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Amenability, Reiter's condition and Liouville property

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ABSTRACT

We show that the Liouville property and Reiter's condition are equivalent for semigroupoids. This result applies to semigroups as well as semigroup actions. In the special case of measured groupoids and locally compact groupoids, our result proves Kaimanovich's conjecture of the equivalence of amenability and the Liouville property.

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1. Introduction

Since the seminal work of von Neumann [38], amenable groups and semigroups have had a profound impact on many areas of mathematics. Amenability of locally compact groups has been shown to be equivalent to many fundamental properties in harmonic

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analysis including the Liouville property which is one of the subjects of the present paper. In operator algebras, amenability plays a pivotal role in their classification (cf. [9,14]) as well as in the recent progress on the Novikov conjecture (see [17,30]). Indeed, amenable groupoids satisfy the Baum–Connes conjecture [37]. Amenability also plays a significant role in the recent development of semigroup C*-algebras relating to some aspects of number theory [10,25,26].

A locally compact group G is amenable if there is a left invariant mean on $L^\infty(G)$. A topological semigroup G is usually called *amenable* if there is a left invariant mean on the algebra $LUC(G)$ of bounded left uniformly continuous functions on G. These two definitions of amenability are equivalent for locally compact groups. The notion of amenability has been extended to group actions by Zimmer [40,41]. For the more general case of groupoids which, among other things, unify both concepts of groups and group actions, it was introduced by Renault [34,1]. Amenable groupoids were defined in terms of Reiter's condition, which stipulates the existence of nets of approximately invariant probability measures and was first formulated by Day [12] for discrete semigroups. For locally compact groups, Reiter's condition is equivalent to amenability and therefore the definition of an amenable groupoid is a natural extension of the group case. However, for topological semigroups, the question of whether Reiter's condition follows from amenability as defined previously appears to be open [23, p.321].

The equivalence of amenability and the Liouville property for σ-compact locally compact groups was first conjectured by Furstenberg [16] and proved by Rosenblatt [36], Kaimanovich and Vershik [21]. More recently, Kaimanovich introduced the Liouville property for groupoids in [20] and conjectured its equivalence to amenability, having proved that the former implies the latter. For semigroups, the Liouville property for abelian semigroups has been studied in [11,24,32], but its connection to amenability has not been the subject of investigation before.

Our main objective in this paper is to clarify the relationships of amenability, Reiter's condition and the Liouville property in the setting of semigroupoids, which subsumes and provides a unified treatment to the important cases of groupoids, semigroups and transformation semigroups. We introduce and study the Liouville property and Reiter's condition for semigroupoids. We prove that a semigroupoid possesses the Liouville property if and only if it satisfies Reiter's condition (Theorems 4.1, 4.2, 5.3, 5.6). An immediate consequence is the equivalence of the Liouville property and Reiter's condition for semigroup actions (Theorem 6.7) as well as the equivalence of amenability and the Liouville property for discrete semigroups (Theorem 6.10) and also, for both measured groupoids and topological groupoids (Theorems 6.1, 6.3), the latter proves a conjecture of Kaimanovich in [20]. We thank Vadim Kaimanovich for informing us, after we have written this paper, that his conjecture for *measured groupoids* has also been proved by Theo Bühler and himself in an unpublished note. Our result includes the case of topological groupoids, which requires some refinements of Reiter's condition.

A Riemannian manifold is said to have the Liouville property if it does not admit non-constant bounded harmonic functions. Examples include complete manifolds with

non-negative Ricci curvature, by a well-known result of Yau [39]. To introduce the Liouville property for groups, we begin with a connected Lie group G. The Laplace operator of G generates a one-parameter convolution semigroup $(\pi_t)_{t>0}$ of probability measures on G such that a function $f \in L^\infty(G)$ is C^2 and harmonic if and only if it satisfies the convolution equations $f = f * \pi_t$ for all $t > 0$ (cf. [18] and [3, Proposition V.6]). This is equivalent to $f = f * \pi_t$ for *some* $t > 0$ if G is of type T [3, p.136] (e.g. G is semisimple with finite centre [3, Théorème II.1]). More generally, given a probability measure π on a locally compact group G, a Borel function $f : G \to \mathbb{C}$ is called *π-harmonic* if it satisfies the convolution equation $f = f * \pi$. The latter condition is an analogue of the mean value property which characterises harmonic functions on manifolds. We say that G has the *Liouville property* if there is an absolutely continuous probability measure π on G such that all bounded π-harmonic functions are constant. In § 2, we extend the definition of Liouville property to semigroupoids, which are algebraic structures that resemble a semigroup, except that multiplication is not globally defined.

We begin our discussion of semigroupoids in the next section, followed by an introduction to Reiter's condition and the Liouville property, in both the measurable and topological contexts. To pave our way, we prove some basic results in § 3 concerning convolution of systems of measures on semigroupoids. In § 4, we show that the Liouville property implies Reiter's condition for semigroupoids, both in the measurable and topological setting. This extends the result in [20] for groupoids equipped with a Haar system. The converse of the previous result for semigroupoids is proved in § 5. In both results, we only make the weaker assumption of a *quasi-Haar system* for semigroupoids so that wider applicability can be achieved. For example, such semigroupoids include discrete semigroups, which need not admit a Haar system. The following section (§ 6) is devoted to the special cases of groupoids, transformation semigroups and semigroups. Our results on semigroupoids apply directly to semigroups equipped with a *quasi-invariant measure* (Theorem 6.9). For semigroups without such a measure and not necessarily locally compact, we discuss briefly the case of metrizable semigroups, for which one can show that Reiter's condition implies the Liouville property, which in turn implies amenability. The proof of the last implication is different from that for semigroupoids. However, a more thorough treatment will be given in another work [7]. We conclude the paper with some examples of semigroups with the Liouville property in § 7.

2. Semigroupoids

We initiate the discussion of semigroupoids with semigroups. By a *topological semigroup*, we mean a semigroup S endowed with a topology such that the multiplication on S is jointly continuous. If moreover, the topology is metrizable, we call S a *metric semigroup*.

To discuss semigroupoids in measurable and topological settings, we need to fix some notations for measures and functions on a topological space X. A positive finite Borel measure μ on X is called *tight* if for each $\varepsilon > 0$, there is a compact set $K \subseteq X$ such that

$\mu(X \setminus K) < \varepsilon$. Every positive finite Borel measure on a *complete separable* metric space is tight.

Let $M(X)$ be the space of complex-valued regular Borel measures on X. For each nonzero $\mu \in M(X)$, the norm of μ is defined by $\|\mu\| = |\mu|(X)$ where $|\mu|$ is the total variation of μ. The *support* of $|\mu|$ is defined by

$$\operatorname{supp}|\mu| = \bigcap \{F \subseteq X : F \text{ is closed}, \ |\mu|(F) = |\mu|(X)\}$$

and the *support* of μ, denoted by $\operatorname{supp}\mu$, is defined to be that of $|\mu|$. A measure $\mu \in M(X)$ is called *tight* if $|\mu|$ is tight. Every Borel measure with compact support is a tight measure. We denote by $M_t(X)$ the subspace of $M(X)$ consisting of tight measures on X. Let $C_b(X)$ be the C^*-algebra of bounded complex continuous functions on X. For a metric space X, the support $\operatorname{supp}\mu$ of each tight measure μ is nonempty and separable. Moreover, for every Borel set $B \subseteq X$ and $\varepsilon > 0$, there is a compact set $K \subseteq B$ satisfying $|\mu|(B \setminus K) < \varepsilon$.

We recall that a topological space Y is a *Polish space* if it is homeomorphic to a complete separable metric space. A subset of Y is called *analytic* if it is of form $f(Z)$ for some continuous function f from a Polish space Z to Y. Every Borel set in a Polish space is analytic.

In the sequel, by an *analytic* Borel space, we mean a measurable space $(\mathcal{G}, \mathcal{B})$ which is isomorphic to an analytic set in a Polish space with the relative Borel structure. The sets in the σ-algebra $\mathcal{B}$ are called the *Borel sets* in $\mathcal{G}$. The σ-algebra $\mathcal{B}$ is often not written explicitly for an analytic Borel space $\mathcal{G}$. We refer to [2] for the basic properties of analytic Borel spaces.

Remark 2.1. Since an uncountable Polish space with its Borel structure is isomorphic to the unit interval $[0, 1]$ with the usual Borel structure [22, p. 451], we see that the Banach space $L^1(Y, \mu)$ of μ-integrable functions on Y is separable for any Borel measure μ on a Polish space Y. The same is true for $L^1(\mathcal{G}, \nu)$, where ν is a Borel measure on an analytic Borel space $\mathcal{G}$.

Let S be a topological semigroup. A function $f \in C_b(S)$ is called *left uniformly continuous* if the mapping $a \in S \mapsto \delta_a * f \in C_b(S)$ is continuous, where $\delta_a * f$ is the left translate of f by a, which is the convolution of the point mass δ_a and f defined below. The space of bounded left uniformly continuous functions on S will be denoted by $LUC(S)$ which forms a sub-C*-algebra of $C_b(S)$. For a discrete semigroup S, we have $LUC(S) = \ell^\infty(S)$.

Given $\pi, \sigma \in M(S)$, we define their convolution $\pi * \sigma$ to be the image of product measure $\pi \times \sigma$ under the map $(x, y) \in S \times S \mapsto xy \in S$, that is,

$$(\pi * \sigma)(E) = \int_{S \times S} \chi_E(xy) d(\pi \times \sigma)(x, y)$$

for each Borel set $E \subseteq S$. For $f \in C_b(S)$, we have

$$\int_S f d(\pi * \sigma) = \int_S \int_S f(xy) d\pi(x) d\sigma(y).$$

In the sequel, we denote by π^n the n-fold convolution $\overbrace{\pi * \cdots * \pi}^{\text{n-times}}$ of π.

Given $\pi \in M(S)$ and a Borel function $f : S \to \mathbb{C}$, we define the *semigroup convolutions* $f * \pi$ and $\pi * f$ by

$$(f * \pi)(x) = \int_S f(xy) d\pi(y) \quad \text{and} \quad (\pi * f)(x) = \int_S f(yx) d\pi(y)$$

if the integrals exist. For a point mass δ_a at $a \in S$, we have $(\delta_a * f)(x) = f(ax)$ and $(f * \delta_a)(x) = f(xa)$ as well as

$$\int_S (\delta_a * f) d\pi = \int_S f d(\delta_a * \pi).$$

Definition 2.2. Let S be a topological semigroup and let $\pi \in M(S)$ be a probability measure. A Borel function $f : S \to \mathbb{C}$ is called π-*harmonic* if $f * \pi = f$.

Definition 2.3. A topological semigroup S is called (*left*) *amenable* if there is a left invariant mean on $LUC(S)$, that is, there exists a norm-one positive linear functional $\varphi : LUC(S) \to \mathbb{C}$ on the C*-algebra $LUC(S)$ satisfying $\varphi(\delta_a * f) = \varphi(f)$ for all $a \in S$.

For a locally compact group S, this definition of amenability agrees with the usual one [29, p.67]. We now turn to semigroupoids, which generalise semigroups and groupoids at the same time, in a very natural way. A *semigroupoid* is a small category. It consists of a set $\mathcal{G}^{(0)}$ of objects (called units), a set $\mathcal{G}$ of morphisms, the surjective *source* and *target* maps $\mathbf{s}, \mathbf{t} : \mathcal{G} \to \mathcal{G}^{(0)}$, and a composition map $(\zeta, \eta) \in \mathcal{G}^{(2)} \mapsto \zeta\eta \in \mathcal{G}$ on $\mathcal{G}^{(2)} := \big\{(\zeta, \eta) \in \mathcal{G}^2 \colon \mathbf{s}(\zeta) = \mathbf{t}(\eta)\big\}$. In the special case where there is only one unit, the concept of semigroupoids reduces to semigroups with identity, which are sometimes called *monoids*. As usual, we identify the units with the corresponding identity morphisms in $\mathcal{G}$ (which exist by definition). In this way, we may consider $\mathcal{G}^{(0)}$ as a subset of $\mathcal{G}$ and by a slight abuse of language, we call $\mathcal{G}$ a *semigroupoid*.

For each $x \in \mathcal{G}^{(0)}$, we write $\mathcal{G}^x := \mathbf{t}^{-1}(x) = \{\gamma \in \mathcal{G} \colon \mathbf{t}(\gamma) = x\}$. One may view the target map $\mathbf{t} : \mathcal{G} = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}^x \to \mathcal{G}^{(0)}$ like a *'bundle projection'*. Actually, we will see that the Liouville property for $\mathcal{G}$ is defined on the *'fibres'* $\mathcal{G}^x$.

A semigroupoid $\mathcal{G}$ is called *Borel* if $\mathcal{G}$ is an analytic Borel space, $\mathcal{G}^{(0)}$ is an analytic Borel space (as a subspace of $\mathcal{G}$) such that $\mathbf{s}$, $\mathbf{t}$ and the composition map are Borel. In this case, the sets $\mathcal{G}^x$ and the maps $\zeta : \eta \in \mathcal{G}^{\mathbf{s}(\zeta)} \mapsto \zeta\eta \in \mathcal{G}^{\mathbf{t}(\zeta)}$ are Borel. We call $\mathcal{G}$ a *topological semigroupoid* if it is endowed with a topology compatible with the semigroupoid structure, that is, the composition map, $\mathbf{s}$ and $\mathbf{t}$ are continuous. We denote by $C_c(\mathcal{G})$ the space of complex continuous functions on $\mathcal{G}$ with compact support.

By a *locally compact semigroupoid*, we mean a topological semigroupoid which is locally compact Hausdorff and second countable. This implies that all our locally compact semigroupoids are σ-compact.

Example 2.4. Groupoids, which are by definition small categories in which every morphism is invertible, form a special class of semigroupoids. We refer to [1,34] for more information about groupoids.

Another class of examples is given by semigroups, or more generally, semigroup actions. Let S be a semigroup with identity e acting on a set X from the right. We denote the action by $(x, s) \in X \times S \mapsto x.s \in X$, where $x.e = x$. The semigroupoid attached to the transformation semigroup $X \curvearrowleft S$ is given by $\mathcal{G} := X \rtimes S := X \times S$ (as a set), with the source and target maps $\mathbf{s}: (x, s) \in X \rtimes S \mapsto x.s \in X$, $\mathbf{t}: (x, s) \in X \rtimes S \mapsto x \in X$, and the composition $(x, s)(x.s, t) = (x, st)$. It is clear that the set $\mathcal{G}^{(0)} = \{(x, e): x \in X\}$ of units of $X \rtimes S$ can be canonically identified with X. By taking $X = \{\mathrm{pt}\}$, one can also view the semigroup S itself as the semigroupoid $\{\mathrm{pt}\} \rtimes S$.

If X and S are analytic Borel spaces, or topological spaces, such that $\mathbf{s}$, $\mathbf{t}$ and the composition map are Borel or continuous respectively, then $X \rtimes S$ becomes a Borel or topological semigroupoid. In particular, this is the case if S is a countable discrete semigroup acting on a Borel space or a topological space by Borel or continuous maps respectively.

Groupoids and transformation semigroups are the two motivating examples for us. We will frequently come back to them.

We will need to consider systems of measures on the fibres $\mathcal{G}^x$ of a semigroupoid $\mathcal{G}$. For groupoids, these are called *kernels* by Connes in [8, p.11].

Definition 2.5. Let $\mathcal{G}$ be a Borel semigroupoid. A *Borel system of measures* on $\mathcal{G}$ is a family $\lambda = (\lambda^x)_{x \in \mathcal{G}^{(0)}}$ of σ-finite positive Borel measures λ^x on $\mathcal{G}^x$ such that for every non-negative Borel function f on $\mathcal{G}$, the map $x \in \mathcal{G}^{(0)} \mapsto \langle \lambda^y, f \rangle := \int_{\mathcal{G}} f d\lambda^x \in [0, \infty]$ is Borel measurable, where we extend λ^x naturally to a measure on $\mathcal{G}$ such that $\lambda^x(\mathcal{G} \setminus \mathcal{G}^x) = 0$, and in particular, the map $x \in \mathcal{G}^{(0)} \mapsto \lambda^x(E)$ is Borel for each Borel set $E \subset \mathcal{G}$.

Definition 2.6. Let $\mathcal{G}$ be a topological semigroupoid. A *continuous system of measures* on $\mathcal{G}$ is a family $\lambda = (\lambda^x)_{x \in \mathcal{G}^{(0)}}$ of (nonzero) σ-finite positive Radon measures λ^x on $\mathcal{G}^x$ such that for every $f \in C_c(\mathcal{G})$, the map $x \in \mathcal{G}^{(0)} \mapsto \langle \lambda^x, f \rangle \in \mathbb{C}$ is continuous.

Given a Borel semigroupoid $\mathcal{G}$ with $x \in \mathcal{G}^{(0)}$ and $\gamma \in \mathcal{G}$ satisfying $\mathbf{s}(\gamma) = x$, and a Borel measure λ^x on $\mathcal{G}^x$, we write $\gamma\lambda^x$ for the pushforward of λ^x under the Borel map $\gamma: \eta \in \mathcal{G}^x \mapsto \gamma\eta \in \mathcal{G}^{\mathbf{t}(\gamma)}$, which is the Borel measure on $\mathcal{G}^{\mathbf{t}(\gamma)}$ induced by the map γ. By definition, we have

$$\langle \gamma\lambda^x, f\rangle = \int_{\mathcal{G}^x} f(\gamma\eta)d\lambda^x(\eta)$$

for each non-negative Borel function f on $\mathcal{G}^{\mathbf{t}(\gamma)}$.

For a locally compact semigroupoid $\mathcal{G}$, the pushforward $\gamma\lambda^x$ is defined analogously. As usual, we write $\mu \prec \nu$ to mean that a measure μ is absolutely continuous with respect to ν.

Definition 2.7. Let $\mathcal{G}$ be a Borel (resp. topological) semigroupoid. A Borel (resp. continuous) system of measures $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ on $\mathcal{G}$ is called a *(left) Haar system* if for all $\gamma \in \mathcal{G}$, we have $\gamma\lambda^{\mathbf{s}(\gamma)} = \lambda^{\mathbf{t}(\gamma)}$ and the map $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\lambda^{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta) \in [0, \infty]$ is Borel, where $d(\gamma\lambda^{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}$ denotes the Radon–Nikodym derivative. In the preceding definition, the system $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ will be called a *(left) quasi-Haar system* if the condition $\gamma\lambda^{\mathbf{s}(\gamma)} = \lambda^{\mathbf{t}(\gamma)}$ is replaced by $\gamma\lambda^{\mathbf{s}(\gamma)} \prec \lambda^{\mathbf{t}(\gamma)}$.

Definition 2.8. A Borel system of measures $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ on a Borel semigroupoid $\mathcal{G}$ is called *proper* if there exists an increasing sequence of Borel subsets A_n of $\mathcal{G}$ with $\mathcal{G} = \bigcup_n A_n$ such that for each $n \in \mathbb{N}$, the map $x \in \mathcal{G}^{(0)} \mapsto \lambda^x(A_n) \in \mathbb{R}$ is bounded.

Definition 2.9. A *measure semigroupoid* is a triple $(\mathcal{G}, \lambda, \mu)$ consisting of a Borel semigroupoid $\mathcal{G}$, a proper quasi-Haar system $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ on $\mathcal{G}$ and a positive Borel measure μ on $\mathcal{G}^{(0)}$.

In the topological setting, we shall denote by $(\mathcal{G}, \lambda)$ a locally compact semigroupoid $\mathcal{G}$ equipped with a quasi-Haar system $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ on $\mathcal{G}$.

Remark 2.10. If λ is a left Haar system on a Borel (or locally compact) groupoid, then λ is a quasi-Haar system since $\gamma\lambda^{\mathbf{s}(\gamma)} = \lambda^{\mathbf{t}(\gamma)}$. An important example of a measure semigroupoid is the *measured groupoid* $(\mathcal{G}, \lambda, \mu)$ studied in [1,20] in which the Borel system λ is assumed to be a left Haar system. Likewise, a continuous system λ of measures on a locally compact groupoid $(\mathcal{G}, \lambda)$ considered in [1,20] is always a left Haar system. Our notion of a quasi-Haar system is a weaker version of a Haar system in the groupoid case. However, this weak version is already sufficient for our purpose and has wider applicability.

Let us now consider the case of the Borel semigroupoid $X \rtimes S$ attached to a Borel transformation semigroup $X \curvearrowleft S$.

Definition 2.11. A positive Borel measure λ on a topological semigroup S is called *quasi-invariant* if $s\lambda \prec \lambda$ for all $s \in S$, where $s\lambda$ is the left translate of λ, which is the measure induced by the left translation $t \in S \mapsto st \in S$.

If S is a group, this is equivalent to saying that all translates of λ are mutually equivalent. Thus our definition extends the classical one (cf. [15, p.58]). Let λ be a

quasi-invariant positive σ-finite Borel measure on S. Suppose that the map $(s,t) \in S \times S \mapsto (d(s\lambda)/d\lambda)(t) \in [0,\infty]$ is Borel. Then $\lambda^x := \delta_x \times \lambda$ is a quasi-Haar system on $X \rtimes S$ as $(x,s)(\delta_{x.s} \times \lambda) = \delta_x \times (s\lambda)$. Here is a concrete class of examples: Suppose that G is a locally compact group, acting on an analytic Borel space X with a Borel action $(x,g) \in X \times G \mapsto x.g \in X$. Let λ_G be the Haar measure on G. If S is a Borel subsemigroup of G containing the identity such that $\lambda_G(S) \neq 0$, then the restriction λ of λ_G to S satisfies $s\lambda \prec \lambda$ for all $s \in S$ and $(d(s\lambda)/d\lambda)(t) = \chi_S(s^{-1}t)$. Therefore $\lambda^x := \delta_x \times \lambda$ gives rise to a quasi-Haar system on $X \rtimes S$. In particular, if $X = \{\text{pt}\}$, this allows us to view the semigroup $S \cong \{\text{pt}\} \rtimes S$ as a measure semigroupoid. Of course, another class of examples is given by the case when S is discrete (and countable) and λ is the counting measure.

2.1. Reiter's condition

Definition 2.12. Let $\mathcal{G}$ be a Borel (resp. topological) semigroupoid, and λ a quasi-Haar system on $\mathcal{G}$. A Borel (resp. continuous) system $\theta = (\theta^x)_{x \in \mathcal{G}^{(0)}}$ of probability measures on $\mathcal{G}^x$ is called λ-*adapted* if

$$\begin{aligned} &\theta^x \prec \lambda^x \text{ for every } x \in \mathcal{G}^{(0)}, \quad \text{and the map} \\ &(\gamma,\eta) \in \mathcal{G}^2 \mapsto \frac{d(\gamma\theta^{\mathbf{s}(\gamma)})}{d\lambda^{\mathbf{t}(\gamma)}}(\eta) \in [0,\infty] \text{ is Borel.} \end{aligned} \tag{1}$$

Note that absolute continuity in (1) and Definition 2.7 imply that $\gamma\theta_n^{\mathbf{s}(\gamma)} \prec \gamma\lambda^{\mathbf{s}(\gamma)} \prec \lambda^{\mathbf{t}(\gamma)}$ and we may consider the Radon–Nikodym derivative in (1).

Given a measure semigroupoid $(\mathcal{G},\lambda,\mu)$, we define a Borel measure $\mu \star \lambda$ on $\mathcal{G}$ by

$$\mu \star \lambda(E) = \int_{\mathcal{G}} \lambda^x(E) d\mu(x) \in [0,\infty]$$

for each Borel set E in $\mathcal{G}$. For $f \in L^1(\mathcal{G}, \mu \star \lambda)$, we write

$$\langle \mu \star \lambda, f \rangle := \int_{\mathcal{G}} f d(\mu \star \lambda) = \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} f d\lambda^x \right) d\mu(x).$$

Definition 2.13. A measure semigroupoid $(\mathcal{G},\lambda,\mu)$ is said to satisfy *Reiter's condition*, or called *Reiter*, if there exists a sequence $(\theta_n)_n$ of λ-adapted Borel systems of probability measures $\theta_n = (\theta_n^x)_{x \in \mathcal{G}^{(0)}}$ such that

$$\lim_{n\to\infty} \int_{\mathcal{G}} \left\| \theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)} \right\| f(\gamma) d(\mu \star \lambda)(\gamma) = 0 \text{ for all } f \in L^1(\mathcal{G}, \mu \star \lambda). \tag{2}$$

Remark 2.14. We observe that $\|\theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)}\| \leq 2$ and by the Lebesgue convergence theorem, the following condition implies (2):

$$\lim_{n\to\infty} \left\|\theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)}\right\| = 0 \text{ for } \mu\text{-a.e. } x \in \mathcal{G}^{(0)} \text{ and } \lambda^x\text{-a.e. } \gamma \in \mathcal{G}^x. \tag{3}$$

Given a continuous system of measures $\theta = (\theta^x)_{x\in\mathcal{G}^{(0)}}$ on a locally compact semigroupoid $\mathcal{G}$ and a compact subset $K \subseteq \mathcal{G}^{(0)}$, we set $\mathrm{supp}_K\theta := \overline{\bigcup_{x\in K} \mathrm{supp}\,\theta^x}$.

Definition 2.15. A locally compact semigroupoid $(\mathcal{G}, \lambda)$ with quasi-Haar system λ is said to satisfy *Reiter's condition* if there exists a sequence $(\theta_n)_n$ of λ-adapted continuous systems of probability measures $\theta_n = (\theta_n^x)_{x\in\mathcal{G}^{(0)}}$ such that

$$\lim_{n\to\infty} \left\|\theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)}\right\| = 0 \text{ for all } \gamma \in \mathcal{G}. \tag{4}$$

We say that $(\mathcal{G}, \lambda)$ satisfies the *uniform Reiter's condition* if the convergence in (4) is uniform on compact subsets of $\mathcal{G}$.

If for every compact subset K of $\mathcal{G}^{(0)}$, the support $\mathrm{supp}_K(\theta_n)$ is compact for each θ_n in (4), we say that $(\mathcal{G}, \lambda)$ satisfies the *tight Reiter's condition*.

We also call $(\mathcal{G}, \lambda)$ *Reiter* (respectively, *uniform Reiter* or *tight Reiter*) if it satisfies Reiter's condition (respectively, the uniform Reiter's condition or the tight Reiter's condition).

Remark 2.16. A measured groupoid $(\mathcal{G}, \lambda, \mu)$, where λ is a left Haar system, is Reiter in our sense if and only if it is amenable (see [1, Chapter 3]). A locally compact groupoid $(\mathcal{G}, \lambda)$, where λ is a left Haar system, is uniform Reiter in our sense if and only if $\mathcal{G}$ is amenable (see [1, Chapter 2]). Moreover, by [35, Theorem 2.14], we know that in the case of locally compact groupoids, $(\mathcal{G}, \lambda)$ is Reiter if and only if it is uniform Reiter. We will see later that for such groupoids, Reiter's condition is also equivalent to the tight Reiter's condition.

Let us now consider the case of the semigroupoid $X \rtimes S$ attached to a transformation semigroup $X \curvearrowleft S$, where S is a countable discrete semigroup. For our quasi-Haar system, we always choose $\lambda^x = \delta_x \times \lambda$, where λ is the counting measure. Going through our definitions, we see that in the Borel case, $(X \rtimes S, \lambda, \mu)$ is Reiter if there exists a sequence $(\theta_n)_n$ of families $\theta_n = (\theta_n^x)_{x\in X}$ of probability measures θ_n^x on S such that for every $n \in \mathbb{N}$ and $s \in S$, the map $x \in X \mapsto \theta_n^x(s) \in \mathbb{R}$ is Borel and

$$\lim_{n\to\infty} \int_X \sum_{s\in S} \|\theta_n^x - s\theta_n^{x.s}\|\, f(x,s)d\mu(x) = 0 \quad \text{for all } f \in L^1(X \times S, \mu \times \lambda).$$

In the locally compact case, our definition says that $(X \rtimes S, \lambda)$ is Reiter if there exists a sequence $(\theta_n)_n$ of families $\theta_n = (\theta_n^x)_{x\in X}$ of probability measures θ_n^x on S such that for every $n \in \mathbb{N}$, the map $x \in X \mapsto \theta_n^x \in \ell^1(S)$ is continuous and

$$\lim_{n\to\infty} \|\theta_n^x - s\theta_n^{x\cdot s}\| = 0 \quad \text{for all } x \in X \text{ and } s \in S.$$

If the convergence is uniform on compact subsets of X, then $(X \rtimes S, \lambda)$ is uniform Reiter, and if $\operatorname{supp}_K \theta_n := \bigcup_{x\in K} \operatorname{supp} \theta_n^x \subseteq S$ is finite for every compact subset K of X, then $(X \rtimes S, \lambda)$ is tight Reiter.

2.2. Liouville property

Let $(\mathcal{G}, \lambda, \mu)$ be a measure semigroupoid and $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ a Borel system of probability measures. By natural extension as before, we often regard the measure π_x on $\mathcal{G}^x$ as a measure on $\mathcal{G}$ vanishing on $\mathcal{G}\backslash\mathcal{G}^x$. Given $\pi_x \prec \lambda^x$ for all $x \in \mathcal{G}^{(0)}$, we can define, for every $x \in \mathcal{G}^{(0)}$, the *fibrewise Markov operator* $P_x : L^\infty(\mathcal{G}^x, \lambda^x) \to L^\infty(\mathcal{G}^x, \lambda^x)$ by

$$P_x(f)(\zeta) = \int\limits_{\mathcal{G}^{\mathbf{s}(\zeta)}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \qquad (\zeta \in \mathcal{G}^x).$$

Extending Definition 2.2, a Borel function $f : \mathcal{G}^x \to \mathbb{C}$ is called P_x-*harmonic* if $P_x(f) = f$. Let $H_{\pi,x}(\mathcal{G}) := \{f \in L^\infty(\mathcal{G}^x, \lambda^x) \colon P_x(f) = f\}$ be the space of bounded P_x-harmonic functions on the fibre $\mathcal{G}^x$.

For a locally compact semigroupid $(\mathcal{G}, \lambda)$ with a λ-adapted continuous system π of probability measures, the space $H_{\pi,x}(\mathcal{G})$ is defined analogously.

Definition 2.17. A measure semigroupoid $(\mathcal{G}, \lambda, \mu)$ is said to have the *Liouville property*, or called *Liouville*, if there exists a λ-adapted Borel system of probability measures $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ such that $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for μ-a.e. $x \in \mathcal{G}^{(0)}$, where $\mathbf{1}$ is the constant function on $\mathcal{G}^x$ with value 1.

Definition 2.18. A locally compact semigroupoid $(\mathcal{G}, \lambda)$ with quasi-Haar system λ is said to have the *Liouville property*, or called *Liouville*, if there exists a λ-adapted continuous system of probability measures $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ such that $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{G}^{(0)}$.

We say that $(\mathcal{G}, \lambda)$ has the *continuous Liouville property*, or is *continuous Liouville*, if the system π above also satisfies the condition that the map $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\pi_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta) \in \mathbb{R}$ is bounded and continuous.

The semigroupoid $(\mathcal{G}, \lambda)$ is said to have the *tight Liouville property* or called *tight Liouville*, if it is Liouville with the system π satisfying the condition that for every compact subset K of $\mathcal{G}^{(0)}$, $\operatorname{supp}_K(\pi)$ is compact.

For groupoids, these definitions coincide with the ones given in [20].

Example 2.19. In the case of the semigroupoid $X \rtimes S$ attached to a transformation semigroup $X \curvearrowleft S$, where S is a discrete semigroup, the fibrewise Markov operator $P_x : \ell^\infty(S) \to \ell^\infty(S)$ is given by $P_x(f)(r) = \sum_{s\in S} f(rs)\pi_{x,r}(s)$. In particular, this is

just the convolution $f * \pi$ on the semigroup S for $X = \{\text{pt}\}$, in which case the Liouville property states that all bounded π-harmonic functions on S are constant.

3. Convolution of systems of measures

In this section, we derive some basic results needed later for the convolution of two systems of measures on semigroupoids. Let $\mathcal{G}$ be a Borel semigroupoid and $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ a Borel system of probability measures on $\mathcal{G}$. Given a positive Borel measure ρ on $\mathcal{G}^x$ for some $x \in \mathcal{G}^{(0)}$, we first define the convolution $\rho * \pi$ of ρ with the system π as the following Borel measure on $\mathcal{G}^x$:

$$\rho * \pi(E) = \int_{\mathcal{G}^x} \pi_{\mathbf{s}(\zeta)}\{\eta \in \mathcal{G}^{\mathbf{s}(\zeta)} : \zeta\eta \in E\} d\rho(\zeta)$$

for each Borel set E in $\mathcal{G}^x$. For $f \in L^1(\mathcal{G}^x, \rho * \pi)$, we have

$$\langle \rho * \pi, f\rangle := \int_{\mathcal{G}^x} f d(\rho * \pi) := \int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^{\mathbf{s}(\zeta)}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \right) d\rho(\zeta).$$

Now for a Borel system $\rho = (\rho_x)_{x\in\mathcal{G}^{(0)}}$ of measures on $\mathcal{G}$, we define $(\rho * \pi)_x := \rho_x * \pi$ and write $\rho * \pi = ((\rho * \pi)_x)_{x\in\mathcal{G}^{(0)}}$ for the family of Borel measures $(\rho * \pi)_x$ on $\mathcal{G}^x$.

For a locally compact semigroupoid $\mathcal{G}$ and a continuous system π of probability measures on $\mathcal{G}$, one can define analogously the convolution $\rho * \pi$ for a Borel measure ρ on some $\mathcal{G}^x$, and for a continuous system ρ of measures on $\mathcal{G}$.

Remark 3.1. The above definition of the convolution $\rho * \pi$ coincides with the one for groupoids given by Connes [8, p.11].

In the case of the semigroupoid $X \rtimes S$ attached to a transformation semigroup $X \curvearrowleft S$, where S is a discrete semigroup, the convolution $\rho * \pi$ is given by $(\rho * \pi)(t) = \sum_{\substack{r,s\in S\\ rs=t}} \pi_{x.r}(s)\rho(r)$.

We collect below some simple facts about convolutions of measures for later use. For a Borel semigroupoid $\mathcal{G}$, we denote by $B_b(\mathcal{G}^x)$ the algebra of bounded Borel functions on a fibre $\mathcal{G}^x$, equipped with the supremum norm $\|\cdot\|_\infty$. We recall that the total variation norm of a finite Borel measure ρ on $\mathcal{G}^x$ is given by $\|\rho\| = \sup\{|\int_{\mathcal{G}^x} f d\rho| : f \in B_b(\mathcal{G}^x), \|f\|_\infty \le 1\}$.

Lemma 3.2. *Let $\mathcal{G}$ be a Borel or locally compact semigroupoid and $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ a Borel system of probability measures on $\mathcal{G}$.*

a) *If ρ is a finite Borel measure on $\mathcal{G}^x$ for some $x \in \mathcal{G}^{(0)}$, then $\|\rho * \pi\| \le \|\rho\|$.*

b) *If ρ is a probability measure on $\mathcal{G}^x$ ($x \in \mathcal{G}^{(0)}$), then $\|\pi_x - \rho * \pi\| \leq \int_{\mathcal{G}^x} \|\pi_x - \zeta\pi_{\mathbf{s}(\zeta)}\| d\rho(\zeta)$.*

Proof.

a) Given $f \in B_b(\mathcal{G}^x)$, the map $f_\pi : \zeta \in \mathcal{G} \mapsto \int_{\mathcal{G}^{\mathbf{s}(\zeta)}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta)$ is a bounded Borel function, with $\|f_\pi\|_\infty \leq \|f\|_\infty$. Hence $|\langle \rho * \pi, f\rangle| = |\langle \rho, f_\pi\rangle| \leq \|\rho\| \cdot \|f_\pi\|_\infty \leq \|\rho\| \cdot \|f\|_\infty$ and the result follows.

b) Let $\varepsilon > 0$. Pick $f \in B_b(\mathcal{G}^x)$ with $\|f\|_\infty = 1$ and $|\langle \pi_x - \rho * \pi, f\rangle| \geq \|\pi_x - \rho * \pi\| - \varepsilon$. As ρ is a probability measure, we have

$$\begin{aligned}
\|\pi_x - \rho * \pi\| &\leq \varepsilon + \left| \int_{\mathcal{G}^x} f(\eta) d\pi_x(\eta) - \int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^{\mathbf{s}(\zeta)}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \right) d\rho(\zeta) \right| \\
&\leq \varepsilon + \int_{\mathcal{G}^x} \left| \int_{\mathcal{G}^x} f(\eta) d\pi_x(\eta) - \int_{\mathcal{G}^{\mathbf{s}(\zeta)}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \right| d\rho(\zeta) \\
&\leq \varepsilon + \int_{\mathcal{G}^x} \left\| \pi_x - \zeta\pi_{\mathbf{s}(\zeta)} \right\| d\rho(\zeta)
\end{aligned}$$

where $\zeta\pi_{\mathbf{s}(\zeta)}$ is a measure on $\mathcal{G}^{\mathbf{t}(\zeta)} = \mathcal{G}^x$. This completes the proof. □

Lemma 3.3. *Let $\mathcal{G}$ be a Borel semigroupoid with Borel systems $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ and $\rho = (\rho_x)_{x\in\mathcal{G}^{(0)}}$ of probability measures on $\mathcal{G}$. Let $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ be a quasi-Haar system on $\mathcal{G}$ in b) and c) below.*

a) *$\rho * \pi$ is a Borel system of probability measures on $\mathcal{G}$.*

b) *If $\pi_y \prec \lambda^y$ for all $y \in \mathcal{G}^{(0)}$, then for every $x \in \mathcal{G}^{(0)}$ and every Borel measure σ on $\mathcal{G}^x$, we have $\sigma * \pi \prec \lambda^x$.*

c) *If $\rho_x \prec \lambda^x$ for all $x \in \mathcal{G}^{(0)}$, and if both maps $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\pi_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$ and $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\rho_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$ are Borel, then so is the map $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma(\rho * \pi)_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$.*

Proof.

a) Given a non-negative Borel function f on $\mathcal{G}$, the non-negative map $\zeta \in \mathcal{G} \mapsto \langle \zeta\pi_{\mathbf{s}(\zeta)}, f\rangle$ is Borel. Therefore the map $x \in \mathcal{G}^{(0)} \mapsto \int_{\mathcal{G}^x} \langle \zeta\pi_{\mathbf{s}(\zeta)}, f\rangle \, d\rho_x(\zeta)$ is Borel since ρ is a Borel system of measures.

b) This follows from $\langle \sigma * \pi, f\rangle = \int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^x} f d(\zeta\pi_{\mathbf{s}(\zeta)}) \right) d\sigma(\zeta)$ for each non-negative Borel function f on $\mathcal{G}^x$, and $\zeta\pi_{\mathbf{s}(\zeta)} \prec \zeta\lambda^{\mathbf{s}(\zeta)} \prec \lambda^{\mathbf{t}(\zeta)} = \lambda^x$.

c) Let $u_\gamma := d(\gamma\pi_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}$ and $v_\gamma := d(\gamma\rho_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}$. Then for each non-negative Borel function f on $\mathcal{G}$ and $\gamma \in \mathcal{G}$ with $\mathbf{s}(\gamma) = x$, we have

$$\begin{aligned}\langle \gamma(\rho * \pi)_x, f\rangle &= \int_{\mathcal{G}^x} \left(\int_{\mathcal{G}} f(\gamma\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \right) d\rho_x(\zeta) = \int_{\mathcal{G}} \left(\int_{\mathcal{G}} f(\xi\eta) d\pi_{\mathbf{s}(\xi)}(\eta) \right) d(\gamma\rho_x)(\xi) \\ &= \int_{\mathcal{G}} \left(\int_{\mathcal{G}} f d(\xi\pi_{\mathbf{s}(\xi)}) \right) d(\gamma\rho_x)(\xi) = \int_{\mathcal{G}} \left(\int_{\mathcal{G}} f u_\xi d\lambda^{\mathbf{t}(\xi)} \right) v_\gamma(\xi) d\lambda^{\mathbf{t}(\gamma)}(\xi) \\ &= \int_{\mathcal{G}} f(\omega) \left(\int_{\mathcal{G}} u_\xi(\omega) v_\gamma(\xi) d\lambda^{\mathbf{t}(\gamma)}(\xi) \right) d\lambda^{\mathbf{t}(\gamma)}(\omega).\end{aligned}$$

Hence we have $\left(d(\gamma(\rho * \pi)_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\omega) = \int_{\mathcal{G}} u_\xi(\omega) v_\gamma(\xi) d\lambda^{\mathbf{t}(\gamma)}(\xi)$. It follows that the map $(\gamma, \omega) \in \mathcal{G}^2 \mapsto \left(d(\gamma(\rho * \pi)_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\omega)$ is Borel. □

Lemma 3.4. *Let $\mathcal{G}$ be a locally compact semigroupoid with two continuous systems $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ and $\rho = (\rho_x)_{x\in\mathcal{G}^{(0)}}$ of probability measures on $\mathcal{G}$. Then*

a) *$\rho * \pi$ is a continuous system of probability measures.*
b) *If π and ρ have the property that for every compact subset $K \subseteq \mathcal{G}^{(0)}$, the supports $\mathrm{supp}_K(\pi)$ and $\mathrm{supp}_K(\rho)$ are compact, then $\rho * \pi$ has this property as well.*
c) *Let $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ be a quasi-Haar system on $\mathcal{G}$. If $\pi_x \prec \lambda^x$ and $\rho_x \prec \lambda^x$ for all $x \in \mathcal{G}^{(0)}$, and if both maps $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\pi_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta) \in \mathbb{R}$ and $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma\rho_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta) \in \mathbb{R}$ are bounded and continuous, then the map $(\gamma, \eta) \in \mathcal{G}^2 \mapsto \left(d(\gamma(\rho * \pi)_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta) \in \mathbb{R}$ is also bounded and continuous.*

Proof.

a) Since $\mathcal{G}$ is locally compact and second countable, we know that the map $x \in \mathcal{G}^{(0)} \mapsto \langle \pi_x, f\rangle \in \mathbb{C}$ is continuous for every $f \in C_b(\mathcal{G})$ (not only for $f \in C_c(\mathcal{G})$), and likewise for ρ. A similar argument as in the proof of [31, Chapter III, Lemma 1.1] shows that for every $f \in C_b(\mathcal{G})$, the map $\zeta \in \mathcal{G} \mapsto \langle \zeta\pi_{\mathbf{s}(\zeta)}, f\rangle \in \mathbb{C}$ is continuous (and obviously bounded, too). Hence, as ρ is a continuous system of measures, we conclude that the map $x \in \mathcal{G}^{(0)} \mapsto \langle (\rho * \pi)_x, f\rangle = \int_{\mathcal{G}^x} \langle \zeta\pi_{\mathbf{s}(\zeta)}, f\rangle \, d\rho_x(\zeta) \in \mathbb{C}$ is continuous.
b) Given two subsets A and B of $\mathcal{G}$, we write $A \cdot B$ for the image of $(A \times B) \cap \mathcal{G}^{(2)}$ under the composition map $\mathcal{G}^{(2)} \to \mathcal{G}$. Obviously, $A \cdot B$ is compact if both A and B are compact. Now given a compact subset K of $\mathcal{G}^{(0)}$, it is obvious that $\mathrm{supp}_K(\rho * \pi) \subseteq \mathrm{supp}_K(\rho) \cdot \mathrm{supp}_K(\pi)$.
c) This follows from the same computation as in the proof of Lemma 3.3 c) and a similar argument to the proof of [31, Chapter III, Lemma 1.1]. □

4. Liouville property implies Reiter's condition

We are now ready to reveal the relationship between Reiter's condition and the Liouville property for semigroupoids. We do so in the measurable and topological setting. In this section, we prove that for semigroupoids equipped with a quasi-Haar system, the Liouville property implies Reiter's condition. This extends two results of Kaimanovich (Theorem 4.2 and Theorem 6.1 in [20]) for groupoids with a Haar system. We first establish the connection between our setting and the one in [20, § 3].

Let $(\mathcal{G}, \lambda, \mu)$ be a measure semigroupoid and $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ a λ-adapted Borel system of probability measures on $\mathcal{G}$. We observe that, for every $x \in \mathcal{G}^{(0)}$, the fibrewise Markov operator $P_x : L^\infty(\mathcal{G}^x, \lambda_x) \to L^\infty(\mathcal{G}^x, \lambda_x)$ has a predual operator given by $\theta \in L^1(\mathcal{G}^x, \lambda_x) \mapsto \theta P_x := \theta * \pi \in L^1(\mathcal{G}^x, \lambda_x)$ since

$$\langle \theta, P_x(f)\rangle = \int_{\mathcal{G}^x} \left(\int_{\mathcal{G}} f(\zeta\eta) d\pi_{\mathbf{s}(\zeta)}(\eta) \right) d\theta(\zeta) = \langle \theta * \pi, f\rangle \qquad (f \in L^\infty(\mathcal{G}^x, \lambda_x))$$

(cf. [20, § 3]). We have the k-th iterate $\theta P_x^k = \theta * \pi^k$.

Theorem 4.1. *Let $(\mathcal{G}, \lambda, \mu)$ be a measure semigroupoid. If $(\mathcal{G}, \lambda, \mu)$ is Liouville. Then it is Reiter.*

Proof. Let $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ be a λ-adapted Borel system of probability measures such that $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for μ-a.e. $x \in \mathcal{G}^{(0)}$. Moreover, let $\theta = (\theta^x)_{x\in\mathcal{G}^{(0)}}$ be any λ-adapted Borel system of probability measures. For instance, we could take $\theta^x = \pi_x$. Set $\theta_n^x := \frac{1}{n+1}\sum_{k=0}^n \theta^x * \pi^k$. This is, for every $n \in \mathbb{N}$, a λ-adapted Borel system of probability measures by Lemma 3.3 a), b) and c). We have for every $\gamma \in \mathcal{G}$:

$$\left\| \theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)} \right\| = \left\| \frac{1}{n+1}\sum_{k=0}^n \theta^{\mathbf{t}(\gamma)} * \pi^k - \gamma(\theta^{\mathbf{s}(\gamma)} * \pi^k) \right\| = \left\| \frac{1}{n+1}\sum_{k=0}^n (\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}) * \pi^k \right\|.$$

By assumption, we have $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for μ-a.e. $x \in \mathcal{G}^{(0)}$. In the language of [19], this means that for μ-a.e. $x \in \mathcal{G}^{(0)}$, the Poisson boundary of P_x is trivial. Hence [19, Theorem 2.8] implies that for μ-a.e. $x \in \mathcal{G}^{(0)}$ and λ^x-a.e. $\gamma \in \mathcal{G}^x$, we have

$$\lim_{n\to\infty} \left\| \frac{1}{n+1}\sum_{k=0}^n (\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}) * \pi^k \right\| = \lim_{n\to\infty} \left\| \frac{1}{n+1}\sum_{k=0}^n (\theta^{\mathbf{t}(\gamma)} P_{\mathbf{t}(\gamma)}^k - \gamma\theta^{\mathbf{s}(\gamma)} P_{\mathbf{t}(\gamma)}^k) \right\| = 0.$$

By Remark 2.14, $(\mathcal{G}, \lambda, \mu)$ is Reiter. □

In the topological setting, we have the following analogous result.

Theorem 4.2. *Let $(\mathcal{G}, \lambda)$ be a locally compact semigroupoid with quasi-Haar system λ.*

1) *If $(\mathcal{G}, \lambda)$ is Liouville, then it is Reiter.*
2) *If $(\mathcal{G}, \lambda)$ is tight Liouville, then it is tight Reiter.*
3) *If $(\mathcal{G}, \lambda)$ is continuous Liouville, then it is uniform Reiter.*

Proof. For 1), let $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ be a λ-adapted continuous system of probability measures on $\mathcal{G}$ such that $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{G}^{(0)}$. Define a new family of measures $\rho = (\rho_x)_{x\in\mathcal{G}^{(0)}}$ by setting $\rho_x := \frac{1}{2}(\pi_x + (\pi * \pi)_x)$. By Lemma 3.3 b), c) and Lemma 3.4 a), ρ is again a λ-adapted continuous system of probability measures on $\mathcal{G}$. Moreover, let $\theta = (\theta^x)_{x\in\mathcal{G}^{(0)}}$ be any λ-adapted continuous system of probability measures. For instance, we could take $\theta^x = \pi_x$. We define, for each $n \in \mathbb{N}$, $\theta_n := \theta * \rho^n$. By our assumption, $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{G}^{(0)}$, or in the language of [19], the Poisson boundary of P_x is trivial for all $x \in \mathcal{G}^{(0)}$. Hence, combining [19, Theorem 2.6] and [19, Theorem 2.7] as in [20, Proof of Theorem 6.1], we obtain for every $\gamma \in \mathcal{G}$:

$$\begin{aligned}\lim_{n\to\infty} \left\|\theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)}\right\| &= \lim_{n\to\infty} \left\|(\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}) * \rho^n\right\| \\ &= \lim_{n\to\infty} \left\|2^{-n}\sum_{k=0}^{n}\binom{n}{k}(\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}) * \pi^{n+k}\right\| = 0. \qquad (5)\end{aligned}$$

This proves 1).

For 2), note that if we choose θ and π with the property that $\mathrm{supp}_K(\theta)$ and $\mathrm{supp}_K(\pi)$ are compact for all compact subsets $K \subseteq \mathcal{G}^{(0)}$, then for every $n \in \mathbb{N}$, θ_n has the same property due to Lemma 3.4 b).

For 3), we choose θ and π such that $\mathcal{G}^2 \to \mathbb{R}$, $(\gamma, \eta) \mapsto \left(d(\gamma\theta_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$ and $\mathcal{G}^2 \to \mathbb{R}$, $(\gamma, \eta) \mapsto \left(d(\gamma\pi_{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$ are bounded and continuous. Then, by Lemma 3.4 c), we know that for every $n \in \mathbb{N}$, θ_n has the same property. Hence for every $n \in \mathbb{N}$, the map $\gamma \in \mathcal{G} \mapsto \left\|\theta_n^{\mathbf{t}(\gamma)} - \gamma\theta_n^{\mathbf{s}(\gamma)}\right\| \in \mathbb{R}$ is continuous. Moreover, these maps form a decreasing sequence (in n) by Lemma 3.2 a). Therefore, Dini's theorem implies that the convergence in (5) is uniform on compact subsets of $\mathcal{G}$. □

5. Reiter's condition implies Liouville property

We are going to prove that Reiter's condition implies the Liouville property for semigroupoids, both in the measurable and topological setting. In the special case of groupoids, this proves Kaimanovich's conjecture and the details will be given in the next section. A crucial construction in the proof is to replace, in Reiter's condition on the semigroupoid $\mathcal{G}$, the Borel systems $(\theta_n)_{n\in\mathbb{N}}$ of approximately invariant measures by a single system $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ so that the convolution powers $(\pi_x^n)_{x\in\mathcal{G}^{(0)}}$ play the role of the sequence $(\theta_n^x)_{x\in\mathcal{G}^{(0)}}$. In the proof of Proposition 5.1, we follow the strategy adopted in [21] for the case of discrete groups. We begin with measure semigroupoids.

Proposition 5.1. *Let $(\mathcal{G},\lambda,\mu)$ be a measure semigroupoid. If $(\mathcal{G},\lambda,\mu)$ is Reiter, then there exists a λ-adapted Borel system of probability measures $\pi=(\pi_x)_{x\in\mathcal{G}^{(0)}}$ on $\mathcal{G}$ such that for μ-a.e. $x\in\mathcal{G}^{(0)}$ and λ^x-a.e. $\gamma\in\mathcal{G}^x$, we have $\lim_{i\to\infty}\left\|\pi^i_{\mathbf{t}(\gamma)}-\gamma\pi^i_{\mathbf{s}(\gamma)}\right\|=0$.*

Proof. Since $L^1(\mathcal{G}^{(0)},\mu)$ is separable (see Remark 2.1), there exists a sequence of finite subsets $(\mathcal{F}_i)_i$ of $L^1(\mathcal{G}^{(0)},\mu)_+$ such that $\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\dots$ and $\bigcup_{i=1}^\infty\mathcal{F}_i$ is dense in $L^1(\mathcal{G}^{(0)},\mu)_+$. Choose sequences $(t_i)_i$ and $(\varepsilon_i)_i$ in $(0,1)$ such that $\sum_{i=1}^\infty t_i=1$, $\varepsilon_1>\varepsilon_2>\varepsilon_3>\dots$ and $\lim_{i\to\infty}\varepsilon_i=0$. As λ is proper (Definition 2.8), we can find an increasing sequence of Borel subsets A_i of $\mathcal{G}$ such that $\mathcal{G}=\bigcup_{i=1}^\infty A_i$ and

$$C_i:=\sup_{x\in\mathcal{G}^{(0)}}\lambda^x(A_i)\in(0,\infty). \tag{6}$$

Choose a sequence $(n_i)_i$ of natural numbers such that $n_1<n_2<\dots$ and $(t_1+\dots+t_{i-1})^{n_i}<\frac{\varepsilon_i}{C_i}$.

Let us now choose inductively λ-adapted Borel systems of probability measures $\theta_i=(\theta_i^x)_{x\in\mathcal{G}^{(0)}}$ on $\mathcal{G}$. Let θ_1 be any λ-adapted Borel system of probability measures on $\mathcal{G}$. Now assume that $\theta_1,\dots,\theta_{m-1}$ have been chosen. Let

$$\Theta_{m-1}:=\left\{\theta_{k_1}*\dots*\theta_{k_j}\colon 1\le j\le n_m-1,\ k_j\in\{1,\dots,m-1\}\right\}.$$

Every $\rho\in\Theta_{m-1}$ is λ-adapted by Lemma 3.3 c). Let $u^x:=d\rho^x/d\lambda^x$. Then $\int_{\mathcal{G}^x}u^x(\zeta)d\lambda^x(\zeta)=\int_{\mathcal{G}^x}d\rho^x(\zeta)=1$ as ρ^x is a probability measure. For $f\in\mathcal{F}_{m-1}$ and $\rho\in\Theta_{m-1}$, let $\varphi_{f,\rho}(\zeta)=u^{\mathbf{t}(\zeta)}f(\mathbf{t}(\zeta))$. Then $\varphi_{f,\rho}$ is Borel because ρ is λ-adapted, and we have $\varphi_{f,\rho}\in L^1(\mathcal{G},\mu\star\lambda)$ since

$$\begin{aligned}\int_{\mathcal{G}^{(0)}}\left(\int_{\mathcal{G}^x}\varphi_{f,\rho}(\zeta)d\lambda^x(\zeta)\right)d\mu(x)&=\int_{\mathcal{G}^{(0)}}\left(\int_{\mathcal{G}^x}u^x(\zeta)d\lambda^x(\zeta)\right)f(x)d\mu(x)\\&=\int_{\mathcal{G}^{(0)}}f(x)d\mu(x)<\infty.\end{aligned}$$

For each $\rho\in\Theta_{m-1}$, let $v_\gamma(\eta):=\left(d(\gamma\rho^{\mathbf{s}(\gamma)})/d\lambda^{\mathbf{t}(\gamma)}\right)(\eta)$. Since $\gamma\rho^{\mathbf{s}(\gamma)}$ is a probability measure, we have

$$\int_{\mathcal{G}^x}v_\gamma(\eta)d\lambda^x(\eta)=\int_{\mathcal{G}^x}d(\gamma\rho^{\mathbf{s}(\gamma)})=1. \tag{7}$$

For $f\in\mathcal{F}_{m-1}$ and $\rho\in\Theta_{m-1}$, set $\phi_{f,\rho}(\eta)=\int_{\mathcal{G}}v_\gamma(\eta)1_{A_m}(\gamma)f(\mathbf{t}(\eta))d\lambda^{\mathbf{t}(\eta)}(\gamma)$. Again, $\phi_{f,\rho}$ is Borel because ρ is λ-adapted, and $\phi_{f,\rho}$ lies in $L^1(\mathcal{G},\mu\star\lambda)$ as (6) and (7) imply

$$\begin{aligned}
&\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \phi_{f,\rho}(\eta) d\lambda^x(\eta) \right) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^{\mathbf{t}(\eta)}} v_\gamma(\eta) 1_{A_m}(\gamma) f(\mathbf{t}(\eta)) d\lambda^{\mathbf{t}(\eta)}(\gamma) \right) d\lambda^x(\eta) \right) d\mu(x) \\
&= \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^x} v_\gamma(\eta) d\lambda^x(\eta) \right) 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \\
&\leq C_m \cdot \int_{\mathcal{G}^{(0)}} f(x) d\mu(x) < \infty.
\end{aligned}$$

Since $(\mathcal{G}, \lambda, \mu)$ is Reiter, we can choose θ_m such that

$$\int_{\mathcal{G}} \left\| \theta_m^{\mathbf{t}(\gamma)} - \gamma \theta_m^{\mathbf{s}(\gamma)} \right\| 1_{A_m}(\gamma) f(\mathbf{t}(\gamma)) d(\mu \star \lambda)(\gamma) < \varepsilon_m \text{ for all } f \in \mathcal{F}_{m-1}, \tag{8}$$

$$\int_{\mathcal{G}} \left\| \theta_m^{\mathbf{t}(\zeta)} - \zeta \theta_m^{\mathbf{s}(\zeta)} \right\| \varphi_{f,\rho}(\zeta) d(\mu \star \lambda)(\zeta) < \tfrac{\varepsilon_m}{C_m} \text{ for all } f \in \mathcal{F}_{m-1} \text{ and } \rho \in \Theta_{m-1} \tag{9}$$

and

$$\int_{\mathcal{G}} \left\| \theta_m^{\mathbf{t}(\eta)} - \eta \theta_m^{\mathbf{s}(\eta)} \right\| \phi_{f,\rho}(\eta) d(\mu \star \lambda)(\eta) < \varepsilon_m \text{ for all } f \in \mathcal{F}_{m-1} \text{ and } \rho \in \Theta_{m-1}. \tag{10}$$

Now set $\pi_x := \sum_{i=1}^\infty t_i \theta_i^x$ for all $x \in \mathcal{G}^{(0)}$. $\pi = (\pi_x)$ is again a λ-adapted Borel system of probability measures.

Take $m \in \mathbb{N}$ and $f \in \mathcal{F}_{m-1}$. Set $n := n_m$. Write the n-fold convolution $\pi_x^n = (\pi_1)_x + (\pi_2)_x$ where

$$(\pi_1)_x = \sum_{\substack{\mathbf{k} \in \mathbb{N}^n \\ \max(\mathbf{k}) < m}} t_{k_1} \cdots t_{k_n} (\theta_{k_1} * \cdots * \theta_{k_n})^x, \tag{11}$$

$$(\pi_2)_x = \pi_x^n - (\pi_1)_x = \sum_{\substack{\mathbf{k} \in \mathbb{N}^n \\ \max(\mathbf{k}) \geq m}} t_{k_1} \cdots t_{k_n} (\theta_{k_1} * \cdots * \theta_{k_n})^x. \tag{12}$$

Then $\|(\pi_1)_x\| \leq (t_1 + \cdots + t_{m-1})^{n_m} < \frac{\varepsilon_m}{C_m}$ for all $x \in \mathcal{G}^{(0)}$ and hence

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\pi_1)_{\mathbf{t}(\gamma)} - \gamma (\pi_1)_{\mathbf{s}(\gamma)} \right\| 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \tag{13}$$

$$< \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} 2\tfrac{\varepsilon_m}{C_m} 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \leq 2\varepsilon_m.$$

For $\mathbf{k} \in \mathbb{N}^n$ with $\max(\mathbf{k}) \geq m$, $\theta_{k_1} * \cdots * \theta_{k_n}$ is of the form $\theta * \tau$ for some $\theta = \theta_k$ with $k \geq m$, or it is of the form $\rho * \theta * \tau$ for some $\rho \in \Theta_{m-1}$ and $\theta = \theta_k$ with $k \geq m$. By (8), we have

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\gamma)} - \gamma \theta^{\mathbf{s}(\gamma)} \right\| 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < \varepsilon_m. \tag{14}$$

For every $x \in \mathcal{G}^{(0)}$, Lemma 3.2 b) implies

$$\|\theta^x - (\rho * \theta)^x\| \leq \int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\zeta)} - \zeta \theta^{\mathbf{s}(\zeta)} \right\| d\rho^x(\zeta). \tag{15}$$

Therefore we have

$$\begin{aligned}
&\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \|\theta^x - (\rho * \theta)^x\| \, 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \qquad (16)\\
&\leq C_m \int_{\mathcal{G}^{(0)}} \|\theta^x - (\rho * \theta)^x\| \, f(x) d\mu(x)\\
&\overset{(15)}{\leq} C_m \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\zeta)} - \zeta \theta^{\mathbf{s}(\zeta)} \right\| d\rho^x(\zeta) \right) f(x) d\mu(x)\\
&= C_m \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\zeta)} - \zeta \theta^{\mathbf{s}(\zeta)} \right\| u^x(\zeta) f(x) d\lambda^x(\zeta) \right) d\mu(x)\\
&= C_m \int_{\mathcal{G}} \left\| \theta^{\mathbf{t}(\zeta)} - \zeta \theta^{\mathbf{s}(\zeta)} \right\| \varphi_{f,\rho}(\zeta) d(\mu \star \lambda)(\zeta) \overset{(9)}{<} \varepsilon_m.
\end{aligned}$$

For every $\gamma \in \mathcal{G}$, Lemma 3.2 b) implies that

$$\left\| \theta^{\mathbf{t}(\gamma)} - \gamma (\rho * \theta)^{\mathbf{s}(\gamma)} \right\| = \left\| \theta^{\mathbf{t}(\gamma)} - (\gamma \rho^{\mathbf{s}(\gamma)}) * \theta \right\| \leq \int_{\mathcal{G}^{\mathbf{t}(\gamma)}} \left\| \theta^{\mathbf{t}(\gamma)} - \eta \theta^{\mathbf{s}(\eta)} \right\| d(\gamma \rho^{\mathbf{s}(\gamma)})(\eta). \tag{17}$$

Using $v_\gamma(\eta) := \left(d(\gamma \rho^{\mathbf{s}(\gamma)}) / d\lambda^{\mathbf{t}(\gamma)} \right)(\eta)$ and $\phi_{f,\rho}(\eta) = \int_{\mathcal{G}} v_\gamma(\eta) 1_{A_m}(\gamma) f(\mathbf{t}(\eta)) d\lambda^{\mathbf{t}(\eta)}(\gamma)$, a computation analogous to (16) gives

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta)^{\mathbf{s}(\gamma)} \right\| 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \tag{18}$$

$$\overset{(17)}{\leq} \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left(\int_{\mathcal{G}^x} \left\| \theta^{\mathbf{t}(\gamma)} - \eta\theta^{\mathbf{s}(\eta)} \right\| d(\gamma\rho^{\mathbf{s}(\gamma)})(\eta) \right) 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x)$$

$$= \int_{\mathcal{G}} \left\| \theta^{\mathbf{t}(\eta)} - \eta\theta^{\mathbf{s}(\eta)} \right\| \phi_{f,\rho}(\eta) d(\mu \star \lambda)(\eta) \overset{(10)}{<} \varepsilon_m.$$

Combining (16) and (18), we get

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\rho * \theta)^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta)^{\mathbf{s}(\gamma)} \right\| 1_{A_m}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < 2\varepsilon_m$$

and also for all $m' \leq m$,

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\rho * \theta)^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta)^{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < 2\varepsilon_m. \tag{19}$$

With the help of Lemma 3.2 a), we deduce from (14) and (19) that

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\theta * \tau)^{\mathbf{t}(\gamma)} - \gamma(\theta * \tau)^{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < \varepsilon_m$$

and

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\rho * \theta * \tau)^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta * \tau)^{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < 2\varepsilon_m.$$

As every summand in π_2 is of the form $\theta * \tau$ or $\rho * \theta * \tau$, we obtain

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| (\pi_2)_{\mathbf{t}(\gamma)} - \gamma(\pi_2)_{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) \tag{20}$$

$$< \sum_{\substack{\mathbf{k} \in \mathbb{N}^n \\ \max(\mathbf{k}) \geq m}} t_{k_1} \cdots t_{k_n} \cdot 2\varepsilon_m \leq \left(\sum_{i=1}^{\infty} t_i \right)^n \cdot 2\varepsilon_m = 2\varepsilon_m.$$

Finally, combining (13) and (20), using Lemma 3.2 a), we obtain, for $m' \leq m$,

$$\int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) < 4\varepsilon_m \text{ for all } i \geq n = n_m.$$

Since $\lim_{m\to\infty} \varepsilon_m = 0$, we have, for every $m' \in \mathbb{N}$ and $f \in \mathcal{F}_{m'-1}$,

$$\lim_{i\to\infty} \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| 1_{A_{m'}}(\gamma) d\lambda^x(\gamma) \right) f(x) d\mu(x) = 0 \tag{21}$$

which gives

$$\lim_{i\to\infty} \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| d\lambda^x(\gamma) \right) f(x) d\mu(x) = 0$$

for all $f \in \bigcup_i \mathcal{F}_i$, and hence for all $f \in L^1(\mathcal{G}^{(0)}, \mu)$. It follows that for μ-a.e. $x \in \mathcal{G}^{(0)}$ and λ^x-a.e. $\gamma \in \mathcal{G}^x$, we have $\lim_{i\to\infty} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| = 0$. Here we have used the fact that the sequence $\left(\left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| \right)_i$ is decreasing by Lemma 3.2 a). □

Lemma 5.2. *Let $(\mathcal{G}, \lambda, \mu)$ be a measure semigroupoid. If there exists a λ-adapted Borel system of probability measures $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ such that $\lim_{i\to\infty} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| = 0$ for λ^x-a.e. $\gamma \in \mathcal{G}^x$ and μ-a.e. $x \in \mathcal{G}^{(0)}$, then we have $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for μ-a.e. $x \in \mathcal{G}^{(0)}$.*

Proof. Take $x \in \mathcal{G}^{(0)}$ such that $\lim_{i\to\infty} \left\| \pi^i_{\mathbf{t}(\gamma)} - \gamma \pi^i_{\mathbf{s}(\gamma)} \right\| = 0$ for λ^x-a.e. $\gamma \in \mathcal{G}^x$. Let $f \in L^\infty(\mathcal{G}^x, \lambda^x)$ satisfy $P_x(f) = f$. Then $f = P^i_x(f)$ and, for λ^x-a.e. $\gamma \in \mathcal{G}^x$, we have $f(\gamma) = \int_{\mathcal{G}^x} f(\gamma\zeta) d\pi^i_{\mathbf{s}(\gamma)}(\zeta)$. Therefore

$$\left| f(\gamma) - \int_{\mathcal{G}^x} f(\zeta) d\pi^i_x(\zeta) \right| = \left| \left\langle \gamma \pi^i_{\mathbf{s}(\gamma)} - \pi^i_{\mathbf{t}(\gamma)}, f \right\rangle \right| \leq \|f\|_\infty \cdot \left\| \gamma \pi^i_{\mathbf{s}(\gamma)} - \pi^i_{\mathbf{t}(\gamma)} \right\| \underset{i\to\infty}{\longrightarrow} 0$$

for λ^x-a.e. $\gamma \in \mathcal{G}^x$. Hence f is constant λ^x-a.e. □

From Proposition 5.1 and Lemma 5.2, we have established the following theorem.

Theorem 5.3. *Let $(\mathcal{G}, \lambda, \mu)$ be a measure semigroupoid. If $(\mathcal{G}, \lambda, \mu)$ is Reiter, then it is Liouville.*

We now discuss topological semigroupoids. In addition to the previous approach to the measure semigroupoid case, we need to make use of the uniform and tight Reiter's condition to deduce the existence of one single continuous system $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ in the following key proposition.

Proposition 5.4. *Let $(\mathcal{G}, \lambda)$ be a locally compact semigroupoid with quasi-Haar system. If $(\mathcal{G}, \lambda)$ is uniform and tight Reiter, then there exists a λ-adapted continuous system of probability measures $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ such that*

$$\lim_{i\to\infty} \left\|\pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}\right\| = 0 \text{ for all } \gamma \in \mathcal{G}.$$

Proof. Since $\mathcal{G}$ is σ-compact (see § 2), there exists a sequence of compact subsets $(C_i)_i$ of $\mathcal{G}$ such that $C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots$ and $\mathcal{G} = \bigcup_{i=0}^{\infty} C_i$. Choose sequences $(t_i)_i$ and $(\varepsilon_i)_i$ in $(0,1)$ such that $\sum_{i=1}^{\infty} t_i = 1$, $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ and $\lim_{i\to\infty} \varepsilon_i = 0$. Furthermore, choose a sequence $(n_i)_i$ of natural numbers such that $n_1 < n_2 < \dots$ and $(t_1 + \dots + t_{i-1})^{n_i} < \varepsilon_i$.

Let us now inductively choose λ-adapted continuous systems of probability measures $\theta_i = (\theta_i^x)_{x\in\mathcal{G}^{(0)}}$. Let θ_1 be such a continuous system of probability measures with $\left\|\theta_1^{\mathbf{t}(\gamma)} - \gamma\theta_1^{\mathbf{s}(\gamma)}\right\| < \varepsilon_1$ for all $\gamma \in C_0$. Now assume that $\theta_1, \dots, \theta_{m-1}$ have been chosen. Let

$$\Theta_{m-1} := \left\{\theta_{k_1} * \dots * \theta_{k_j} \colon 1 \le j \le n_m - 1,\ k_j \in \{1, \dots, m-1\}\right\}.$$

Lemma 3.3 c) implies that every $\rho \in \Theta_{m-1}$ is λ-adapted. Moreover, for $K \subseteq \mathcal{G}^{(0)}$ compact, set $\operatorname{supp}_K(\Theta_{m-1}) := \bigcup_{\rho\in\Theta_{m-1}} \operatorname{supp}_K(\rho)$. Then, by Lemma 3.4 b), $\operatorname{supp}_K(\Theta_{m-1})$ is compact for every compact subset $K \subseteq \mathcal{G}^{(0)}$. Now choose θ_m such that

$$\left\|\theta_m^{\mathbf{t}(\gamma)} - \gamma\theta_m^{\mathbf{s}(\gamma)}\right\| < \varepsilon_m \tag{22}$$

for all $\gamma \in \operatorname{supp}_{\mathbf{t}(C_{m-1})}(\Theta_{m-1}) \cup (C_{m-1} \cdot \operatorname{supp}_{\mathbf{s}(C_{m-1})}(\Theta_{m-1})) \cup C_{m-1}$. Such θ_m exist since $\mathcal{G}$ is uniform and tight Reiter.

Now set $\pi_x := \sum_{i=1}^{\infty} t_i\theta_i^x$ for all $x \in \mathcal{G}^{(0)}$. $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ is again a λ-adapted continuous system of probability measures. Take $m \in \mathbb{N}$ and $\gamma \in C_{m-1}$. Set $n := n_m$. Write $\pi_x^n = (\pi_1)_x + (\pi_2)_x$ where $(\pi_1)_x$ and $(\pi_2)_x$ are defined as in (11) and (12). Then $\|(\pi_1)_x\| \le (t_1 + \dots + t_{m-1})^{n_m} < \varepsilon_m$ for all $x \in \mathcal{G}^{(0)}$. Therefore, $\left\|\gamma(\pi_1)_{\mathbf{s}(\gamma)}\right\| < \varepsilon_m$. Thus we obtain

$$\left\|(\pi_1)_{\mathbf{t}(\gamma)} - \gamma(\pi_1)_{\mathbf{s}(\gamma)}\right\| < 2\varepsilon_m. \tag{23}$$

For $\mathbf{k} \in \mathbb{N}^n$ with $\max(\mathbf{k}) \ge m$, $\theta_{k_1} * \dots * \theta_{k_n}$ is of the form $\theta * \tau$ for some $\theta = \theta_k$ with $k \ge m$, or it is of the form $\rho * \theta * \tau$ for some $\rho \in \Theta_{m-1}$ and $\theta = \theta_k$ with $k \ge m$. By choice of θ_i, we have

$$\left\|\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}\right\| \overset{(22)}{<} \varepsilon_m. \tag{24}$$

Moreover, it follows that from Lemma 3.2 b) that

$$\left\|\theta^{\mathbf{t}(\gamma)} - (\rho * \theta)^{\mathbf{t}(\gamma)}\right\| \leq \int\limits_{\mathrm{supp}\,(\rho^{\mathbf{t}(\gamma)})} \left\|\theta^{\mathbf{t}(\zeta)} - \zeta\theta^{\mathbf{s}(\zeta)}\right\| d\rho^{\mathbf{t}(\gamma)}(\zeta) \overset{(22)}{<} \varepsilon_m. \tag{25}$$

Similarly, Lemma 3.2 b) implies that

$$\begin{aligned}\left\|\theta^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta)^{\mathbf{s}(\gamma)}\right\| &= \left\|\theta^{\mathbf{t}(\gamma)} - (\gamma\rho^{\mathbf{s}(\gamma)}) * \theta\right\| \leq \int\limits_{\mathcal{G}} \left\|\theta^{\mathbf{t}(\gamma)} - \zeta\theta^{\mathbf{s}(\zeta)}\right\| d(\gamma\rho^{\mathbf{s}(\gamma)})(\zeta) \\ &= \int\limits_{\mathrm{supp}\,(\rho^{\mathbf{s}(\gamma)})} \left\|\theta^{\mathbf{t}(\gamma\eta)} - \gamma\eta\theta^{\mathbf{s}(\gamma\eta)}\right\| d\rho^{\mathbf{s}(\gamma)}(\eta) \overset{(22)}{<} \varepsilon_m. \end{aligned} \tag{26}$$

Combining (25) and (26), we get

$$\left\|(\rho * \theta)^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta)^{\mathbf{s}(\gamma)}\right\| < 2\varepsilon_m. \tag{27}$$

Therefore, because of (24) and (27), Lemma 3.2 a) implies $\left\|(\theta * \tau)^{\mathbf{t}(\gamma)} - \gamma(\theta * \tau)^{\mathbf{s}(\gamma)}\right\| < \varepsilon_m$ and $\left\|(\rho * \theta * \tau)^{\mathbf{t}(\gamma)} - \gamma(\rho * \theta * \tau)^{\mathbf{s}(\gamma)}\right\| < 2\varepsilon_m$. As every summand in π_2 is of the form $\theta * \tau$ or $\rho * \theta * \tau$, we obtain

$$\left\|(\pi_2)_{\mathbf{t}(\gamma)} - \gamma(\pi_2)_{\mathbf{s}(\gamma)}\right\| < \sum_{\substack{\mathbf{k}\in\mathbb{N}^n \\ \max(\mathbf{k})\geq m}} t_{k_1} \cdots t_{k_n} \cdot 2\varepsilon_m \leq \left(\sum_{i=1}^{\infty} t_i\right)^n \cdot 2\varepsilon_m = 2\varepsilon_m. \tag{28}$$

Finally, combining (23) and (28), we get using Lemma 3.2 a)

$$\left\|\pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}\right\| < 4\varepsilon_m \text{ for all } i \geq n.$$

On the whole, we deduce that for every $\gamma \in \mathcal{G}$,

$$\lim_{i\to\infty} \left\|\pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}\right\| = 0. \quad \square$$

Lemma 5.5. *Let $\mathcal{G}$ be a topological semigroupoid with a quasi-Haar system λ. Suppose that there exists a λ-adapted continuous system of probability measures $\pi = (\pi_x)_{x\in\mathcal{G}^{(0)}}$ such that $\lim_{i\to\infty} \left\|\pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}\right\| = 0$ for all $\gamma \in \mathcal{G}^x$. Then we have $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{G}^{(0)}$.*

Proof. Take $x \in \mathcal{G}^{(0)}$ and $f \in L^\infty(\mathcal{G}^x, \lambda^x)$ with $P_x(f) = f$. Then $f = P^i_x(f)$ and $f(x) = \int_{\mathcal{G}^x} f(\zeta) d\pi^i_x(\zeta)$ and also, $f(\gamma) = \int_{\mathcal{G}^x} f(\gamma\zeta) d\pi^i_{\mathbf{s}(\gamma)}(\zeta)$ for all $\gamma \in \mathcal{G}^x$. Therefore $|f(x) - f(\gamma)| = \left|\left\langle \pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}, f\right\rangle\right| \leq \|f\|_\infty \cdot \left\|\pi^i_{\mathbf{t}(\gamma)} - \gamma\pi^i_{\mathbf{s}(\gamma)}\right\| \underset{i\to\infty}{\longrightarrow} 0$. This means that for all $\gamma \in \mathcal{G}^x$, we have $f(\gamma) = f(x)$ and hence f is constant. This proves $H_{\pi,x}(\mathcal{G}) = \mathbb{C}\mathbf{1}$. $\square$

Finally, Proposition 5.4 and Lemma 5.5 together yield the following result.

Theorem 5.6. *Let $(\mathcal{G}, \lambda)$ be a locally compact semigroupoid with quasi-Haar system. If $(\mathcal{G}, \lambda)$ is uniform and tight Reiter, then it is Liouville.*

6. Special cases

In this section, we discuss three important special cases, namely groupoids, transformation semigroups and semigroups. We begin with groupoids.

6.1. Groupoids

An important consequence of our results is the equivalence of amenability and the Liouville property for groupoids, in both measurable and topological settings. In particular, we settle a conjecture of Kaimanovich [20, Conjecture 4.6] by showing that amenable groupoids admit the Liouville property. To be precise, combining Theorem 4.1 and Theorem 5.3, we obtain the following result for measured groupoids.

Theorem 6.1. *Let $(\mathcal{G}, \lambda, \mu)$ be a measured groupoid, where λ is a left Haar system. Then $(\mathcal{G}, \lambda, \mu)$ is Liouville if and only if it is Reiter.*

Since amenability and Reiter's condition are equivalent for groupoids, this theorem establishes the equivalence of the Liouville property and amenability for measured groupoids. This result has also been proved by Bühler and Kaimanovich in an unpublished note. However, a similar result for topological groupoids is more subtle, which requires a deeper analysis of Reiter's condition, given below.

Lemma 6.2. *Let $(\mathcal{G}, \lambda)$ be a locally compact groupoid, where λ is a left Haar system. The following conditions are equivalent:*

(i) *$(\mathcal{G}, \lambda)$ is Reiter,*
(ii) *$(\mathcal{G}, \lambda)$ is uniform Reiter,*
(iii) *$(\mathcal{G}, \lambda)$ is tight Reiter.*

Proof. (i) $\Rightarrow$ (ii) follows from [35, Theorem 2.14]. (iii) $\Rightarrow$ (i) is obvious. It remains to prove (ii) $\Rightarrow$ (iii).

We show that for every $\varepsilon > 0$ and $C \subseteq \mathcal{G}$ compact, there exists a λ-adapted continuous system of probability measures $\theta = (\theta^x)_{x \in \mathcal{G}^{(0)}}$ on $\mathcal{G}$ such that $\left\|\theta^{\mathbf{t}(\gamma)} - \gamma\theta^{\mathbf{s}(\gamma)}\right\| < \varepsilon$ for all $\gamma \in C$, and in addition, $\mathrm{supp}_K(\theta)$ is compact for every compact subset $K \subseteq \mathcal{G}$.

Given $\varepsilon > 0$ and $C \subseteq \mathcal{G}$ compact, since $\mathcal{G}^{(0)}$ is locally compact and σ-compact, there exist $K_n \subseteq \mathcal{G}^{(0)}$ compact with $\mathbf{s}(C) \cup \mathbf{t}(C) \subseteq K_1$, $K_n \subseteq \overset{\circ}{K}_{n+1}$ for all $n \in \mathbb{N}$ and $\mathcal{G}^{(0)} = \bigcup_{n=1}^{\infty} K_n$.

Since $(\mathcal{G},\lambda)$ is Reiter, $\mathcal{G}$ is amenable and hence there exists a *topological approximate invariant density* $(g_j)_{j=1}^{\infty}$ in $C_c(\mathcal{G})$, as defined in [35, Definition 2.7]). Normalizing $g_j\cdot\lambda^x$ for some j and for each $x\in K_2$, we obtain a λ-adapted continuous system of measures $\vartheta_2=(\vartheta_2^x)_{x\in\mathcal{G}^{(0)}}$ such that

$$\left\|\vartheta_2^{\mathbf{t}(\gamma)}-\gamma\vartheta_2^{\mathbf{s}(\gamma)}\right\|<\varepsilon \text{ for all } \gamma\in C \tag{29}$$

with the additional properties that $\mathrm{supp}_{K_2}(\vartheta_2)$ is compact and ϑ_2^x is a probability measure for all $x\in K_2$. Likewise, for all $n\geq 3$, one can find λ-adapted continuous systems of measures $\vartheta_n=(\vartheta_n^x)_{x\in\mathcal{G}^{(0)}}$ such that $\mathrm{supp}_{K_n}(\vartheta_n)$ is compact and ϑ_n^x is a probability measure for all $x\in K_n$.

Now define $U_2:=\overset{\circ}{K_2}$ and $U_n:=\overset{\circ}{K_n}\setminus K_{n-2}$ for all $n\geq 3$. By construction, $\mathcal{G}^{(0)}=\bigcup_{n=2}^{\infty}U_n$. Since $\mathcal{G}^{(0)}$ is locally compact and σ-compact, $\mathcal{G}^{(0)}$ is paracompact and one can find a partition of unity $\{h_n\}_{n\geq 2}$ subordinate to $\{U_n\}_{n\geq 2}$. Define $\theta_n^x:=h_n(x)\vartheta_2^x$ for $x\in U_n$ and $\theta_n^x:=0$ for $x\notin U_n$. By construction, these θ_n are λ-adapted continuous systems of measures such that $\mathrm{supp}\,(\theta_n):=\mathrm{supp}_{\mathcal{G}^{(0)}}(\theta_n)\subseteq\mathrm{supp}_{K_n}(\vartheta_n)$. In particular, $\mathrm{supp}\,(\theta_n)$ is compact.

Set $\theta^x:=\sum_{n=2}^{\infty}\theta_n^x$ which is a finite sum since the cover $\{U_n\}$ is locally finite. As each θ_n is λ-adapted, so is the continuous system $\theta=(\theta^x)_{x\in\mathcal{G}^{(0)}}$ where $\|\theta^x\|=\sum_n\|\theta_n^x\|=\sum_n h_n(x)=1$.

We claim that θ has the desired properties. Indeed, given $\gamma\in C$, both $\mathbf{t}(\gamma)$ and $\mathbf{s}(\gamma)$ lie in K_1, hence $h_n(\mathbf{t}(\gamma))=h_n(\mathbf{s}(\gamma))=0$ for all $n\geq 3$, and so $h_2(\mathbf{t}(\gamma))=h_2(\mathbf{s}(\gamma))=1$. Thus $\theta^{\mathbf{t}(\gamma)}=\theta_2^{\mathbf{t}(\gamma)}=\vartheta_2^{\mathbf{t}(\gamma)}$, and also $\theta^{\mathbf{s}(\gamma)}=\theta_2^{\mathbf{s}(\gamma)}=\vartheta_2^{\mathbf{s}(\gamma)}$. It follows that

$$\left\|\theta^{\mathbf{t}(\gamma)}-\gamma\theta^{\mathbf{s}(\gamma)}\right\|=\left\|\vartheta_2^{\mathbf{t}(\gamma)}-\gamma\vartheta_2^{\mathbf{s}(\gamma)}\right\|\overset{(29)}{<}\varepsilon.$$

Finally, given $K\subseteq\mathcal{G}^{(0)}$ compact, there exists $N\in\mathbb{N}$ with $K\subseteq\bigcup_{n=2}^{N}U_n$, so that $h_n|_K\equiv 0$ for all $n\geq N+2$. Hence $\mathrm{supp}_K(\theta)\subseteq\bigcup_{n=2}^{N+1}\mathrm{supp}\,(\theta_n)$ which is compact. □

Combining Theorem 4.2 and Theorem 5.6 with the previous lemma, we obtain the equivalence of the Liouville property and Reiter's condition in the topological setting. This proves the topological analogue of [20, Conjecture 4.6].

Theorem 6.3. *Let $(\mathcal{G},\lambda)$ be a locally compact groupoid, where λ is a left Haar system. Then $(\mathcal{G},\lambda)$ is Liouville if and only if it is Reiter.*

Remark 6.4. By convention, all our locally compact groupoids are locally compact and second countable. However, the same proofs as above show that Lemma 6.2 and Theorem 6.3 also hold for topological groupoids which are locally compact and σ-compact.

6.2. Transformation semigroups

We now show the equivalence of the Liouville property and Reiter's condition for transformation semigroups. First, combining Theorem 4.1 and Theorem 5.3, one obtains the following result in the measure setting.

Theorem 6.5. *Let (X, μ) be an analytic Borel space. Let S be a discrete semigroup acting on X by Borel maps. Let $(X \rtimes S, \lambda, \mu)$ be the corresponding measure semigroupoid. Then $(X \rtimes S, \lambda, \mu)$ is Liouville if and only if it is Reiter.*

For the topological setting, we prove a lemma first.

Lemma 6.6. *Let X be a locally compact and second countable Hausdorff space, and S a countable discrete semigroup acting on X by continuous maps. Let $(X \rtimes S, \lambda)$ be the corresponding locally compact semigroupoid with quasi-Haar system. The following are equivalent:*

(i) *$(X \rtimes S, \lambda)$ is uniform Reiter,*
(ii) *$(X \rtimes S, \lambda)$ is uniform and tight Reiter,*
(iii) *For each $\varepsilon > 0$, finite set $F \subseteq S$ and compact set $C \subseteq X$, there exists a family $\theta = (\theta^x)_{x \in X}$ of probability measures on S satisfying the following conditions:*
  (1) *the map $x \in X \mapsto \theta^x \in \ell^1(S)$ is continuous;*
  (2) *$\|\theta^x - s\theta^{x \cdot s}\| < \varepsilon$ for all $s \in F$ and $x \in C$;*
  (3) *$\operatorname{supp}(\theta) := \bigcup_{x \in X} \operatorname{supp}(\theta^x)$ is finite.*

Proof. Clearly, we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). It remains to prove (i) $\Rightarrow$ (iii). Let $\varepsilon > 0$ with a finite set $F \subseteq S$ and compact set $C \subseteq X$ given in (iii). Since $(X \rtimes S, \lambda)$ is uniform Reiter, there is a family of probability measures $\vartheta = (\vartheta^x)_{x \in X}$ on S such that the map $x \in X \mapsto \vartheta^x \in \ell^1(S)$ is continuous and $\|\vartheta^x - \delta_s * \vartheta^{x \cdot s}\| < \frac{\varepsilon}{2}$ for all $s \in F$ and $x \in C$. Since X is locally compact and σ-compact, there exist compact subsets K_n of X with $C \cup C.F \subseteq K_1$, $K_n \subseteq \overset{\circ}{K}_{n+1}$ for all $n \in \mathbb{N}$ and $X = \bigcup_{n=1}^{\infty} K_n$.

Let us now define probability measures ϑ_2^x on S for all $x \in K_2$ such that the map $x \in K_2 \mapsto \vartheta_2^x \in \ell^1(S)$ is continuous. For $1 > \varepsilon' > 0$ and $x \in K_2$, there exists a finite set $E_x \subseteq S$ such that $\left\|\sum_{s \in E_x} \vartheta^x(s)\delta_s - \vartheta^x\right\| < \varepsilon'$. As $y \in X \mapsto \vartheta^y \in \ell^1(S)$ is continuous, the map $y \in X \mapsto \sum_{s \in E_x} \vartheta^y(s)\delta_s - \vartheta^y \in \ell^1(S)$ is continuous as well. Hence there exists an open neighbourhood U_x of x such that $\left\|\sum_{s \in F_x} \vartheta^y(s)\delta_s - \vartheta^y\right\| < \varepsilon'$ for all $y \in U_x$. Since K_2 is compact, there exist finitely many $x_1, \ldots, x_N$ in K_2 such that $K_2 \subseteq \bigcup_{i=1}^N U_{x_i}$. Define $E := \bigcup_{i=1}^N E_{x_i}$ and for $x \in K_2$, set $\tilde{\vartheta}_2^x := \sum_{s \in E} \vartheta^x(s)$. Obviously, the map $x \in K_2 \mapsto \tilde{\vartheta}_2^x \in \ell^1(S)$ is continuous. For all $x \in K_2$, we have $\left\|\tilde{\vartheta}_2^x\right\| \geq \|\vartheta^x\| - \left\|\tilde{\vartheta}_2^x - \vartheta^x\right\| > 1 - \varepsilon'$. Thus we may form $\vartheta_2^x := \left\|\tilde{\vartheta}_2^x\right\|^{-1} \cdot \tilde{\vartheta}_2^x$ for all $x \in K_2$. By construction, ϑ_2^x are probability measures on S such that the map $x \in K_2 \mapsto \vartheta_2^x \in \ell^1(S)$ is continuous. Moreover, for all $x \in K_2$, we have

$$\|\vartheta_2^x - \vartheta^x\| \le \|\vartheta_2^x - \tilde{\vartheta}_2^x\| + \|\tilde{\vartheta}_2^x - \vartheta^x\| = \left(\|\tilde{\vartheta}_2^x\|^{-1} - 1\right)\|\tilde{\vartheta}_2^x\| + \|\tilde{\vartheta}_2^x - \vartheta^x\|$$
$$< (1-\varepsilon')^{-1} - 1 + \varepsilon' < \tfrac{\varepsilon}{4}$$

for sufficiently small ε'. Hence it follows that for all $s \in E$ and $x \in C$, we have

$$\|\vartheta_2^x - \delta_s * \vartheta_2^{x.s}\| \le \|\vartheta_2^x - \vartheta^x\| + \|\vartheta^x - \delta_s * \vartheta^{x.s}\| + \|\delta_s * \vartheta^{x.s} - \delta_s * \vartheta_2^{x.s}\| \quad (30)$$
$$< \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{4} = \varepsilon.$$

Now define $U_2 := \overset{\circ}{K}_2$ and $U_n := \overset{\circ}{K}_n \setminus K_{n-2}$ for all $n \ge 3$. By construction, we have $\overset{\circ}{K}_n \setminus \overset{\circ}{K}_{n-1} \subseteq U_n$ for all $n \ge 3$, and $X \subseteq \bigcup_{n=2}^\infty K_{n-1} \subseteq \bigcup_{n=2}^\infty \overset{\circ}{K}_n \subseteq \bigcup_{n=2}^\infty U_n$. Since X is locally compact and σ-compact, we can find a partition of unity $\{h_n\}_{n\ge2}$ subordinate to $\{U_n\}_{n\ge2}$. Define $\theta_2^x := h_2(x)\vartheta_2^x$ for $x \in U_2$ and $\theta_2^x := 0$ for $x \notin U_2$. Then the map $x \in X \mapsto \theta_2^x \in \ell^1(S)$ is continuous. Now fix $t \in S$ and define, for $n \ge 3$, $\theta_n^x := h_n(x)\delta_t$. Set $\theta^x := \sum_{n=2}^\infty \theta_n^x$.

We claim that θ has the desired properties. For every $x \in X$, $\|\theta_n^x\| = h_n(x)$ implies $\|\theta^x\| = \sum_n h_n(x) = 1$ and θ^x are probability measures. Moreover, given $x \in X$, there exist $N \ge 2$ such that x lies in U_N. But by construction, $U_N \subseteq K_N$ implies $U_N \cap U_n = \emptyset$ for all $n \ge N+2$. Therefore, $\theta^x = \sum_{n=2}^{N+1} \theta_n^x$ for all $x \in U_N$. This shows continuity of the map $x \in U_N \mapsto \theta^x \in \ell^1(S)$ and $x \in X \mapsto \theta^x \in \ell^1(S)$. Furthermore, it is clear that for all $x \in X$, we have $\operatorname{supp}(\theta^x) \subseteq E \cup \{t\}$. Finally, given $s \in F$ and $x \in C$, both x and $x.s$ lie in K_1 and hence $h_n(x) = h_n(x.s) = 0$ for all $n \ge 3$, and $h_2(x) = h_2(x.s) = 1$. Thus $\theta^x = \theta_2^x = \vartheta_2^x$ and $\theta^{x.s} = \theta_2^{x.s} = \vartheta_2^{x.s}$. We conclude that

$$\|\theta^x - \delta_s * \theta^{x.s}\| = \|\vartheta_2^x - \delta_s * \vartheta_2^{x.s}\| \overset{(30)}{<} \varepsilon. \quad \square$$

We observe from the definition that the transformation semigroupoid $(X \rtimes S, \lambda)$ in the above lemma is Liouville if and only if it is continuous Liouville, and now, using Theorem 4.2 and Theorem 5.6, we can conclude with the following result.

Theorem 6.7. *Let X be a locally compact and second countable Hausdorff space, and S a countable discrete semigroup acting on X by continuous maps. Let $(X \rtimes S, \lambda)$ be the corresponding locally compact semigroupoid with quasi-Haar system. Then $(X \rtimes S, \lambda)$ is Liouville if and only if it is uniform Reiter.*

6.3. Locally compact semigroups

Let S be a second countable locally compact semigroup equipped with a positive quasi-invariant Borel measure λ such that the map $(s,t) \in S \times S \mapsto (d(s\lambda)/d\lambda)(t)$ is Borel. As explained in Remark 2.10, by identifying S with $\{\mathrm{pt}\} \rtimes S$, we may view (S, λ) as a locally compact semigroupoid with a quasi-Haar system. In this case, (S, λ) is Reiter, in the sense of Definition 2.15, if there exists a sequence (θ_n) of Borel probability measures on S with $\theta_n \prec \lambda$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|\theta_n - s\theta_n\| = 0$ for all $s \in S$.

Remark 6.8. In the above setting, (S, λ) is Reiter if and only if there exists a left invariant mean on $L^\infty(S, \lambda)$ since the proof in [25, § 4.2] carries over. Therefore there is no difference between Reiter's condition and amenability for (S, λ).

To describe the Liouville property for (S, λ) according to Definition 2.18, let π be a probability measure on S with $\pi \prec \lambda$. In this case, the fibrewise Markov operators reduce to a single one $P : L^\infty(S, \lambda) \to L^\infty(S, \lambda)$ with $P(f)(s) = \int_S f(st)d\pi(t)$, and $H_\pi(S) := \{f \in L^\infty(S, \lambda) \colon P(f) = f\}$ is the space of π-harmonic functions in $L^\infty(S, \lambda)$. The locally compact semigroup (S, λ) is Liouville if there exists a probability measure π on S with $\pi \prec \lambda$ such that $H_\pi(S) = \mathbb{C}\mathbf{1}$.

Theorem 6.9. *In the situation described above, (S, λ) is Liouville if and only if it is Reiter.*

Proof. The key observation is that, by definition, (S, λ) is Liouville if and only if $(\{\mathrm{pt}\} \rtimes S, \delta \times \lambda, \delta)$ is Liouville as a measure semigroupoid (where δ is the point mass). By Theorem 4.1, we know that (S, λ) is Reiter if it is Liouville. For the converse, let (S, λ) be Reiter. This means that (S, λ) is Reiter as a locally compact semigroupoid with quasi-Haar system. Again by definition, this implies that $(\{\mathrm{pt}\} \rtimes S, \delta \times \lambda, \delta)$ is Reiter as a measure semigroupoid. Hence Theorem 5.3 implies that $(\{\mathrm{pt}\} \rtimes S, \delta \times \lambda, \delta)$ is Liouville as a measure semigroupoid. Hence the locally compact semigroup (S, λ) is Liouville. □

The question remains which semigroups S admit a quasi-invariant measure λ with a Borel map $(s, t) \in S \times S \mapsto (d(s\lambda)/d\lambda)(t)$. This is for instance the case if S is a locally compact subsemigroup of a second countable locally compact group G, that is, S is a subsemigroup of G as well as a locally compact subspace. If the Haar measure λ_G on G satisfies $\lambda_G(S) \neq 0$, then the restriction λ of λ_G to S is a measure with the desired properties.

Of course, another class of examples is given by countable discrete semigroups S with identity and counting measure λ. For these semigroups, we have the following result from Remark 6.8 and Theorem 6.9. In the next subsection, we discuss the case of metric semigroups, which need not be locally compact nor support a quasi-invariant measure.

Theorem 6.10. *Let S be a countable discrete semigroup with identity. The following conditions are equivalent.*

(i) *S is Liouville, that is, there exists a probability measure π on S such that every bounded π-harmonic function is constant;*
(ii) *S is Reiter, that is, there exists a sequence $(\theta_n)_n$ of probability measures on S such that $\lim_{n\to\infty} \|\theta_n - s\theta_n\| = 0$ for all $s \in S$;*
(iii) *S is amenable.*

In the last section, we give some examples of discrete semigroups with the Liouville property for all non-degenerate probability measures, i.e., with the property that every bounded π-harmonic function is constant for every non-degenerate probability measure π.

6.4. Metric semigroups

Following the previous results for semigroups equipped with a quasi-invariant measure, it is natural to enquire about semigroups without such a measure. We present the case of metric semigroups in this subsection, but refer to [7] for further discussion of topological semigroups.

Given a probability measure π on a metric semigroup S, we denote by $H_\pi(S)$ the closed subspace of $LUC(S)$ consisting of π-harmonic functions on S. We say that S has the *Liouville property* if there is a tight probability measure π on S such that $H_\pi(S) = \mathbb{C}\mathbf{1}$ where $\mathbf{1}$ denotes the constant function on S with value 1.

A metric semigroup S is said to satisfy *Reiter's condition* if for every $\varepsilon > 0$ and compact set $K \subseteq S$, there is a probability measure θ on S with compact support such that $\|\theta - \delta_s * \theta\| < \varepsilon$ for all $s \in K$.

For a countable discrete semigroup S, the notion of the Liouville property and Reiter's condition just introduced agrees with the one previously defined.

Given a σ-compact metric semigroup S satisfying Reiter's condition, a construction analogous to that in the proof of Proposition 5.4 yields a tight probability measure π on S such that

$$\lim_{n\to\infty} \|\delta_s * \pi^n - \pi^n\| = 0 \tag{31}$$

for all $s \in S$. In fact, this is also true for topological semigroups and a detailed proof will be given in [7]. Consequently, we have the following result.

Theorem 6.11. *Let S be a σ-compact metric semigroup with identity e satisfying Reiter's condition. Then S enjoys the Liouville property.*

Proof. By the previous observation, there is a tight probability measure π on S satisfying (31) for all $s \in S$. Hence for each π-harmonic function $f \in LUC(S)$, we have

$$\begin{aligned} |f(x) - f(e)| &= |\int_S (f(xy) - f(y)) d\pi^n(y)| = |\int_S (\delta_x * f - f) d\pi^n| \\ &= |\int_S f d(\delta_x * \pi^n - \pi^n)| \le \|f\|_\infty \|\delta_x * \pi^n - \pi^n\| \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

and it follows that f is constant. This completes the proof. □

We conclude this section by showing that a metric semigroup S with the Liouville property, but not necessarily σ-compact, must be amenable. This will follow readily from the fact that the space $H_\pi(S)$ of bounded π-harmonic functions on S is the range of a contractive projection on $LUC(S)$, which commutes with left translations. To show this, we need the following two lemmas which are straightforward extension of similar results in [6] for groups.

Let ρ denote the topology of pointwise convergence in $LUC(S)$ and let τ be the topology of uniform convergence on compact sets in S.

Lemma 6.12. *Let S be a metric semigroup and let $f \in LUC(S)$. Denote by K_f the ρ-closure $\overline{\text{co}}^{\,\rho}\{f * \delta_s : s \in S\}$ of the convex hull of $\{f * \delta_s : s \in S\}$. Then the topologies ρ and τ coincide on K_f which is compact in these topologies.*

Lemma 6.13. *Let $f \in LUC(S)$ and $\pi \in M(S)$ be a probability measure. Then $f * \pi \in K_f$.*

Proposition 6.14. *Let S be a metric semigroup and let $\pi \in M_t(S)$ be a probability measure. Then there exists a contractive projection $P : LUC(S) \to LUC(S)$ with range equal to $H_\pi(S)$. Moreover, P commutes with left translations on S.*

Proof. The arguments are similar to those given in [4] and [6] for groups, but we include the proof for completeness. Let $LUC(S)$ be equipped with the topology ρ of pointwise convergence and let the Cartesian product $LUC(S)^{LUC(S)}$ be equipped with the product topology. Define a linear map $L : LUC(S) \to LUC(S)$ by $L(f) = f * \pi$ $(f \in LUC(S))$.

Consider L as an element in $LUC(S)^{LUC(S)}$, and so are the n-th iterates $L^n = \overbrace{L \circ \cdots \circ L}^{\text{n-times}}$. By Lemma 6.13, $L^n \in \prod_{f \in LUC(S)} K_f \subseteq LUC(S)^{LUC(S)}$ for all n. It follows that the closed convex hull $K := \overline{\text{co}}\,\{L^n : n = 1, 2, \ldots\} \subseteq \prod_f K_f$ is compact in the product topology by Lemma 6.12.

Define an affine map $T : K \to K$ by $T(\Lambda)(f) = \Lambda(f) * \pi$ $(\Lambda \in K, f \in LUC(S))$. Then T is continuous. Indeed, given a net $(\Lambda_\alpha)_\alpha$ in K converging to $\Lambda \in K$, then $\Lambda_\alpha(f) \longrightarrow \Lambda(f)$ pointwise on S for each $f \in LUC(S)$. By Lemma 6.13, $\Lambda(f), \Lambda_\alpha(f) \in K_f$ and hence Lemma 6.12 implies that $(\Lambda_\alpha(f))_\alpha$ converges to $\Lambda(f)$ uniformly on compact sets in S. It follows that $(\Lambda_\alpha(f) * \pi)_\alpha$ converges pointwise to $\Lambda(f) * \pi$. This proves that $(T(\Lambda_\alpha))_\alpha$ converges to $T(\Lambda)$ in the product topology.

By the Markov–Kakutani fixed-point theorem, T admits a fixed point $P \in K$ which gives

$$P(f) = P(f) * \pi \qquad (f \in LUC(S)).$$

It is easy to see that $P^2 = P$ and $P(LUC(S)) = H_\pi(S)$. The last assertion follows from the fact that left translations on S commute with the operator L. □

Theorem 6.15. *Let S be a metric semigroup with the Liouville property. Then S is amenable.*

Proof. Let π be a tight probability measure on S such that $H_\pi(S) = \mathbb{C}\mathbf{1}$. Let $P : LUC(S) \to LUC(S)$ be the contractive projection constructed in Proposition 6.14. Since $P(LUC(S)) = H_\pi(S)$, we have $P(f) = \varphi(f)\mathbf{1}$ $(f \in LUC(S))$ for a unique functional $\varphi : LUC(S) \to \mathbb{C}$. Evidently, φ is a left invariant mean. □

The above approach using the contractive projection P gives an alternative proof of the implication (i) ⇒ (iii) in Theorem 6.10.

7. Examples

A measure on a semigroup is called *non-degenerate* if its support generates the semigroup. It is known that if a locally compact abelian semigroup S supports a non-degenerate probability measure π, then $H_\pi(S) = \mathbb{C}\mathbf{1}$ (cf. [24,33]). We now present a class of discrete non-abelian semigroups with the Liouville property for all non-degenerate probability measures.

Definition 7.1. Let S be a discrete semigroup with identity e. A *central series* of S is a finite chain

$$\{e\} = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = S,$$

of subsemigroups in S such that

(i) S_m is *right reversible* for all $0 \leq m \leq n$, that is, for all x, y in S_m, we have $S_m x \cap S_m y \neq \emptyset$;
(ii) For every $1 \leq m \leq n$, $a \in S_m$ and $s \in S$, there exist x, y in S_{m-1} with $xas = ysa$.

We call n the *length* of the above central series.

Semigroups admitting central series are a natural generalisation of nilpotent groups. We should note, however, that the concept of nilpotent semigroups is not defined in terms of the existence of central series (cf. [27]). Evidently, every abelian semigroup admits a central series of length 1.

Example 7.2. The Heisenberg semigroup

$$S = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R}) \colon x, y, z \geq 0 \right\}$$

admits a central series of length 2.

To see this, set

$$S_0 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, S_1 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \geq 0 \right\} \text{ and } S_2 = S.$$

Condition (i) is obviously satisfied for S_0 and S_1 (S_1 is abelian), and it holds for $S_2 = S$ as well because of

$$\begin{pmatrix} 1 & 0 & \zeta+x\eta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi & 0 \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z+\xi y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi & \zeta \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}$$

for all x, y, z and ξ, η, ζ.

Condition (ii) is true for $m = 1$ since S_1 is abelian. For $m = 2$, condition (ii) holds because we have for all x, y, z and ξ, η, ζ:

$$\begin{pmatrix} 1 & 0 & x\eta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi & \zeta \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \xi y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi & \zeta \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}.$$

Actually, exactly the same computations show that for every subring R of $\mathbb{R}$, the semigroup

$$S = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R}) \colon x, y, z \in R \cap [0, \infty) \right\}$$

admits a central series of length 2.

We show that semigroups with central series have the Liouville property for all non-degenerate probability measures. This generalizes the corresponding result for nilpotent groups [13,28] (see also [5]). First, we sketch a proof of the following lemma, using similar arguments in [5].

Lemma 7.3. *Let S be a discrete semigroup with identity e and π a non-degenerate probability measure on S. Then for each $f \in H_\pi(S)$, we have $f(ax) = f(x)$ for all $x \in S$ and a in the centre of S.*

Proof. Considering the real and imaginary parts separately, we may assume that f is real-valued. Fix a in the centre of S. Set $g(x) = f(x) - f(ax)$, where $g \in \ell^\infty(S)$. Let $\alpha := \sup_{x \in S} g(x)$. Then $g \in H_\pi(S)$. Choose a sequence $(x_n)_n$ in S with $\alpha = \lim_{n\to\infty} g(x_n)$. Define $g_n \in \ell^\infty(S)$ by $g_n(x) := g(x_n x)$. Then $g_n \in H_\pi(S)$ and $\lim_{n\to\infty} g_n(e) = \alpha$, where $g_n(x) = g(x_n x) \leq \alpha$ for all $x \in S$. We also know that $\|g_n\|_\infty \leq \|g\|_\infty$.

Let h be a w*-limit point of g_n in $\ell^\infty(S)$. Since $g_n \in H_\pi(S)$ and the map $\varphi \in \ell^\infty(S) \mapsto \varphi * \pi \in \ell^\infty(S)$ is w*-continuous, we have $h = h * \pi$ in $\ell^\infty(S)$ and therefore $h \in H_\pi(S)$. As w*-convergence in $\ell^\infty(S)$ implies pointwise convergence, $g_n(x) \leq \alpha$ implies $h(x) \leq \alpha$ for all $x \in S$. Since $\lim_{n\to\infty} g_n(e) = \alpha$, we have $h(e) = \alpha$.

We show $h(a) = \alpha$. Since $S = \bigcup_{n=1}^{\infty} (\operatorname{supp} \pi)^n = \bigcup_{n=1}^{\infty} \operatorname{supp}\, \pi^n$, one can choose $n \in \mathbb{N}$ such that $\pi^n(a) > 0$. If $h(a) < \alpha$, then

$$\alpha = h(e) = h * \pi^n(e) = \sum_{y \in S} h(y)\pi^n(y) = h(a)\pi^n(a) + \sum_{y \neq a} h(y)\pi^n(y)$$
$$< \alpha\pi^n(a) + \sum_{y \neq a} \alpha\pi^n(y) = \alpha$$

which is a contradiction. Hence we must have $h(a) = \alpha$. Similarly, $h(a^p) = \alpha$ for all $p \in \mathbb{N}$.

Now if $\alpha > 0$, then we can choose $m \in \mathbb{N}$ with $\|f\|_\infty \leq \frac{1}{4} m\alpha$, and $n \in \mathbb{N}$ with $g_n(a^p) > \frac{1}{2}\alpha$ for all $1 \leq p \leq m$. This gives

$$\tfrac{1}{2} m\alpha < \sum_{p=1}^{m} g_n(a^p) = \sum_{p=1}^{m} f(x_n a^p) - f(a x_n a^p) = \sum_{p=1}^{m} f(x_n a^p) - f(x_n a^{p+1})$$
$$= f(x_n a) - f(x_n a^{p+1}) \leq 2 \cdot \tfrac{1}{4} m\alpha = \tfrac{1}{2} m\alpha$$

which is impossible. Therefore $\alpha \leq 0$ and we have $g(x) \leq 0$ for all $x \in S$, that is, $f(x) \leq f(ax)$ for $x \in S$. Repeating the previous argument for the function $f(ax) - f(x)$, we get $f(ax) \leq f(x)$ and hence $f(x) = f(ax)$ for all $x \in S$. □

Theorem 7.4. *Let S be a discrete semigroup with identity and a central series. Then for every non-degenerate probability measure π on S, we have $H_\pi(S) = \mathbb{C}\mathbf{1}$.*

Proof. We proceed inductively on the length n of the central series of S. The case $n = 0$ is trivial. For the inductive step, assume the assertion is true for length n and let

$$\{e\} = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_n \subseteq S_{n+1} = S$$

be a central series of length $n+1$. Since S has an identity in S_1 which is right reversible, we can define an equivalence relation $\sim$ on S by setting $x \sim y$ if there exist $a, b \in S_1$ satisfying $ax = by$. The set $S/_\sim$ of equivalence classes $\dot{x}$ is a semigroup structure with product $\dot{x} \cdot \dot{y} := (xy)^{\cdot}$, which is well-defined because S_1 is in the centre of S. Observe that

$$\{\dot{e}\} = S_1/_\sim \subsetneq S_2/_\sim \subsetneq \ldots \subsetneq S_n/_\sim \subseteq S_{n+1}/_\sim = S/_\sim$$

is a central series of length at most n for $S/_\sim$.

Now let π be a probability measure on S such that supp π generates S as a semigroup. Let $q : S \to S/_\sim$ be the quotient map. Define a probability measure $\dot{\pi}$ on $S/_\sim$ by setting $\dot{\pi}(\dot{x}) := \sum_{x \in q^{-1}(\dot{x})} \pi(x)$. Then supp $\dot{\pi}$ generates $S/_\sim$ as a semigroup. By the inductive hypothesis, we have $H_{\dot{\pi}}(S/_\sim) = \mathbb{C}\mathbf{1}$.

Let $f \in H_\pi(S)$. Define $\dot{f}: S/_\sim \to \mathbb{C}$ by $\dot{f}(\dot{x}) := f(x)$. This is well-defined. Indeed, if $x \sim y$, then there exist $a, b \in S_1$ with $ax = by$. As S_1 is contained in the centre of S, Lemma 7.3 yields $f(x) = f(ax) = f(by) = f(y)$. One verifies readily that $\dot{f} * \dot{\pi} = \dot{f}$. Therefore $\dot{f}$, and hence f, must be constant. □

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