ROYAL INSTITUTION MASTERCLASS

INFINITY and Beyond

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Sets and notation

A set is a collection of objects (or elements).

Let us introduce symbols so that we can describe such a structure. If the set X is the collection of four objects a, b, c and d. We write

$$X = \{a, b, c, d\}.$$

This is called a **listing** of the set X.
The listing within the "curly brackets" is the set of **elements** of X.

If x is an element of the set X, we write $x \in X$.

If, for example, $X = \{a, b, c, d\}$, then we can write $a \in X$.
Similarly, $b \in X$, $c \in X$ and $d \in X$.

The set with no elements is called the *empty set* and denoted by $\emptyset$.

So $\emptyset = \{\ \}$.

Examples of sets are obvious

1. $X_1 = \{1, 2, 3, 4\}$
2. $X_2 = \{1, 2, 3, 4, 10\}$
3. $X_3 = \{Monday, Tuesday, Wednesday\}$
4. $X_4 = \{Mercury, Venus, Earth, Mars, Jupiter\}$
5. $X_5 = \{Mercury, Monday\}$

Exercise

Distinguish between the following four statements:
$5,\ \{5\},\ \{\{5\}\},\ \{5, \{5\}\}$.

Further set notation

If every element of a set A is also an element of the set X, then A is said to be a *subset* of the set X, or equivalently X is a *superset* of A.

We write $A \subseteq X$ or $X \supseteq A$.

For example: $A = \{1, 2\}$ is a subset of $X = \{1, 2, 3\}$ or, EQUIVALENTLY, X is a superset of A.

If we know that $A \neq X$, and $A \subseteq X$, we say A is a *proper* subset of X and we write $A \subset X$

Note X is also a subset of X, i.e. a set is always a subset of itself, but it not a proper subset!

Some KEY sets

Natural numbers $\mathbb{N}$

the set of **natural** or **counting** numbers $\mathbb{N}$ which can be listed as $\{1, 2, 3, \ldots\}$;

Integers $\mathbb{Z}$

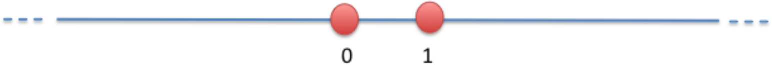
the set of **integers** $\mathbb{Z}$ which can be listed as $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$ or $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

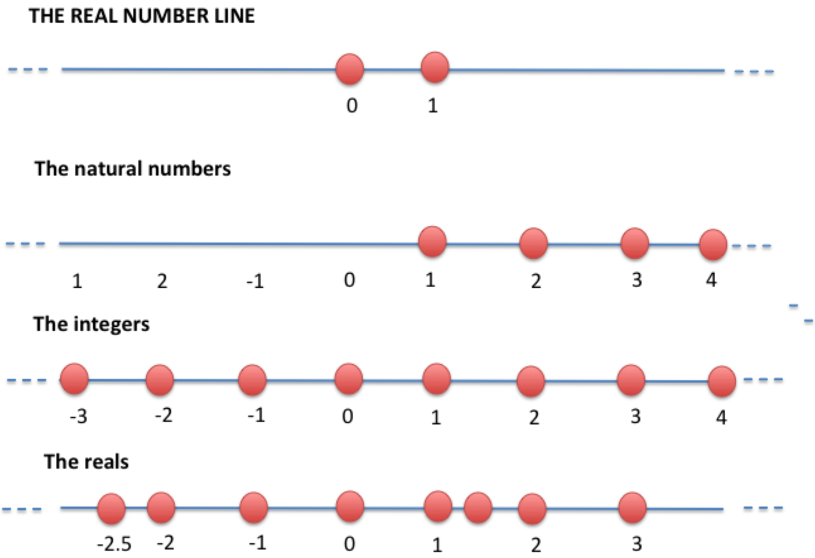
Real numbers $\mathbb{R}$

the set of **reals** $\mathbb{R}$ which is the continuum of positive, negative and zero distances on the real number line.

All three sets can be represented as points on the **real number line** $\mathbb{R}$.

THE REAL NUMBER LINE

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The reals
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Listing and describing sets

Listing sets

This is when we write out the list of elements separated by commas which form the set X. For example $X = \{a, b, c\}$.

Describing sets and qualifiers

Sometimes, instead of listing all the elements, we write a set in the form $X = \{x|p(x)\}$, which reads "X is the set of elements x from some universal set U for which the statement $p(x)$ is true". The statement $p(x)$ is the qualifier for the element x to belong to the set X.

Example Let $U = \mathbb{N}$ and give the listing of the set
$X = \{x \mid x^2 \text{ is less than } 100\}$.
$X = \{1, 2, 3, \ldots, 9\}$.

Describing sets

Exercise - list the elements of each set X described below

1. Let $U = \mathbb{Z}$ and give the listing of the set
   $X = \{x \,|\, x^2 \text{ is less than } 100\}$.
   $X = \{0, \pm 1, \pm 2, \pm 3, \dots, \pm 9\}$.
2. Let $U = \mathbb{N}$ and give the listing of the set
   $X = \{x \,|\, x^2 \text{ is less than } 100; \}$
3. Let $U = \mathbb{Z}$ and give the listing of
   $X = \{x \,|\, \mathrm{x} \text{ is one of the first four positive integers}\}$;
4. Let $U = \mathbb{Z}$ and give the listing of
   $X = \{x \,|\, x \text{ is an even integer such that } 17 < |x| < 28\}$;

Some more examples

1. X is the set of the possible outcomes of tossing a coin three times (heads(H) or tails(T));
2. X is the set of the integer solutions x of the equation $x^2 - 3x + 2 = 0$;
3. X is the set of the integer solutions of $x^3 - 3x + 2 = 0$;
4. X is the set of the integer solutions of $x^3 - 5x + 4 = 0$;
5. X is the set of the possible outcomes of tossing a pair of coins twice;
6. X is the set of the integer solutions of $x + 1 = x$.

What are some universal sets that could be chosen in each of these cases?

Write the following sets in "statement" form

1. $\{1, 3, 5, 7, 9\}$; U is the set of integers; OR ....
2. $\{Jack, Queen, King\}$;U is a suite of 13 playing cards, e.g. Diamonds
3. {cube, tetrahedron, dodecahedron}; U is the set of regular solids

Function

A function relates elements of two sets , say X and Y, in a special way. A function f is often denoted by $f : X \to Y$, meaning that for each element $x \in X$, the function f produces an element $y \in Y$.

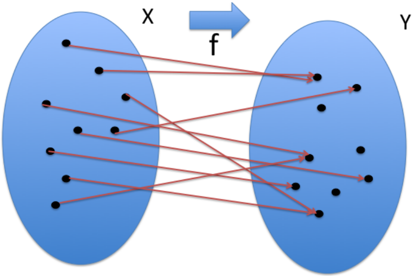
We write $f(x) = y$.

You can think of a function as behaving like a black (or BLUE!) box with an input x and an output y.

You input an element of X into the blue box, and it outputs an element of Y. It produces an element of Y for every element of X, and every time you input a particular element of X, you always obtain the same element of Y.

Venn diagram illustration of a function

The red arrows indicate that an element x of the set X is mapped to a element y of the set Y. Note for every $x \in X$ there is exactly one arrow which goes to a point of Y. This is showing that given x, the function f produces just one y. Also, note that not all elements $y \in Y$ are realised as an $f(x)$.



Sometimes the function f can be described by a formula, but not always.

Examples

(i) $X = \mathbb{Z}$ and $Y = \mathbb{N}$. Define $f : X \to Y$ by $x \mapsto x^2$. Thus $f(-2) = 4$; $f(9) = 81$.
(ii) $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$, and $f : X \to Y$ defined by $f(1) = c$, $f(2) = b, f(3) = c$, $f(4) = a$.
Note in example (i), not all elements of Y arise as *images* of elements of X by the function f. In example (ii), all elements of Y are obtained by applying f.

onto function

A function $f : X \to Y$ is said to be **onto** if for every element of $y \in Y$, there is an $x \in X$ such that $f(x) = y$. In the example above (ii) is onto, and (i) is not onto. The idea of the word *onto* is that f maps X onto the *whole* set of elements of Y.

1-1 function

A function is said to be **one-to one** or 1-1 if for every pair of distinct points $x, x' \in X$, the points $f(x)$ and $f(x')$ are also distinct.

Exercise

Can you find examples of functions $f : X \to Y$ which are different combinations of onto and 1-1.
Give 4 examples as Venn diagrams: f is (i) onto and 1-1, (ii) not onto but 1-1, (iii) not 1-1, but onto, and (iv) neither onto nor 1-1?
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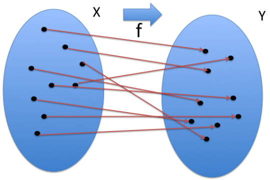
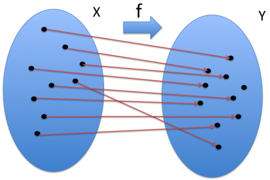
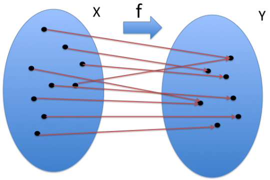


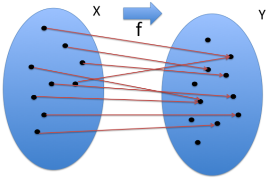
(i) function; 1-1 ; onto



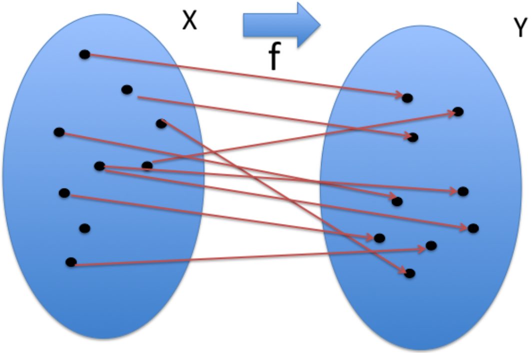
(ii) function; 1-1, not onto



(iii) function; not 1-1; onto



(iv) function; not onto ; not 1-1



not a function - give 2 reasons why it is not a function

1-1 and onto

If $f : X \to Y$ is both 1-1 and onto, then it is called a **1-1 correspondence**.

Why call f a 1-1 correspondence?

Consider the set of pairs $(x, f(x))$ for each $x \in X$. Then for every $x \in X$, there is a unique $y = f(x) \in Y$. Given $y \in Y$, since f is onto, there is an $x \in X$ such that $f(x) = y$. So for every y there is an x. Moreover the x is unique, there is only one such x since f is 1-1.

Therefore the pairs $(x, f(x))$ form a correspondence of unique pairings of ALL the elements of X with ALL those of Y.

In that sense, the sets X and Y are seen to have the number of elements or *cardinality*

Cardinal number of a set

Consider the set $S_n = \{1, 2, \ldots, n\}$, the first n-integers. We say that the **cardinal number** of the set S_n, that is $\sharp(S_n)$ is the number of elements in the set S_n and that is denoted by n. Every set in 1-1 correspondence with S_n has cardinal number n.

We write $\sharp(S_n) = n$ which reads "the cardinal number of S_n is n", or the number of elements in the set S_n is n.

Exercise

Show that $T = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$ has cardinal number n. To do this you need to find a 1-1 correspondence between the sets S_n and T.

the 1-1 correspondence

the 1-1 correspondence that we need is $f : S_n \to T$ given by $k \mapsto 1/k$, i.e. $f(k) = 1/k$, for $k = 1, \dots, n$. So $\sharp(T) = n$

Infinite sets

An **infinite** set is one that is not finite! - that is a set that is not in 1-1 correspondence with S_n for some n.

The set $\mathbb{N}$ is an infinite set

What is its cardinality? - well it is NOT any integer n.

Infinite cardinal

Let us use the symbol $\aleph_0$ for the cardinality of $\mathbb{N}$. The set $\mathbb{N}$ is said to **countably infinite** because the element can be counted in a single list.

Hilbert's paradox-or "infinite sets are not what they seem!"

From Wikipedia

Hilbert's paradox of the Grand Hotel is a *veridical* paradox (a valid argument with a seemingly absurd conclusion), as opposed to a *falsidical* paradox, (a seemingly valid demonstration of an actual contradiction) about infinite sets presented by David Hilbert in the 1920s, meant to illustrate certain counterintuitive properties of infinite sets.

Finitely many new guests

Suppose the hotel is FULL - that is every room has a guest. Then a new guest arrives and wishes to be accommodated in the hotel. Because the hotel has infinitely many rooms, we can move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3 and so on, and fit the newcomer into room 1. By repeating this procedure, it is possible to make room for any finite number of new guests.

Exercise With Infinitely many new guests, can we accommodate them?

Suppose the list of guests already in the hotel is $\{g_1, g_2, \ldots, g_n, \ldots\}$ and the infinite list of new guests is $\{ng_1, ng_2, ng_3, \ldots\}$.
How do we reshuffle the room allocation to accommodate the new guests?

Arithmetic of infinities

Is there an arithmetic for cardinal numbers? Can we add and multiply cardinal numbers?

How do we add finite cardinals?

Let X and Y be sets with an empty intersection then

$$\sharp(X \cup Y) = \sharp(X) + \sharp(Y).$$

Exercise

Can you prove that $\aleph_0 + 1 = \aleph_0$?
Can you prove that $\aleph_0 + \aleph_0 = \aleph_0$?

Exercise

Show that if X is a finite set and $A \subset X$, then $\sharp(A) < \sharp(X)$.

The cartesian product $X \times Y$ of two sets X and Y

$X \times Y = \{(x, y) | x \in X, y \in Y\}$.

Example Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$.
The set $X \times Y$ can be written as the array

$$\left\{ \begin{array}{cc} (1, a), & (1, b), \\ (2, a), & (2, b), \\ (3, a), & (3, b) \end{array} \right\}$$

Note $\sharp(X \times Y) = \sharp(X) \times \sharp(Y)$, i.e. $6 = 3 \times 2$!!

Exercise What is the set $X \times Y$?

(i) $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3\}$;
(ii) $X = \mathbb{N}$ and $Y = \{1, 2, 3\}$.

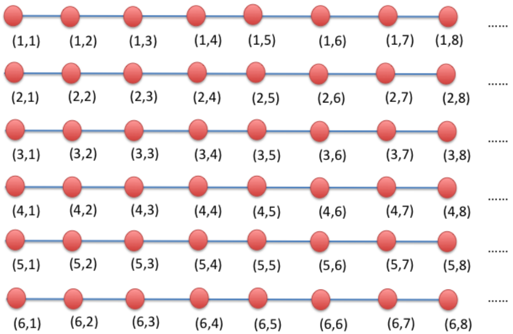
How do we multiply cardinals?

Definition: Let X and Y be sets then

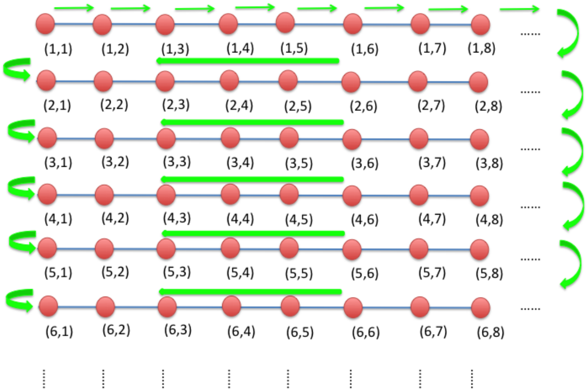
$\sharp(X) \; x \; \sharp(Y) = \sharp(X \times Y)$

$\sharp(\mathbb{N} \times \mathbb{N}) = \sharp(\mathbb{N}) \times \sharp(\mathbb{N}) = \aleph_0 \times \aleph_0 = \aleph_0^2$

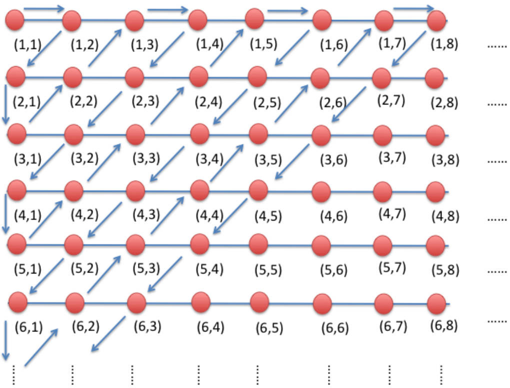
Exercise. Can you prove $\aleph_0^2 = \aleph_0$? Here is $\mathbb{N} \times \mathbb{N}$.



Exercise Is this a way to count $\mathbb{N} \times \mathbb{N}$?



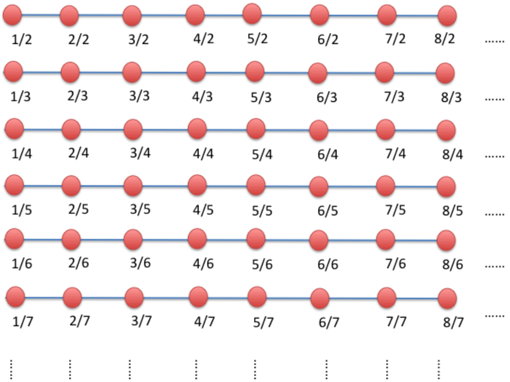
Exercise So what is the best way to count $\mathbb{N} \times \mathbb{N}$?

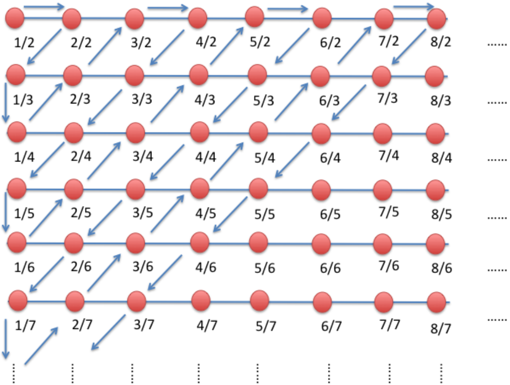


Can you think of another way to count $\mathbb{N} \times \mathbb{N}$?

Exercise What is the cardinality of the rational numbers $\mathbb{Q}$?

Can you show how the rationals can be put into a single list so as to be counted?

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The Power set $P(X)$ of the set X is the set of all subsets of X

Exercise How many elements are there in $P(X)$ if $\sharp(X) = n$. Try for $n = 0, 1, 2, 3 \ldots$

The power set $P(X)$ is larger than the set X for finite sets

$\sharp P(X) = 2^n$ if $\sharp(X) = n$ and $2^n > n$ for $n = 0, 1, 2, \ldots$

CANTOR's THEOREM

$\sharp P(X) > \sharp X$ for INFINITE sets too
So there are different infinities - there is, in fact
AN INFINITY of INFINITIES.

Is $\sharp P(\mathbb{N}) = \sharp\mathbb{N}$?

Suppose there is 1-1 correspondence between $\mathbb{N}$ and $P(\mathbb{N})$

$$\mathbb{N} \longleftrightarrow P(\mathbb{N})$$

$$1 \longleftrightarrow \{1, 2, 3\}$$

$$2 \longleftrightarrow \{1, 3, 4\}$$

$$3 \longleftrightarrow \{3, 4, 5\}$$

$$4 \longleftrightarrow \{1, 3, 5, 7, \dots\}$$

$$5 \longleftrightarrow \{3, 4\}$$

$$\vdots \longleftrightarrow \vdots$$

$$\vdots \longleftrightarrow \vdots$$

If $n \in \mathbb{N}$ corresponds with a subset of $\mathbb{N}$ containing n ,
e.g. $1 \longleftrightarrow \{1, 2, 3\}$
the integer n is called SELFISH ,

so $n = 1$ is SELFISH;

$n = 2$ is NON-SELFISH;

$n = 3$ is . . . , etc.;

Let D be the set of all NON-SELFISH(blue) integers from the previous slide and let $d \longleftrightarrow D$ in the 1-1 correspondence between $\mathbb{N}$ and $P(\mathbb{N})$.

Is d selfish or not?

OK, so assume d is selfish - then $d \longleftrightarrow D$ and so $d \in D$, therefore d is non-selfish. !!!!

CONTRADICTIONS every way we go !!!

What is the way out of this?

There is **NOT** a 1-1 correspondence between $\sharp P(\mathbb{N})$ and $\sharp\mathbb{N}$

So $\sharp P(\mathbb{N}) > \sharp\mathbb{N}$ and $\sharp P(\mathbb{N})$ is a NEW infinite cardinal.

$2^{\aleph_0} = \aleph_1 \neq \aleph_0$

$2^{\aleph_1} = \aleph_2 \neq \aleph_1$

$2^{\aleph_2} = \aleph_3 \neq \aleph_2$

So why is $\sharp(\mathbb{R}) = \aleph_1$? Think DECIMALS!!!!!!!

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