzmann factor in the partition function. The number of maximal Fermi configurations contributes to the coefficient of the highest highest power of p in the pair connectedness polynomial. Corresponding formulae for the coefficients of the other terms of the polynomial would require similar successful analysis for the non-maximal Fermi-walk configurations.

We have also obtained an alternative and more direct proof of the Inui-Katori expression (5.2) for the connectedness function in terms of Fermi partition functions. The proof applies to the more general graphs H considered in this paper and in particular the star graph formulae determine the percolation probability.

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