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Spectrum of a homogeneous graph

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ABSTRACT

We describe the spectrum of the Laplacian for a homogeneous graph acted on by a discrete group. This follows from a more general result which describes the spectrum of a convolution operator on a homogeneous space of a locally compact group. We also prove a version of Harnack inequality for a Schrödinger operator on an invariant homogeneous graph.

1. Introduction

In a weighted graph (V, E), finite or infinite, let d_v and $w : V \times V \to [0, \infty)$ denote respectively the degree of a vertex $v \in V$ and the weight $w(v, u) = w(u, v)$, satisfying $d_v = \sum_u w(v, u) < \infty$. The *Laplacian* $\mathcal{L}$, acting on real or complex functions f on V, is defined by

$$\mathcal{L} f(v) = f(v) - \sum_{\substack{u \\ (v,u)\in E}} \frac{f(u)w(v,u)}{\sqrt{d_v d_u}} \quad (v \in V).$$

An important problem in spectral geometry is the estimation of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$. It is known, for instance, that $1 - \sqrt{1 - h^2}$ is a lower bound for the positive eigenvalues where h is the Cheeger constant of the graph [5,10,12,14,16].

In this paper, we give a full description of the spectrum $\sigma(\mathcal{L})$ for a homogeneous graph under some weight condition.

We call (V, E) a *homogeneous graph* (cf. [5]), if the vertex set V is a homogeneous space of a discrete group G with a graph condition, by which we mean G acts transitively on V by a right action $(v, g) \in V \times G \mapsto vg \in V$ so that V is represented as a right coset space G/H of G by a finite subgroup H and the edge set E is described by a finite subset $K = K^{-1} \subset G$ in that $(v, u) \in E$ if and only if $u = va$ for some $a \in K$. Henceforth we denote a homogeneous graph by (V, K), with the edge generating set K having finite cardinality $|K|$. We note that (V, K) is a Cayley graph if H reduces to the identity of G, in which case we write (G, K) for the graph. Although one can consider a more general notion of a homogeneous graph $(G/H, K)$ in which the isotropy subgroup H can be infinite, we only consider this case in the last section of the paper.

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The Laplacian for a weighted homogeneous graph (V, K) can be written as

$$\mathcal{L} f(v) = f(v) - \frac{1}{|K|}\sum_{a\in K} f(va)w(v, va) = \frac{1}{|K|}\sum_{a\in K}\big(f(v) - f(va)\big)w(v, va) \quad (v \in V).$$

We describe the spectrum of $\mathcal{L}$ completely in terms of irreducible representations of G when the weight w is given by a measure μ on G which is symmetric and constant on each set aHb, that is, $w(Ha, Hb) = \mu(a^{-1}b) = \mu(b^{-1}a)$ and $\mu(acb) = \mu(ab)$ for all $c \in H$. A weight w is given by such a measure μ if $w(v, va) = w(u, ua)$ for $u, v \in V$ and $a \in K$, in which case μ is a measure supported by K. For instance, for unweighted graphs, we have $w(v, va) = 1$.

In fact, we prove a more general result for the L^2-spectrum of a convolution operator on the homogeneous space of a locally compact group G by a compact subgroup H, which is of independent interest and includes the above Laplacian as a special case. We note that the connection between a finite homogeneous graph Laplacian and group representations has been discussed in [5, p. 117] and [6]. Our result for convolution operators involves group C^*-algebras and applies to infinite graphs as well.

A homogeneous graph (V, K) is called *invariant* in [7] if G acts on V as automorphisms of V and $aK = Ka$ for all $a \in K$. We characterize the invariance of (V, K) in terms of group structures and show that all positive $\mathcal{L}$-harmonic functions on a connected invariant graph are constant. A Harnack inequality has been proved in [7] for the Laplacian $\mathcal{L}$ of an invariant unweighted homogeneous graph. We extend this Harnack inequality for a Schrödinger operator $\mathcal{L} + \varphi$ on an invariant homogeneous graph.

2. Convolution operators on homogeneous spaces

Let G be a locally compact group with identity e and a right invariant Haar measure λ. Let G act transitively on a locally compact Hausdorff space V by a (continuous) right action

$$(v, g) \in V \times G \mapsto vg \in V$$

such that V is represented as a *right* coset space G/H of G by a compact subgroup H of G and the action identifies with the natural action of G on G/H by right multiplication. In this case, $V = G/H$ admits a G-invariant measure ν satisfying $\nu = \lambda \circ q^{-1}$ where $q : G \to G/H$ denotes the quotient map throughout (cf. [11, p. 58]).

For $1 \leqslant p \leqslant \infty$, let $L^p(G/H)$ be the complex Lebesgue space of p-integrable functions on G/H with respect to ν, and write $L^p(G)$ for $H = \{e\}$, also $\ell^p(G)$ for a discrete group G. We note that $L^1(G)$ has an involution

$$f^*(x) = \overline{f\big(x^{-1}\big)}\Delta\big(x^{-1}\big) \quad (x \in G)$$

where Δ is the modular function of G.

Let $M(G)$ be the Banach algebra of complex Borel measures on G, with the total variation norm, in which the product of two measures $\mu, \mu' \in M(G)$ is given by convolution:

$$\int_G f\, d(\mu * \mu') = \int_G \int_G f(xy)\, d\mu(x)\, d\mu'(y)$$

for each continuous function f on G vanishing at infinity. The convolution $h * \mu$ for $h \in L^p(G)$ is defined by $h * \mu(x) = \int_G h(xy^{-1})\, d\mu(y)$.

A measure $\mu \in M(G)$ is called *absolutely continuous* if its total variation $|\mu|$ is absolutely continuous with respect to the Haar measure λ, in which case μ has a density $f \in L^1(G)$ so that $\mu = f \cdot \lambda$. We call μ *symmetric* if $d\mu(x) = d\mu(x^{-1})$. The unit mass at a point $a \in G$ is denoted by δ_a.

Given $\mu \in M(G)$, we define the convolution operator $L_\mu : L^p(G/H) \to L^p(G/H)$ by

$$(L_\mu f)(Hx) = \int_G f\big(Hxy^{-1}\big)\, d\mu(y) \quad \big(f \in L^p(G/H)\big).$$

This operator is well defined by G-invariance of the measure ν and we have $\|L_\mu\| \leqslant \|\mu\|$. We note that L_μ is a self-adjoint operator on the Hilbert space $L^2(G/H)$ if μ is symmetric.

Our first task is to describe the spectrum of $L_\mu : L^2(G/H) \to L^2(G/H)$ for an absolutely continuous symmetric measure μ. For this, we develop a device to identify L_μ as an element in a quotient of the group C^*-algebra $C^*(G)$ which then enables us to use spectral theory of C^*-algebras to conclude the result.

We recall that the group C^*-algebra $C^*(G)$ of G is the completion of $L^1(G)$ with respect to the norm

$$\|f\|_c = \sup_\pi\big\{\big\|\pi(f)\big\|\big\}$$

where the supremum is taken over all $*$-representations $\pi : L^1(G) \to B(H_\pi)$, the latter denotes the algebra of all bounded operators on the Hilbert space H_π. If G is discrete, then $C^*(G)$ contains an identity.

Let $\rho : C^*(G) \to B(L^2(G))$ be the right regular representation given by

$$\rho(f)h = h * f \quad \big(f \in L^1(G),\ h \in L^2(G)\big)$$

which is an extension of the right regular representation $a \in G \mapsto \rho(a) \in B(L^2(G))$ of G, where $\rho(a)h = h * \delta_a$. The reduced group C^*-algebra $C_r^*(G)$ is the norm closure $\overline{\rho(L^1(G))} = \rho(C^*(G))$.

We have two natural well-defined continuous linear maps $j : L^2(G/H) \to L^2(G)$ and $Q : L^2(G) \to L^2(G/H)$ given by

$$j(f) = f \circ q, \qquad Qg(Hx) = \int_H g(\xi x)\,d\xi \quad \big(f \in L^2(G/H),\ g \in L^2(G)\big)$$

where $d\xi$ is the normalized Haar measure on the compact group H (cf. [3]).

There is a natural continuous linear map $\Phi : B(L^2(G)) \to B(L^2(G/H))$ given by the following diagram:

$$\begin{tikzcd} L^2(G) \arrow[r, "L"] & L^2(G) \arrow[d, "Q"] \\ L^2(G/H) \arrow[u, "j"] \arrow[r, "\Phi(L)"] & L^2(G/H) \end{tikzcd}$$

that is,

$$\Phi(L) = Q \circ L \circ j \tag{1}$$

for each $L \in B(L^2(G))$. We define a unitary representation $\tau : G \to B(L^2(G/H))$ by right translation:

$$\tau(a)f(Hx) = f\big(Hxa^{-1}\big) \quad \big(a, x \in G,\ f \in L^2(G/H)\big).$$

We can extend τ to a representation $\rho_H : C^*(G) \to B(L^2(G/H))$ in the usual way (cf. [13, p. 229]).

Lemma 2.1. *Let $\rho : C^*(G) \to B(L^2(G))$ be the right regular representation and let $\Phi : B(L^2(G)) \to B(L^2(G/H))$ be the map defined in* (1)*. Then the diagram*

$$\begin{tikzcd} C^*(G) \arrow[rr, "\rho_H"] \arrow[dr, "\rho"'] & & B(L^2(G/H)) \\ & B(L^2(G)) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

is commutative.

Proof. For $f \in L^1(G)$ and $g \in L^2(G/H)$, we have

$$\Phi(\rho f)(g) = Q(\rho f)j(g) = Q\big(\rho f(g \circ q)\big) = Q\big((g \circ q) * f\big)$$

and

$$\begin{aligned} Q\big((g \circ q) * f\big)(Hx) &= \int_H (g \circ q) * f(\xi x)\,d\xi = \int_H \int_G (g \circ q)\big(\xi x y^{-1}\big) f(y)\,d\lambda(y)\,d\xi = \int_H \int_G g\big(Hxy^{-1}\big) f(y)\,d\lambda(y)\,d\xi \\ &= \int_G g\big(Hxy^{-1}\big) f(y)\,d\lambda(y) = g * f(Hx) = \rho_H(f)(g)(Hx). \end{aligned}$$

Hence $\Phi(\rho f) = \rho_H(f)$. $\square$

Lemma 2.2. *Let $\mu \in M(G)$ be absolutely continuous with $\mu = f \cdot \lambda$ and $f \in L^1(G)$. Then $\rho_H(f) = L_\mu \in B(L^2(G/H))$.*

Proof. We have

$$\rho_H(f)h = \int_G (h * \delta_x) f(x)\,d\lambda(x) \in L^2(G/H) \quad \big(h \in L^2(G/H)\big)$$

and

$$\rho_H(f)h(Hy) = \int_G (h * \delta_x)(Hy) f(x)\,d\lambda(x) = \int_G h\big(Hyx^{-1}\big) f(x)\,d\lambda(x) = (h * f)(Hy) = L_\mu(h)(Hy). \qquad \square$$

Let $\widehat{G}$ be the dual space of G, consisting of (equivalence classes of) continuous irreducible unitary representations of G. If G is abelian, then $\widehat{G}$ is the character group of G.

The *spectrum* of a C^*-algebra A is defined to be the space $\widehat{A}$ of (equivalence classes) of irreducible representations $\pi : A \to B(H_\pi)$ of A [9, 3.1.5]. The spectrum $\widehat{C^*(G)}$ identifies with $\widehat{G}$ [9, 13.93] where each $\pi \in \widehat{G}$ is identified as the irreducible representation of $C^*(G)$ satisfying

$$\pi(f) = \int_G f(x)\pi(x)\,d\lambda(x) \quad \big(f \in L^1(G) \subset C^*(G)\big).$$

The spectrum $\widehat{C_r^*(G)}$ identifies with the following closed subset of $\widehat{G}$, the *reduced dual* of G:

$$\widehat{G}_r = \{\pi \in \widehat{G}\colon\ \ker\pi \supset \ker\rho\}$$

(cf. [9, 18.3]). We note that $\widehat{G}_r = \widehat{G}$ if G is amenable.

We define the Fourier transform $\widehat{\mu}$ of a measure $\mu \in M(G)$ by

$$\widehat{\mu}(\pi) = \int_G \pi\big(x^{-1}\big)\,d\mu(x) \quad (\pi \in \widehat{G})$$

which is an operator in $B(H_\pi)$, with spectrum denoted by $\sigma(\widehat{\mu}(\pi))$.

The spectrum $\sigma(a)$ of a self-adjoint element a in a C^*-algebra A with identity is given by

$$\sigma(a) = \bigcup_{\pi \in \widehat{A}} \sigma\big(\pi(a)\big)$$

where $\sigma(\pi(a))$ is the spectrum of $\pi(a)$ in $B(H_\pi)$ (cf. [9, 3.3.5]).

If A is without identity, we adjoin an identity to A as usual to obtain $A_1 = A \oplus \mathbb{C}$, then we have the identification $\widehat{A}_1 = \widehat{A} \cup \{\omega\}$ where ω is the one-dimensional irreducible representation of A_1 annihilating A (cf. [9, 3.2.4]). The quasi-spectrum $\sigma'(a)$ of a self-adjoint element $a \in A$ is the spectrum of a in A_1 and we have

$$\sigma'(a) = \sigma_{A_1}(a) = \bigcup_{\pi \in \widehat{A}_1} \sigma\big(\pi(a)\big) = \bigcup_{\pi \in \widehat{A}} \sigma\big(\pi(a)\big) \cup \{0\}.$$

Theorem 2.3. *Let $\mu \in M(G)$ be symmetric and absolutely continuous and let $\sigma(L_\mu)$ be the spectrum of the convolution operator $L_\mu : L^2(G/H) \to L^2(G/H)$. Then we have*

$$\sigma(L_\mu) \cup \{0\} = \bigcup \big\{\sigma\big(\widehat{\mu}(\pi)\big)\colon\ \pi \in \widehat{G}_r,\ \ker\pi \supset \ker\rho_H\big\} \cup \{0\}.$$

In particular, $\sigma(L_\mu) \cup \{0\} = \bigcup\{\sigma(\widehat{\mu}(\pi)) : \pi \in \widehat{G}_r\} \cup \{0\}$ if $H = \{e\}$. If G is discrete, then $\{0\}$ can be removed from both sides of the above equations.

Proof. Let $\mu = f \cdot \lambda$ with $f \in L^1(G)$. By Lemma 2.2, we have $L_\mu = \rho_H(f) \in \rho_H(C^*(G)) \cong C^*(G)/\ker\rho_H$. We consider the quasi-spectrum $\sigma'(\rho_H(f))$ of $\rho_H(f)$ in $\rho_H(C^*(G))$ which may not have an identity.

Let $\sigma'(L_\mu)$ be the quasi-spectrum of the self-adjoint operator L_μ in $B(L^2(G/H))$. Then we have

$$\begin{aligned}
\sigma(L_\mu) \cup \{0\} &= \sigma'(L_\mu) = \sigma'\big(\rho_H(f)\big) = \sigma'(f + \ker\rho_H)\\
&= \bigcup\big\{\sigma\big(\pi(f + \ker\rho_H)\big)\colon\ \pi \in \widehat{C^*(G)/\ker\rho_H}\big\} \cup \{0\}\\
&= \bigcup\big\{\sigma\big(\pi(f)\big)\colon\ \pi \in \widehat{C^*(G)},\ \ker\pi \supset \ker\rho_H\big\} \cup \{0\}\\
&= \bigcup\big\{\sigma\big(\pi(f)\big)\colon\ \pi \in \widehat{G},\ \ker\pi \supset \ker\rho_H\big\} \cup \{0\}\\
&= \bigcup\big\{\sigma\big(\pi(f)\big)\colon\ \pi \in \widehat{G}_r,\ \ker\pi \supset \ker\rho_H\big\} \cup \{0\}
\end{aligned}$$

where, by Lemma 2.1, $\ker\rho_H \supset \ker\rho$ which gives the last equality, and

$$\pi(f) = \int_G \pi(x) f(x)\,d\lambda(x) = \int_G \pi(x)\,d\mu(x) = \widehat{\mu}(\pi)$$

by symmetry of μ. This proves the first assertion.

If G is discrete, then $C^*(G)$ has an identity and one can dispense with the quasi-spectrum and remove $\{0\}$. □

Remark 2.4. If G is abelian, the above result can be deduced directly from the Plancherel theorem instead, without the assumption of compactness of H and absolute continuity of μ.

Corollary 2.5. *If H is a normal subgroup of G in Theorem* 2.3*, then*

$$\sigma(L_\mu)\cup\{0\}=\bigcup\big\{\sigma\big(\widehat{\mu}(\pi)\big)\colon \pi\in\widehat{G}_r,\ \pi(H)=\pi\{e\}\big\}\cup\{0\}.$$

Proof. By composing with the quotient map $q:G\to G/H$, the dual space $\widehat{G/H}$ identifies with $\{\pi\in\widehat{G}\colon \pi(H)=\pi\{e\}\}$, and also $\rho_H=\rho_{G/H}\circ q$ where $\rho_{G/H}$ is the right regular representation of the group G/H. It follows that the reduced dual $\widehat{G/H}_r$ identifies with $\{\pi\in\widehat{G}_r\colon \pi(H)=\pi\{e\}\}$. $\square$

We now consider homogeneous graphs. Let (V,K) be a homogeneous graph with $V=G/H$ and let μ be a positive symmetric measure on G, supported by K, satisfying

$$\mu(xcy)=\mu(xy)\quad (x,y\in G,\ c\in H).$$

We can define a weight w on $V\times V$ by

$$w(Hx,Hy)=\mu\big(x^{-1}y\big).$$

In this case and in the sequel, $w(v,va)=\mu(a)$ and the Laplacian has the form

$$(\mathcal{L}f)(v)=\frac{1}{|K|}\sum_{a\in K}\big(f(v)-f(va)\big)\mu(a)=f*\left(\delta_e-\frac{\mu}{|K|}\right)(v) \tag{2}$$

which is a convolution operator $L_{\mu'}:L^2(G/H)\to L^2(G/H)$ with $\mu'=\delta_e-\mu/|K|$, where $\mu/|K|$ is a probability measure. For unweighted graphs, we have $\mu(a)=1$ for all $a\in K$.

We note that $\mathcal{L}:\ell^2(V)\to\ell^2(V)$ is a positive operator since the inner product

$$\langle\mathcal{L}f,f\rangle=\frac{1}{2|K|}\sum_{v\in V}\sum_{a\in K}\big(f(v)-f(va)\big)^2\mu(a)\quad \big(f\in\ell^2(V)\big)$$

is nonnegative. Hence we always have $\sigma(\mathcal{L})\subset[0,2]$ as $\|\mathcal{L}\|\leqslant\|\delta_e-\frac{\mu}{|K|}\|\leqslant 2$.

Since $\mu=\sum_{a\in K}\mu(a)\delta_a$ and $\widehat{\delta_a}(\pi)=\pi(a)$, we have the following description of the spectrum $\sigma(\mathcal{L})$.

Corollary 2.6. *Let (V,K) be a homogeneous graph with $V=G/H$ and weight w given by a measure μ as above. The spectrum of the Laplacian in* (2) *is given by*

$$\sigma(\mathcal{L})=1-\bigcup\left\{\sigma\left(\sum_{a\in K}\mu(a)|K|^{-1}\pi(a)\right)\colon \pi\in\widehat{G}_r,\ \ker\pi\supset\ker\rho_H\right\}.$$

Remark 2.7. In [6], a Laplacian acting on vector valued functions $f:G/H\to X$ has been considered and the resulting spectrum is called the *vibrational spectrum*. For the vector space X of $n\times n$ matrices, the spectrum of a convolution operator acting on X-valued functions on a group G has been described in [3], which yields the vibrational spectrum of a Cayley graph (G,K) in this case.

Example 2.8. Let $V=\mathbb{Z}^2/n\mathbb{Z}\times m\mathbb{Z}$ with a finite generating set $K=-K\subset\mathbb{Z}^2$. The character group $\widehat{\mathbb{Z}^2}$ is the product $\mathbb{T}\times\mathbb{T}$ of two copies of the circle group $\mathbb{T}$. Each $\pi\in\widehat{\mathbb{Z}^2}$ identifies with $(\pi(1,0),\pi(0,1))\in\mathbb{T}\times\mathbb{T}$, and $\pi(n\mathbb{Z}\times m\mathbb{Z})=\{1\}$ if and only if $\pi=(e^{2\pi ik/n},e^{2\pi i\ell/m})$ for $(k,\ell)\in\{0,\dots,n-1\}\times\{0,\dots,m-1\}$. For such π, we have

$$\pi(a,b)=e^{2\pi i(ka/n+\ell b/m)}\quad \big((a,b)\in K\big).$$

Hence

$$\sigma(\mathcal{L})=\left\{1-\left(\sum_{(a,b)\in K}\frac{\mu(a,b)}{|K|}\cos 2\pi(ka/n+\ell b/m)\right)\colon (k,\ell)\in\mathbb{Z}_n\times\mathbb{Z}_m\right\}.$$

Example 2.9. Let G be the discrete Heisenberg group

$$\left\{\begin{pmatrix}1&m&p\\0&1&n\\0&0&1\end{pmatrix}\colon m,n,p\in\mathbb{Z}\right\}$$

which is amenable. The characters of G are known (cf. [1,11,15]). Let $\mathbb{R}/\mathbb{Z}$ be the real numbers mod $\mathbb{Z}$ and denote an element of G by (m,n,p). As in [11, Corollary 6.5] or [15], $\widehat{G}$ contains, among others, the one-dimensional unitary representations

$$\{\chi_{\alpha,\beta}\colon\ \alpha,\beta\in\mathbb{R}/\mathbb{Z}\}$$

where

$$\chi_{\alpha,\beta}(m,n,p)=e^{2\pi i(\alpha m+\beta n)}.$$

Consider the Cayley graph (G,K) with $K=\{(\pm m,0,0),(0,\pm n,0)\}$ and $m,n\neq 0$. Let μ be the following measure on G supported by K:

$$\mu=\frac{1}{2}\delta_{(m,0,0)}+\frac{1}{2}\delta_{(-m,0,0)}+\frac{3}{2}\delta_{(0,n,0)}+\frac{3}{2}\delta_{(0,-n,0)}.$$

We have

$$\begin{aligned}\sigma(\mathcal{L})&=1-\bigcup_{\pi\in\widehat{G}}\sigma\left(\frac{1}{4}\sum_{a\in K}\mu(a)\pi(a)\right)\supset 1-\bigcup\left\{\frac{1}{4}\sum_{a\in K}\mu(a)\chi_{\alpha,\beta}(a)\colon\ \alpha,\beta\in\mathbb{R}/\mathbb{Z}\right\}\\&=\left\{1-\left(\frac{1}{4}\cos(2\pi\alpha m)+\frac{3}{4}\cos(2\pi\beta n)\right)\colon\ \alpha,\beta\in\mathbb{R}/\mathbb{Z}\right\}=[0,2].\end{aligned}$$

It follows that $\sigma(\mathcal{L})=[0,2]$.

3. Harnack inequality

In this section, we prove a version of Harnack inequality for an invariant homogeneous graph. We do not assume that the isotropy group H is finite in a homogeneous graph $(G/H,K)$, but we let G act as graph automorphisms of G/H, that is, two vertices Hx and Hy are adjacent if and only if Hxg and Hyg are adjacent for all $g\in G$. A homogeneous graph (V,K) is called *invariant* in [7] if the edge generating set K satisfies $aK=Ka$ for each $a\in K$. This condition imposes some structure on the group G acting on V. It turns out that a connected Cayley graph (G,K) is invariant if and only if G is an $[\mathrm{IN}_0]$-group as defined in [4]. A locally compact group G is called an *$[\mathrm{IN}_0]$-group* if $G=\bigcup_{n=1}^{\infty}C^n$ for some compact neighborhood C of the identity satisfying $gC=Cg$ for each $g\in G$. We first show the relationship between graph invariance and group structures.

Proposition 3.1. *Let $V=G/H$ be a homogeneous space of a discrete group G. The following conditions are equivalent.*

(i) *(V,K) is a connected invariant homogeneous graph for some finite set $K\subset G$.*
(ii) *$G=\bigcup_{n=0}^{\infty}HK^n$ with $K^0=\{e\}$ for some finite set $K=K^{-1}$ satisfying $aK=Ka$ and $HgK=HKg$ for $a\in K$ and $g\in G$.*

In particular, (G,K) is a connected invariant Cayley graph for some finite set $K\subset G$ if and only if G is an $[\mathrm{IN}_0]$-group.

Proof. (i) $\Rightarrow$ (ii). Denote by $v\sim u$ the adjacency of two points in V. We first show $G=\bigcup_{n=0}^{\infty}HK^n$. Let $g\in G$ and $g\notin H$. Then $Hg\neq H$. Since V is connected, we have $Hg\sim Hg_1\sim\cdots\sim Hg_n\sim H$ for some $g_1,\dots,g_n\in G$, and hence $Hg=(Hg_1)a_1=(Hg_2)a_2a_1=\cdots=(Hg_n)a_n\cdots a_1=Ha_{n+1}a_n\cdots a_1$ where $a_1,\dots,a_{n+1}\in K$. So $g\in HK^{n+1}$. This proves $G=H\cup HK\cup HK^2\cup\cdots$.

Next, let $a\in K$ and $g\in G$. Then $H\sim Ha$ which implies $Hg\sim Hag$ since G acts on V as automorphisms of V. Hence $Hag=Hga_1$ for some $a_1\in K$, and we have $HKg\subset HgK$. Similarly, $HgK\subset HKg$ using $Hg\sim Hga$ implies $H\sim Hgag^{-1}$.

(ii) $\Rightarrow$ (i). Define adjacency $\sim$ in V by K. Given $v\sim u$ in V with $u=va$ for some $a\in K$, we have, for each $g\in G$, that $ug=vag=vga'$ for some $a'\in K$, that is, $ug\sim vg$. Hence (V,K) is a homogeneous graph which is clearly invariant and connected.

Finally, if (G,K) is an invariant connected Cayley graph, then $C=K\cup\{e\}$ is an invariant neighborhood of the identity by (ii) and $G=\bigcup_{n=1}^{\infty}C^n$ is an $[\mathrm{IN}_0]$-group.

Conversely, if G is an $[\mathrm{IN}_0]$-group with $G=\bigcup_{n=1}^{\infty}C^n$, then (G,K) is a connected invariant graph with $K=C\cup C^{-1}$. $\square$

The product $O(n)\times\mathbb{R}$ of the orthogonal group $O(n)$ and the additive group $\mathbb{R}$ is an $[\mathrm{IN}_0]$-group [4]. Evidently, a homogeneous graph $(G/H,K)$ is invariant if G is abelian or K is a subgroup of G. We refer to [5] for more examples of invariant homogeneous graphs.

A Harnack inequality for eigenfunctions of the Laplacian on a finite unweighted invariant homogeneous graph has been shown in [7]. This inequality can be proved similarly for the Laplacian in (2) for weighted graphs. We will extend the idea in [7] to deduce a version of Harnack inequality for a Schrödinger operator $\mathcal{L}+\varphi$. We first prove that the positive $\mathcal{L}$-harmonic functions, that is, the positive 0-eigenfunctions of $\mathcal{L}$, are constant.

Let (V, K) be an invariant homogeneous graph with $V = G/H$ and the quotient map $q : G \to G/H$. Let $C = K \cup \{e\}$ which is an invariant neighborhood of $e \in G$. The discrete subgroup

$$G_0 = \bigcup_{n=1}^{\infty} C^n \subset G$$

is an $[\mathrm{IN}_0]$-group. The measure $\mu/|K|$ in the Laplacian $\mathcal{L}$ has support $K \subset G_0$ and restricts to a probability measure μ_0 on G_0. A real function h on G_0 is called *μ_0-harmonic* if $h = h * \mu_0$. Given an $\mathcal{L}$-harmonic function $f : V \to \mathbb{R}$, the equation $\mathcal{L}f = 0$ gives

$$f(Hx) = \left(f * \frac{\mu}{|K|}\right)(Hx) = \int_G f\left(Hxy^{-1}\right)\frac{d\mu}{|K|}(y) = \int_{G_0} f\left(Hxy^{-1}\right)d\mu_0(y)$$

and hence $f \circ q$ restricts to a μ_0-harmonic function on G_0.

A function $\varphi : G_0 \to (0, \infty)$ is called *exponential* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G_0$.

Proposition 3.2. *Let (V, K) be a connected invariant homogeneous graph with the Laplacian $\mathcal{L}$ in* (2)*. Then all positive $\mathcal{L}$-harmonic functions on V are constant.*

Proof. Let f be a positive function on $V = G/H$ satisfying $\mathcal{L}f = 0$. By the above remark, the quotient map $q : G \to G/H$ lifts f to a positive μ_0-harmonic function $f \circ q$ on G_0. Since G_0 is an $[\mathrm{IN}_0]$-group and the support of μ_0 generates G_0, it follows from [4, Theorem 9] that $f \circ q|_{G_0}$ is an integral

$$f \circ q(x) = \int_{\mathcal{E}} h(x)\,dP(h) \quad (x \in G_0)$$

of (constant multiples of) exponential functions with respect to a probability measure P on $\mathcal{E}$, where each $h \in \mathcal{E}$ is a constant multiple $\alpha\varphi$ of an exponential function φ on G_0 satisfying

$$\int_{G_0} \varphi\left(x^{-1}\right)d\mu_0(x) = 1.$$

We show $\varphi = 1$ for all such φ. Indeed, if $\varphi(a) \neq 1$ for some $a \in K$, then $\varphi(a) + \varphi(a^{-1}) = \varphi(a) + \varphi(a)^{-1} > 2$ and $1 = \int_{G_0} \varphi(x^{-1})\,d\mu_0(x) = \sum_{b \in K} \varphi(b)\mu(b)/|K|$ implies

$$|K| = \varphi(a)\mu(a) + \varphi(a)^{-1}\mu(a) + \sum_{b \in K\setminus\{a,a^{-1}\}} \varphi(b)\mu(b) > 2\mu(a) + \sum_{b \in K\setminus\{a,a^{-1}\}} \varphi(b)\mu(b) \geqslant \sum_{b \in K} \mu(b) = \sum_{b \in K} w(v, vb) = |K|$$

which is impossible. Hence $\varphi = 1$ on $C = K \cup \{e\}$ and therefore, on $\bigcup_{n=1}^{\infty} C^n = G_0$.

It follows that $f \circ q$ is constant on G_0. Since $G = \bigcup_{n=1}^{\infty} HC^n$ by connectedness of the graph and Proposition 3.1, we have $f(Hx) = f(H)$ for all $x \in G$. $\square$

Let (V, K) be a weighted invariant homogeneous graph in which the weight is given by a symmetric measure μ satisfying

$$\mu(a) = \mu\left(bab^{-1}\right) > 0 \quad (a, b \in K). \tag{3}$$

Let $w_a = \mu(a)/|K|$ for $a \in K$ so that the Laplacian in (2) is written

$$\mathcal{L}f(v) = \sum_{a \in K}\left(f(v) - f(va)\right)w_a.$$

Chung and Yau [7] have proved a Harnack inequality for eigenfunctions of $\mathcal{L}$ on unweighted (V, K) where $\mu(a) = 1$ for all $a \in K$. By Proposition 3.2, the positive eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 0$ are constant. By [2, Corollary 3.14], the ℓ^p-eigenfunctions of $\mathcal{L}$ for $\lambda = 0$ and $1 \leqslant p < \infty$ are also constant. Extending the idea in [7], we consider below eigenfunctions corresponding to eigenvalues $\lambda > 0$ for a Schrödinger operator $\mathcal{L} + \varphi$ which is a positive operator on the Hilbert space $\ell^2(V)$ if $\varphi \geqslant 0$, but may be unbounded if V is infinite.

We note that if K is a subgroup of G in an invariant homogeneous graph (V, K), then $V = \bigcup_{v \in V}\{v\} \cup vK$ is a disjoint union of connected components. The vertex set S of a union of these components satisfies $SK \subset S$.

Theorem 3.3. *Let (V, K) be an invariant homogeneous graph. Let $\varphi \geqslant 0$ be a function on V and let f be a real function on V satisfying*

$$\mathcal{L}f + \varphi f = \lambda f \quad (\lambda > 0).$$

Then on any finite subgraph with vertex set S satisfying $SK \subset S$, we have

$$\sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 + \alpha\lambda f^2(v) \leqslant \left(\frac{\alpha^2\lambda}{\alpha-2} + \frac{4}{(\alpha-2)\lambda}\sup_S \varphi\right)\sup_S f^2$$

for $v \in S$ and $\alpha > 2$. In particular, the inequality holds for all $v \in V$ if V is finite, with $S = V$.

Proof. We extend the arguments in [7] and include the details for later reference. Define

$$\rho(v) = \sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 \quad (v \in S)$$

and let $\mathcal{L}$ act on the functions ρ and f^2. First consider

$$\begin{aligned}
\mathcal{L}\rho(v) &= \sum_{b\in K} w_b \sum_{a\in K} w_a\big\{\big[f(v) - f(va)\big]^2 - \big[f(vb) - f(vba)\big]^2\big\} \\
&= -\sum_{b\in K} w_b \sum_{a\in K} w_a\big[f(v) - f(va) - f(vb) + f(vba)\big]^2 \\
&\quad + 2\sum_{b\in K} w_b \sum_{a\in K} w_a\big[f(v) - f(va) - f(vb) + f(vba)\big]\big[f(v) - f(va)\big].
\end{aligned}$$

Let X denote the second term above. We have

$$\begin{aligned}
X &= 2\sum_{b\in K} w_b \sum_{a\in K} w_a\big[f(v) - f(va) - f(vb) + f(vba)\big]\big[f(v) - f(va)\big] \\
&= 2\sum_{a\in K} w_a\left(\sum_{b\in K} w_b\big[f(v) - f(va) - f(vb) + f(vab)\big]\right)\big[f(v) - f(va)\big] \\
&\quad + 2\sum_{a\in K} w_a\left(\sum_{b\in K} w_b\big[f(vba) - f(vab)\big]\right)\big[f(v) - f(va)\big] \\
&= 2\lambda\sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 + 2\sum_{a\in K} w_a\big[\varphi(va)f(va) - \varphi(v)f(v)\big]\big[f(v) - f(va)\big]
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{b\in K} w_b\big[f(v) - f(vb)\big] = \lambda f(v) - \varphi(v)f(v), \\
&\sum_{b\in K} w_b\big[f(va) - f(vab)\big] = \lambda f(va) - \varphi(va)f(va)
\end{aligned}$$

and $\sum_{b\in K} w_b[f(vba) - f(vab)] = 0$ follows from the symmetry of μ and (3).

It follows that

$$\begin{aligned}
\mathcal{L}\rho(v) \leqslant X &= 2\lambda\sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 + 2\sum_{a\in K} w_a\big[\varphi(va)f(va) - \varphi(v)f(v)\big]\big[f(v) - f(va)\big] \\
&\leqslant 2\lambda\sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 + 2\sum_{a\in K} w_a\big[\varphi(va)f(va)f(v) + \varphi(v)f(v)f(va)\big].
\end{aligned}$$

Next we consider

$$\begin{aligned}
\mathcal{L}f^2(v) &= \sum_{a\in K} w_a\big[f^2(v) - f^2(va)\big] = 2\sum_{a\in K} w_a f(v)\big[f(v) - f(va)\big] - \sum_{a\in K} w_a\big[f(v) - f(va)\big]^2 \\
&= 2\big(\lambda - \varphi(v)\big)f^2(v) - \sum_{a\in K} w_a\big[f(v) - f(va)\big]^2.
\end{aligned}$$

Putting the last two inequalities above together, we arrive at

$$\mathcal{L}\big(\rho(v)+\alpha\lambda f^2(v)\big)\leqslant 2\alpha\lambda\big(\lambda-\varphi(v)\big)f^2(v)-(\alpha-2)\lambda\sum_{a\in K}w_a\big[f(v)-f(va)\big]^2 + 2f(v)\sum_{a\in K}w_a\varphi(va)f(va)+2\varphi(v)f(v)\sum_{a\in K}w_a f(va).$$

We can find $s\in S$ such that

$$\rho(s)+\alpha\lambda f^2(s)=\sup\big\{\rho(v)+\alpha\lambda f^2(v)\colon\ v\in S\big\}.$$

Since $SK\subset S$, we have

$$\begin{aligned}&0\leqslant\mathcal{L}\big(\rho(s)+\alpha\lambda f^2(s)\big)\\&\quad\leqslant 2\alpha\lambda\big(\lambda-\varphi(s)\big)f^2(s)-(\alpha-2)\lambda\sum_{a\in K}w_a\big[f(s)-f(sa)\big]^2+2f(s)\sum_{a\in K}w_a\varphi(sa)f(sa)+2\varphi(s)f(s)\sum_{a\in K}w_a f(sa).\end{aligned}\tag{4}$$

This implies

$$\sum_{a\in K}w_a\big[f(s)-f(sa)\big]^2\leqslant\frac{1}{(\alpha-2)\lambda}\bigg(2\alpha\lambda\big(\lambda-\varphi(s)\big)f^2(s)+2f(s)\sum_{a\in K}w_a\varphi(sa)f(sa)+2\varphi(s)f(s)\sum_{a\in K}w_a f(sa)\bigg).$$

Hence for every $v\in S$, we have

$$\begin{aligned}&\sum_{a\in K}w_a\big[f(v)-f(va)\big]^2+\alpha\lambda f^2(v)\\&\quad\leqslant\frac{1}{(\alpha-2)\lambda}\bigg(2\alpha\lambda\big(\lambda-\varphi(s)\big)f^2(s)+2f(s)\sum_{a\in K}w_a\varphi(sa)f(sa)+2\varphi(s)f(s)\sum_{a\in K}w_a f(sa)+\alpha\lambda(\alpha-2)\lambda f^2(s)\bigg)\\&\quad\leqslant\frac{1}{(\alpha-2)\lambda}\bigg(\alpha^2\lambda^2 f^2(s)+2f(s)\sum_{a\in K}w_a\varphi(sa)f(sa)+2\varphi(s)f(s)\sum_{a\in K}w_a f(sa)\bigg)\\&\quad\leqslant\frac{1}{(\alpha-2)\lambda}\bigg(\alpha^2\lambda^2 f^2(s)+\sum_{a\in K}w_a\varphi(sa)\big(f^2(s)+f^2(sa)\big)+\sum_{a\in K}w_a\varphi(s)\big(f^2(s)+f^2(sa)\big)\bigg)\\&\quad\leqslant\frac{\alpha^2\lambda}{\alpha-2}\sup_S f^2+\frac{4}{(\alpha-2)\lambda}\sup_S\varphi\sup_S f^2.\qquad\square\end{aligned}$$

Remark 3.4. For $\varphi=0$ and $w_a=\frac{1}{|K|}$ in Theorem 3.3, the inequality is identical with the Harnack inequality for finite V in [7].

Finally we derive a similar Harnack inequality for Dirichlet eigenfunctions on a finite convex subgraph of an invariant homogeneous graph (V,K), extending the result in [8]. The *boundary* δS of a subgraph of (V,K) with vertex set S is defined by $\delta S=\{v\in V\setminus S\colon\ v\sim\text{some }u\in S\}$ where $\sim$ denotes adjacency. A subgraph of (V,K) with vertex set S is called *convex* [8] if, for any subset $Y\subset\delta S$, its neighborhood $N(Y)=\{v\in V\colon\ v\sim\text{some }u\in Y\}$ satisfies the boundary expansion property:

$$\big|N(Y)\setminus(S\cup\delta S)\big|=\big|\{v\notin S\cup\delta S\colon\ v\sim\text{some }u\in Y\}\big|\geqslant|Y|.$$

An eigenfunction f on $S\cup\delta S$ of a Schrödinger operator $\mathcal{L}+\varphi$ is said to satisfy the *Dirichlet boundary condition* if $f(v)=0$ for $v\in\delta S$.

Theorem 3.5. *Let (V,K) be an invariant homogeneous graph and let S be the vertex set of a finite convex subgraph of (V,K). Let $\varphi\geqslant 0$ and let f be a real function on $S\cup\delta S$ satisfying*

$$\mathcal{L}f(v)+\varphi(v)f(v)=\lambda f(v)\quad(\lambda>0)\tag{5}$$

for $v\in S$ and $f(v)=0$ for $v\in\delta S$. Then we have the inequality

$$w_a\big[f(v)-f(va)\big]^2+\alpha\lambda f^2(v)\leqslant\bigg(\frac{\alpha^2\lambda}{\alpha-2}+\frac{4}{(\alpha-2)\lambda}\sup_S\varphi\bigg)\sup_S f^2$$

for $v\in S$, $a\in K$ and $\alpha>2,|K|/\lambda k$ where $k=\inf\{w_a\colon\ a\in K\}$.

Proof. As in the proof of [8, Theorem 1], convexity of S enables one to extend the function f to all vertices of V adjacent to $S \cup \delta S$ so that Eq. (5) also holds on δS, and as in the proof of Theorem 3.3, one can apply similar arguments to the function

$$\rho_a(v) = w_a\big[f(v) - f(va)\big]^2 + \alpha\lambda f^2(v) \quad (v \in S \cup \delta S,\ a \in K)$$

and find some $s \in S$ and $b \in K$ satisfying

$$\rho_b(s) = \sup\big\{\rho_a(v):\ v \in S,\ a \in K\big\}.$$

We have $\rho_b(s) \geqslant \rho_b(sa)$ for each $a \in K$, given $\alpha > |K|/\lambda k$. It follows that $\mathcal{L}(\rho_b(s)) \geqslant 0$ as in (4) in the proof of Theorem 3.3. From this, one obtains the required inequality as before. $\square$

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