J. reine angew. Math. **778** (2021), 145–169
DOI 10.1515/crelle-2021-0027

Journal für die reine und angewandte Mathematik

Siegel domains over Finsler symmetric cones

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Abstract. Let Ω be a proper open cone in a real Banach space V. We show that the tube domain $V \oplus i\Omega$ over Ω is biholomorphic to a bounded symmetric domain if and only if Ω is a normal linearly homogeneous Finsler symmetric cone, which is equivalent to the condition that V is a unital JB-algebra in an equivalent norm and Ω is the interior of $\{v^2 : v \in V\}$.

1. Introduction

Let $V \oplus i\Omega$ be a Siegel domain of the first kind over a proper open cone Ω in a real Banach space V, often called a *tube domain*. If V is finite-dimensional, it is well known from the seminal works of Koecher [24] and Vinberg [33] that $V \oplus i\Omega$ is biholomorphic to a bounded symmetric domain if and only if Ω is a linearly homogeneous self-dual cone, or equivalently, the closure $\overline{\Omega}$ is the cone $\{a^2 : a \in \mathcal{A}\}$ in a formally real Jordan algebra $\mathcal{A}$, in which case Ω carries the structure of a Riemannian symmetric space (see also [5, 15, 29]). This result has an infinite-dimensional extension by the work of Braun, Kaup and Upmeier in [8, 20], which shows that $V \oplus i\Omega$ of any dimension is biholomorphic to a bounded symmetric domain if and only if $\overline{\Omega} = \{a^2 : a \in \mathcal{A}\}$ in a unital JB-algebra $\mathcal{A}$. In both cases, V is the underlying vector space of $\mathcal{A}$. However, in contrast to the finite-dimensional case, the question of characterising all tube domains $V \oplus i\Omega$ which are biholomorphic to a bounded symmetric domain in terms of the geometric structure of Ω has been open. The question amounts to extending Koecher and Vinberg's condition of a linearly homogeneous self-dual cone to infinite-dimensional Banach spaces. A fundamental obstacle is that the concept of a self-dual cone is unavailable in infinite-dimensional Banach spaces from want of a positive definite quadratic form. Nevertheless, using Finsler structure, we are able to circumvent this difficulty and address the above question affirmatively.

We show that the tube domain $V \oplus i\Omega$ is biholomorphic to a bounded symmetric domain if and only if Ω is a normal linearly homogeneous Finsler symmetric cone. The latter can be viewed as an infinite-dimensional generalisation of the notion of a linearly homogeneous self-dual cone. Further details are given below.

This research was partly supported by the Engineering and Physical Sciences Research Council (UK) grant EP/R044228/1.

Let Ω be an open cone in a real Banach space V. Then Ω is a real Banach manifold modelled on V. Let $L(V)$ be the Banach algebra of bounded linear operators on V, which is a real Banach Lie algebra in the Lie brackets

$$[S,T] := ST - TS \quad (S,T \in L(V)).$$

Let $\mathrm{GL}(V)$ be the open subgroup of $L(V)$ consisting of invertible elements in $L(V)$. It is a real Banach Lie group with Lie algebra $L(V)$. The linear maps $g \in \mathrm{GL}(V)$ satisfying $g(\Omega) = \Omega$ form a subgroup of $\mathrm{GL}(V)$ and will be denoted by

$$G(\Omega) = \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}.$$

We shall call $G(\Omega)$ the *linear automorphism group* of Ω. An element $g \in \mathrm{GL}(V)$ belongs to $G(\Omega)$ if and only if $g(\overline{\Omega}) = \overline{\Omega}$, the latter denotes the closure of Ω. Hence $G(\Omega)$ is a closed subgroup of $\mathrm{GL}(V)$ and can be topologised to a real Banach Lie group with Lie algebra

$$\mathfrak{g}(\Omega) = \{X \in L(V) : \exp tX \in G(\Omega) \text{ for all } t \in \mathbb{R}\} \tag{1.1}$$

(cf. [32, p. 387]).

An open cone Ω in V can be homogeneous under various group actions. The terminology *linearly homogeneous* throughout the paper is defined below.

Definition 1.1. An open cone Ω in a real Banach space is called *linearly homogeneous* if the linear automorphism group $G(\Omega)$ acts transitively on Ω, that is, given $a, b \in \Omega$, there is a continuous linear isomorphism $g \in G(\Omega)$ such that $g(a) = b$.

An open cone Ω in a real Hilbert space V with an inner product $\langle \cdot, \cdot \rangle$ is called *self-dual* if $\Omega = \Omega^*$, where

$$\Omega^* = \{v \in V : \langle v, x \rangle > 0 \text{ for all } x \in \overline{\Omega} \backslash \{0\}\}$$

denotes the dual cone of Ω.

Remark. Linearly homogeneous self-dual cones are often called *symmetric cones* in literature. In this paper, we adopt the former terminology to avoid the latter being confused with the notion of *symmetric domains*.

Recently, the result of Koecher [24] and Vinberg [33] has been extended to infinite-dimensional Hilbert spaces in [13] (cf. Corollary 4.4), where it has been shown that an open cone Ω in a real Hilbert space V, with inner product $\langle \cdot, \cdot \rangle$, is a linearly homogeneous self-dual cone if and only if V carries the structure of a Jordan algebra with identity and $\overline{\Omega} = \{x^2 : x \in V\}$, in which the Jordan product satisfies

$$\langle ab, c \rangle = \langle b, ac \rangle \quad (a, b, c \in V).$$

Such a real Jordan algebra, with or without identity, is called a *JH-algebra*. Together with the result of [8] mentioned before, the above assertion implies that the tube domain $V \oplus i\Omega$ over an open cone Ω in a Hilbert space V is biholomorphic to a bounded symmetric domain if and only if Ω is linearly homogeneous and self-dual. In this case, Ω is also a Riemannian symmetric space [12].

In finite-dimensional Euclidean spaces, it has been shown by Shima [30] and Tsuji [31] that if an open cone Ω is linearly homogeneous, and if Ω is a symmetric space in some

Riemannian metric, then it is self-dual and hence $V \oplus i\,\Omega$ is indeed biholomorphic to a bounded symmetric domain. We extend this result to Hilbert spaces in Corollary 4.4, as a direct consequence of our main result for Banach spaces.

In the absence of Riemannian structures and self-duality in Banach spaces, we establish an equivalent geometric condition on Ω for $V \oplus i\,\Omega$ to be biholomorphic to a bounded symmetric domain for Banach spaces V, namely, that Ω be a normal linearly homogeneous Finsler symmetric cone.

Definition 1.2. By a *Finsler symmetric cone*, we mean an open cone Ω in a real Banach space, which is a symmetric Banach manifold in a *compatible* $G(\Omega)$*-invariant tangent norm* (defined in Section 2).

Normal cones are defined in Section 3. In finite dimensions, proper open cones are normal. Self-dual cones in Hilbert spaces are also normal. We prove the following main result in Theorem 4.2, which resolves the aforementioned question.

Main Theorem. *Let Ω be a proper open cone in a real Banach space V. The following conditions are equivalent:*

(i) *The Siegel domain $V \oplus i\,\Omega$ is biholomorphic to a bounded symmetric domain.*

(ii) *Ω is a normal linearly homogeneous Finsler symmetric cone.*

Condition (ii) in this theorem also provides a simple order-geometric characterisation of unital JB-algebras as it is equivalent to V being a unital JB-algebra in an equivalent norm and Ω the interior of $\{a^2 : a \in V\}$. Hence Finsler symmetric cones abound. The well-known characterisation of unital JB-algebras by geometric properties of the state space has been established by Alfsen and Schultz in [2], which is the culmination of a noncommutative spectral theory developed in a series of papers [1, 3, 4].

To prove the Main Theorem, we first give, in the next two sections, the definition of symmetric Banach manifolds and JB-algebras, together with some relevant results on cones and hermitian operators, which will be used, in tandem with Jordan and Lie theory, to establish the theorem in the last section.

2. Symmetric Banach manifolds

Let M be a Banach manifold (with an analytic structure), modelled on a real or complex Banach space $(V, \|\cdot\|_V)$, with tangent bundle $TM = \{(p, v) : p \in M,\ v \in T_pM\}$. A mapping

$$\nu : TM \to [0, \infty)$$

is called a *tangent norm* if $\nu(p, \cdot)$ is a norm on the tangent space $T_pM \approx V$ for each $p \in M$. We call ν a *compatible tangent norm* if it satisfies the following two conditions:

(i) ν is continuous.

(ii) For each $p \in M$, there exist a local chart $\varphi : \mathcal{U} \to V$ at p, and constants $0 < r < R$ such that

$$r\|d\varphi_a(v)\|_V \le \nu(a, v) \le R\|d\varphi_a(v)\|_V \quad (a \in \mathcal{U} \subset M,\ v \in T_aM).$$

The integrated distance d_ν of the tangent norm ν on M is given by

$$d_\nu(x, y) = \inf_\gamma \left\{ \int_0^1 \nu(\gamma(t), \gamma'(t))\, dt : \gamma(0) = x,\ \gamma(1) = y \right\},$$

where $\gamma : [0, 1] \to M$ is a piecewise smooth curve in M.

Remark. In finite dimensions, a compatible tangent norm satisfying certain smoothness and convexity conditions is known as a *Finsler metric* [11]. Nevertheless, a Banach manifold with a compatible tangent norm is also called a *Finsler manifold* in literature (e.g. [27]) and this nomenclature has been adopted in Definition 1.2.

Given a Banach manifold M with a compatible tangent norm ν, a bianalytic map

$$f : M \to M$$

is called a ν-*isometry* if it satisfies

$$\nu(f(p), df_p(\cdot)) = \nu(p, \cdot) \quad \text{for all } (p, \cdot) \in TM$$

in which case, we have $d_\nu(f(x), f(y)) = d_\nu(x, y)$ for all $x, y \in M$.

Definition 2.1. Let Ω be an open cone in a real Banach space V, equipped with a tangent norm ν. We say that ν is $G(\Omega)$-*invariant* if each $g \in G(\Omega)$ is a ν-isometry.

Example 2.2. A Riemannian manifold (M, g) modelled on a real Hilbert space V, with Riemannian metric g, admits a compatible tangent norm $\nu : TM \to [0, \infty)$ defined by

$$\nu(p, v) := g_p(v, v)^{\frac{1}{2}} \quad (p \in M,\ v \in T_pM \approx V).$$

The ν-isometries of M are exactly the isometries of M with respect to the Riemannian metric g.

Example 2.3. Let D be a bounded domain in a complex Banach space V. Then the Carathéodory differential metric, defined below, is a compatible tangent norm on D.

$$\mathcal{C}(p, v) = \sup\{|f'(p)(v)| : f \in H(D, \mathbb{D}) \text{ and } f(p) = 0\} \quad ((p, v) \in TM),$$

where $H(D, \mathbb{D})$ is the set of all holomorphic maps from D to $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In this case, all biholomorphic maps on D are $\mathcal{C}$-isometries.

An open cone Ω in a real Banach space V is a real connected Banach manifold modelled on V. A homogeneous polynomial $p : V \to V$ of degree n is of the form

$$p(v) = f(v, \dots, v) \quad (v \in V)$$

where $f : V^n \to V$ is a continuous n-linear map. In particular, each $f \in L(V)$ is a polynomial of degree 1, and polynomials of degree 0 are the constant maps on V.

To each homogeneous polynomial p on V, we associate an analytic vector field $p\frac{\partial}{\partial x}$ on V. If $X = h\frac{\partial}{\partial x}$ is a linear vector field on Ω, that is, h is (the restriction of) a continuous linear map $f \in L(V)$, we identify X with f. Conversely, each $f \in L(V)$ identifies with the vector field $f\frac{\partial}{\partial x}$ on Ω.

Let $I \in L(V)$ be the identity map. If X is a linear vector field on Ω, then evidently $[I, X] = 0$. The converse is also true. We sketch a proof for completeness. Let $X = h\frac{\partial}{\partial x}$ be an analytic vector field and $[I, X] = 0$, and let

$$h(x) = \sum_{n=-1}^{\infty} p_n(x - e)$$

be the power series expansion of h in a neighbourhood of a point $e \in \Omega$, where

$$p_n(v) = f_n(v, \dots, v)$$

is a homogeneous polynomial of degree $n + 1$ with $f_n : V^{n+1} \to V$, and $p_{-1} = h(e)$. We have

$$X = \sum_{n=-1}^{\infty} X_n, \quad X_n = p_n(x - e)\frac{\partial}{\partial x}$$

in a local chart at e and

$$0 = [I, X] = \sum_{n=-1}^{\infty} (\operatorname{ad} I)X_n = \sum_{n=-1}^{\infty} q_n \frac{\partial}{\partial x}.$$

implies

$$\sum_{n=-1}^{\infty} q_n(x) = 0, \tag{2.1}$$

where $q_{-1} = -h(e)$, $q_0(x) = p_0(e)$ and $q_1(x) = f_1(x - e, x) + f_1(x, x - e) - p_1(x - e)$. This gives $-h(e) + p_0(e) = 0$ and

$$h(x) = p_0(x) + p_1(x - e) + \cdots .$$

Differentiating (2.1) twice, we obtain

$$q_1''(e) = q_1''(e) + q_2''(e) + \cdots = 0,$$

where $q_1''(e)(x) = f_1(x, \cdot) + f_1(\cdot, x) - f_1(e, \cdot) - f_1(\cdot, e) \in L(V)$ for $x \in V$. It follows that $p_1(x) = f_1(x, x) = 0$. Differentiating repeatedly then gives $p_2 = p_3 = \cdots = 0$ and $h = p_0$ is linear.

To introduce the concept of a symmetric Banach manifold, we begin with the notion of a *symmetry* of a manifold. Let M be a Banach manifold endowed with a compatible tangent norm ν and let $p \in M$. A ν-*symmetry* (or *symmetry*, if ν is understood) at p is a ν-isometry

$$s : M \to M$$

satisfying the following two conditions:

(i) s is involutive, that is, s^2 is the identity map on M.

(ii) p is an isolated fixed-point of s, in other words, p is the only point in some neighbourhood of p satisfying $s(p) = p$.

Definition 2.4. By a *symmetric Banach manifold* (with a tangent norm ν), we mean a *connected* Banach manifold M, equipped with a compatible tangent norm ν, such that there is a unique ν-symmetry $s_p : M \to M$ at each $p \in M$ (see also [19, 32]).

By definition, a *Finsler symmetric cone* Ω in a real Banach space V is a symmetric Banach manifold of which the tangent norm is $G(\Omega)$-invariant.

Example 2.5. Riemannian symmetric spaces are (real) symmetric Banach manifolds (in the Riemannian metric). A *bounded symmetric domain* is a bounded domain D in a complex Banach space such that for each $p \in D$, there is an involutive biholomorphic map $s_p : D \to D$ (necessarily unique) of which p is an isolated fixed-point. Equipped with the Carathéodory metric, a bounded symmetric domain is a complex symmetric Banach manifold and s_p is the symmetry at p. Finite-dimensional Hermitian symmetric spaces of non-compact type are exactly the bounded symmetric domains in $\mathbb{C}^d$ via the Harish-Chandra realisation and have been classified by É. Cartan [10].

Example 2.6. A concept of a symmetric manifold has been introduced by Loos in [26] (see also [6]), where a connected (real) smooth manifold M is called a *symmetric space* if there is a smooth map

$$\mu : (x, y) \in M \times M \mapsto x \cdot y \in M$$

satisfying the following axioms for all $x, y, z \in M$:

(i) $x \cdot x = x$,

(ii) $x \cdot (x \cdot y) = y$,

(iii) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,

(iv) there is a neighbourhood U of x such that $x \cdot y = y \in U$ implies $x = y$.

We call (M, μ) a *Loos symmetric space*. The *left multiplication* $S(x) : y \in M \mapsto x \cdot y \in M$ is called a *symmetry around* x in [26]. A diffeomorphism $f : M \to M$ is called a *μ-automorphism* if $f(x \cdot y) = f(x) \cdot f(y)$.

Lemma 2.7. *Let $f, g : M \to M$ be two μ-automorphisms on a Loss symmetric space (M, μ) such that $f(x) = g(x)$ and $f'(x) = g'(x)$ at some point $x \in M$. Then we have $f = g$.*

Proof. This follows from [27, Lemma 3.5, Theorem 3.6] since M is a connected manifold with spray. □

Given a (real) symmetric Banach manifold M, one can define $\mu : M \times M \to M$ by

$$\mu(x, y) = s_x(y) \quad (s_x \text{ is the symmetry at } x)$$

which makes (M, μ) into a Loos symmetric space and $s_x = S(x)$.

A Loos symmetric space (M, μ) is equipped with a canonical affine connection (see [26, p. 83] and [6, Theorem 26.3]; see also Appendix), which is geodesically complete (see [27, Theorem 3.6]). The derivative $S(p)'(p) : T_p M \to T_p M$ of the symmetry $S(p)$ equals $-\mathrm{id}$, where id is the identity map (see [27, Lemma 3.2]). Given a geodesic $\gamma : \mathbb{R} \to M$ through p with $\gamma(0) = p$, the symmetry $S(p)$ reverses γ in that $S(p)(\gamma(t)) = \gamma(-t)$.

3. Jordan algebras and order structures

For later applications, we review some basics of Jordan algebras, first introduced in [18], and refer to [12, 32] for more details. We also prove some relevant order-theoretical results in this section. In what follows, a Jordan algebra $\mathcal{A}$ is a real vector space, which can be infinite-dimensional, equipped with a bilinear product $(a,b) \in \mathcal{A} \times \mathcal{A} \mapsto ab \in \mathcal{A}$ that is commutative, but not necessarily associative, and satisfies the *Jordan identity*

$$a(ba^2) = (ab)a^2 \quad (a, b \in \mathcal{A}).$$

A vector space $\mathcal{A}$ equipped with a bilinear product will be called an *algebra*. For each element a in an algebra $\mathcal{A}$, we define inductively

$$a^1 = a, a^{n+1} = aa^n \quad (n = 1, 2, \ldots)$$

and call $\mathcal{A}$ *power associative* if

$$a^m a^n = a^{m+n} \quad (m, n = 1, 2, \ldots).$$

We call $\mathcal{A}$ *unital* if it contains an identity. Evidently, if $\mathcal{A}$ is unital and power associative, then the subalgebra $\mathcal{J}(a, e)$ in $\mathcal{A}$ generated by a and the identity e is associative.

A linear map $\delta : V \to V$ on an algebra V is called a *derivation* if it satisfies

$$\delta(ab) = \delta(a)b + a\delta(b) \quad (a, b \in V),$$

which can be rephrased as

$$[\delta, L_a] = L_{\delta(a)} \quad (a \in V) \tag{3.1}$$

where $L_a : V \to V$ is the *left multiplication* $L_a(x) = ax$ for $x \in V$, and $[\delta, L_a] = \delta L_a - L_a\delta$ is the usual *commutator*. Given a derivation δ on V and $a \in V$, a simple induction shows that

$$\delta(a) = 0 \implies \delta(a^n) = 0 \quad (n = 2, 3, \ldots). \tag{3.2}$$

Further, if V is commutative, then $\delta(a^2) = 0$ implies

$$2a\delta(a) = \delta(a^2) = 0. \tag{3.3}$$

We will make use of the following result, which follows from [9, Lemma 2.4.4].

Lemma 3.1. *Let V be a commutative algebra on which the commutator $[L_x, L_y]$ is a derivation for all $x, y \in V$. Then for all $a \in V$, we have*

(i) $[L_a, L_{a^3}] = 3L_a[L_a, L_{a^2}]$,

(ii) $[[L_a, L_{a^2}], [[L_a, L_{a^2}], L_{a^2}]] = 0$.

Proof. (i) This is proved in [9, Lemma 2.4.5]. (ii) Using (i), a simple argument in [9, Lemma 2.4.4] gives $[L_a, L_{a^2}]^2(a^2) = 0$. Applying (3.1) twice yields

$$[[L_a, L_{a^2}], [[L_a, L_{a^2}], L_{a^2}]] = [[L_a, L_{a^2}], L_{[L_a, L_{a^2}](a^2)}] = L_{[L_a, L_{a^2}][L_a, L_{a^2}](a^2)} = 0.$$

The lemma is proved. □

Jordan algebras are power associative. An element a in a Jordan algebra $\mathcal{A}$ with identity e is called *invertible* if there exists an element $a^{-1} \in \mathcal{A}$ (which is necessarily unique) such that $aa^{-1} = e$ and $(a^2)a^{-1} = a$. A Jordan algebra $\mathcal{A}$ is called *formally real* if $a_1^2 + \cdots + a_n^2 = 0$ implies $a_1 = \cdots = a_n = 0$ for any $a_1, \ldots, a_n \in \mathcal{A}$ (see [18]). A finite-dimensional formally real Jordan algebra $\mathcal{A}$ is necessarily unital (cf. [12, Proposition 1.1.13]).

On a Jordan algebra $\mathcal{A}$, one can define a *Jordan triple product* by

$$\{a, b, c\} = (ab)c + a(bc) - b(ac) \quad (a, b, c \in \mathcal{A})$$

which plays an important role in the structures of $\mathcal{A}$.

A real Jordan algebra $\mathcal{A}$ is called a *JB-algebra* if it is also a Banach space and the norm satisfies

$$\|ab\| \le \|a\|\|b\|, \quad \|a^2\| = \|a\|^2, \quad \|a^2\| \le \|a^2 + b^2\|$$

for all $a, b \in \mathcal{A}$. A JB-algebra $\mathcal{A}$ admits a natural order structure determined by the set

$$\mathcal{A}_+ = \{x^2 : x \in \mathcal{A}\}$$

which forms a closed cone [16, Lemmas 3.3.5 and 3.3.7] and satisfies $\mathcal{A}_+ \cap -\mathcal{A}_+ = \{0\}$. In finite dimensions, JB-algebras are exactly the formally real Jordan algebras [12, Lemma 2.3.7].

Let V be a real Banach space. By a *cone* Ω in V, we mean a *nonempty* subset of V satisfying (i) $\Omega + \Omega \subset \Omega$ and (ii) $\alpha\Omega \subset \Omega$ for all $\alpha > 0$. We note that a cone is necessarily convex. Trivially, V itself is a cone. In the sequel, we shall exclude this case. If Ω is an open cone properly contained in V, then we must have $0 \notin \Omega$ although the closure $\overline{\Omega}$ contains 0.

Let Ω be an open cone properly contained in a real Banach space V with norm $\|\cdot\|$, and let $\le$ be the partial order defined by the closure $\overline{\Omega}$, which is a cone, so that

$$x \le y \iff y - x \in \overline{\Omega}.$$

We also write $y \ge x$ for $x \le y$. Let V^* be the dual Banach space of V, consisting of continuous linear functionals on V. As usual, a linear functional $f : V \to \mathbb{R}$ is called *positive* if $f(\overline{\Omega}) \subset [0, \infty)$. By the Hahn–Banach separation theorem, we have

$$\overline{\Omega} = \{v \in V : f(v) \ge 0 \text{ for each } f \in V^* \text{ satisfying } f(\overline{\Omega}) \subset [0, \infty)\}.$$

We note that each element $e \in \Omega$ is an *order unit*, that is, for each $v \in V$, we have

$$-\alpha v \le v \le \alpha e$$

for some $\alpha > 0$. Indeed, since Ω is open, $e - \Omega$ is a neighbourhood of $0 \in V$ and therefore one can find $\lambda > 0$ such that $\pm\lambda v \in e - \Omega$, which gives $\lambda v = e - a_1$ and $-\lambda v = e - a_2$ for some $a_1, a_2 \in \Omega$. In other words,

$$-\frac{1}{\lambda} e \le v \le \frac{1}{\lambda} e.$$

The preceding argument also implies

$$V = \Omega - \Omega. \tag{3.4}$$

An order unit $e \in \Omega$ induces a semi-norm $\|\cdot\|_e$ on V, defined by

$$\|x\|_e = \inf\{\lambda > 0 : -\lambda e \le x \le \lambda e\} \quad (x \in V)$$

which satisfies

$$-\|x\|_e e \le x \le \|x\|_e e$$

and

$$\{x \in V : \|x\|_e \le 1\} = \{x \in V : -e \le x \le e\}. \tag{3.5}$$

Since $\{x \in V : \|x\|_e = 0\} = \overline{\Omega} \cap -\overline{\Omega}$, the semi-norm $\|\cdot\|_e$ is a norm if and only if

$$\overline{\Omega} \cap -\overline{\Omega} = \{0\}$$

in which case Ω is called a *proper cone* and $\|\cdot\|_e$ is called the *order-unit norm* induced by e. All order-unit norms induced by elements in Ω are mutually equivalent.

Henceforth, let Ω be a proper open cone in V. It follows from (3.5) that every linear map $\psi : V \to V$ which is *positive*, meaning $\psi(\overline{\Omega}) \subset \overline{\Omega}$, is continuous with respect to the order-unit norm $\|\cdot\|_e$ and moreover, $\|\psi\|_e = \|\psi(e)\|_e$, where the former denotes the norm of ψ with respect to $\|\cdot\|_e$. In particular, if $\psi : V \to \mathbb{R}$ is a positive linear functional, then $\|\psi\|_e = \psi(e)$.

Let $(V, \|\cdot\|_e)$ denote the vector space V equipped with the order-unit norm $\|\cdot\|_e$, and $(V, \|\cdot\|_e)^*$ its dual space. A positive linear map $\psi : (V, \|\cdot\|_e) \to (V, \|\cdot\|_e)$ is an isometry if and only if $\psi(e) = e$ (see [13, Proposition 2.3]). By [13, Lemma 2.5], there is a positive constant $c > 0$ such that

$$\|\cdot\|_e \le c\|\cdot\|. \tag{3.6}$$

It follows that every $\|\cdot\|_e$-continuous linear functional on V is also $\|\cdot\|$-continuous. On the other hand, given $f \in V^*$ satisfying $f(e) = 1 = \|f\|_e$, then f is positive and hence continuous with respect to the norm $\|\cdot\|_e$.

Denote the *state space* (with respect to the order unit e) by

$$\begin{aligned} S_e &= \{f \in (V, \|\cdot\|_e)^* : f(e) = 1 = \|f\|_e\} \\ &= \{f \in V^* : f(e) = 1, f \text{ is positive}\}, \end{aligned} \tag{3.7}$$

which is a weak* compact convex set in the dual V^* and we have

$$\|x\|_e = \sup\{|f(v)| : f \in S_e\} \quad (x \in V)$$

(cf. [16, Lemma 1.2.5]).

Lemma 3.2. *Let Ω be a proper open cone in a real Banach space V and let $e \in \Omega$, which induces an order-unit norm $\|\cdot\|_e$ on V. Then we have*

$$\Omega = \bigcap_{f \in S_e} f^{-1}(0, \infty).$$

Proof. Given that V is partially ordered by the closure $\overline{\Omega}$, we have

$$\overline{\Omega} = \bigcap_{f \in S_e} f^{-1}[0, \infty) \tag{3.8}$$

since $\frac{f}{f(e)} \in S_e$ for each nonzero positive linear functional $f \in V^*$.

Let $a \in \Omega$. Then for each $f \in S_e$, we have $f(a) > 0$ since a is an order unit, which implies $e \leq \lambda a$ for some constant $\lambda > 0$ and hence $1 \leq \lambda f(a)$. This proves

$$\Omega \subset \bigcap_{f \in S_e} f^{-1}(0, \infty).$$

Conversely, let $a \in V$ and $f(a) > 0$ for all $f \in S_e$. Then $a \in \overline{\Omega}$ and by weak* compactness of S_e, one can find some $\delta > 0$ such that $f(a) \geq \delta$ for all $f \in S_e$. Let

$$N = \left\{x \in V : \|x - a\| < \frac{\delta}{2c}\right\} \subset \left\{x \in V : \|x - a\|_e < \frac{\delta}{2}\right\},$$

where $c > 0$ is given in (3.6). Then N is an open neighbourhood of a and, $N \subset \overline{\Omega}$ since

$$x \in N \implies -\frac{\delta}{2}e \leq x - a \implies a - \frac{\delta}{2}e \leq x \implies \frac{\delta}{2} \leq f(x)$$

for all $f \in S_e$. Hence a belongs to the interior $\overline{\Omega}^0$ of $\overline{\Omega}$ and, as Ω is open and convex, we have $\Omega = \overline{\Omega}^0$ and $a \in \Omega$. □

We see from (3.6) that if $\dim V < \infty$, then the order-unit norm $\|\cdot\|_e$ is equivalent to the norm of V by the open mapping theorem. In fact, the equivalence of the two norms is related to the basic concept of a normal cone in the theory of partially ordered topological vector spaces.

Lemma 3.3. *Let Ω be a proper open cone in a real Banach space V with norm $\|\cdot\|$. Then the order-unit norm $\|\cdot\|_e$ induced by $e \in \Omega$ is equivalent to $\|\cdot\|$ if and only if Ω is a normal cone in V, that is, there is a constant $\gamma > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \gamma\|y\|$ for all $x, y \in V$. In particular, $(V, \|\cdot\|_e)$ is a Banach space if Ω is a normal cone.*

Proof. By the definition of the order-unit norm $\|\cdot\|_e$, we have $0 \leq x \leq y$ in V implies $\|x\|_e \leq \|y\|_e$. Hence Ω is normal in $(V, \|\cdot\|_e)$. If $\|\cdot\|$ is equivalent to $\|\cdot\|_e$, then evidently Ω is also normal in $(V, \|\cdot\|)$.

Conversely, let Ω be normal in $(V, \|\cdot\|)$. We have already noted in (3.6) that $\|\cdot\|_e \leq c\|\cdot\|$ for some constant $c > 0$. By (3.5) and normality of Ω, there is a constant $\gamma > 0$ such that $\|x\|_e \leq 1$ implies

$$-e \leq x \leq e \implies 0 \leq x + e \leq 2e \implies \|x + e\| \leq 2\gamma\|e\| \implies \|x\| < 2(\gamma + 1)\|e\|,$$

which yields $\|\cdot\| \leq 2(\gamma + 1)\|e\|\|\cdot\|_e$ and the equivalence of $\|\cdot\|$ and $\|\cdot\|_e$. □

We note that a self-dual cone Ω in a Hilbert space H is a proper cone, and also normal since it has been shown in [13, Lemma 2.6] that the order-unit norms induced by elements in Ω are all equivalent to the norm of H.

Let $L(W)$ be the Banach algebra of bounded linear operators on a complex Banach space W and $I \in L(W)$ the identity operator. We recall that an element $T \in L(W)$ is called *hermitian* if its numerical range $\mathsf{V}(T)$ is contained in $\mathbb{R}$, where

$$\mathsf{V}(T) = \{\psi(T) : \psi \in L(W)^* \text{ satisfies } \|\psi\| = 1 = \psi(I)\},$$

which is equivalent to

$$\|\exp itT\| = \left\|I + itT + \frac{(itT)^2}{2!} + \cdots\right\| = 1 \quad (t \in \mathbb{R})$$

(cf. [7, Chapter 2]). If $T_0 \in L(W)$ is hermitian, then the left multiplication

$$L_{T_0} : S \in L(W) \mapsto T_0 S \in L(W)$$

is a hermitian operator in L(L(W)) because the linear map $T \in L(W) \mapsto L_T \in L(L(W))$ is an isometry.

Lemma 3.4. *Let $\eta : L(W) \to L(W)$ be a hermitian operator. Then for all $T \in L(W)$, we have $\|\eta(T)\|^2 \leq 4\|T\|\|\eta^2(T)\|$.*

Proof. This is proved in [7, p. 95]. □

Given a real Banach space V, one can equip its complexification $V_c = V \otimes \mathbb{C} = V \oplus iV$ with a norm $\|\cdot\|_c$ so that $(V_c, \|\cdot\|_c)$ is a complex Banach space and

(i) the isometric embedding $v \in V \mapsto (v, 0) \in V \oplus iV$ identifies V as a real closed subspace of V_c,

(ii) the map $T \in L(V) \mapsto T_c \in L(V_c)$ is isometric, where T_c is the complexification of T defined by $T_c(x + iy) = T(x) + iT(y)$ for $x, y \in V$.

Moreover, if V is an algebra satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in V$, the norm $\|\cdot\|_c$ can be chosen so that $\|ab\|_c \leq \|a\|_c\|b\|_c$ for all $a, b \in V_c$. In this case, the linear map

$$a \in V_c \mapsto L_a \in L(V_c) \tag{3.9}$$

is an isometry, where L_a is the left multiplication. In the sequel, we will make use of this construction.

In the preceding construction, if the norm of V is an order-unit norm $\|\cdot\|_e$, one can also define a notion of *numerical range* $\mathsf{v}(a)$ of an element $a \in V_c$ by

$$\mathsf{v}(a) = \{f(a) : f \in V_c^* \text{ satisfies } \|f\| = 1 = f(e)\}.$$

If V is an algebra and the order unit e is an algebra identity, then an application of the isometry in (3.9) implies $\mathsf{V}(L_a) \subset \mathsf{v}(a)$ and therefore L_a is hermitian if $\mathsf{v}(a) \subset \mathbb{R}$.

4. Tube domains over Finsler symmetric cones

We prove the main theorem in this section. Let Ω be a proper open cone in a real Banach space $(V, \|\cdot\|)$. Then it is a real connected Banach manifold modelled on V. Let $(V_c, \|\cdot\|_c)$ be a complexification of V. The domain

$$D(\Omega) := V \oplus i\Omega = \{v + i\omega : v \in V, \omega \in \Omega\} \subset V_c = V \oplus iV$$

in V_c is called a *tube domain* over Ω.

Let $V \oplus i\Omega$ be biholomorphic to a bounded domain (this is always the case if dim $V < \infty$, see [23, Chapter II, Section 5]). On $D(\Omega) = V \oplus i\Omega$, the Carathéodory distance ρ is defined, in terms of the Poincaré distance $\rho_{\mathbb{D}}$ on $\mathbb{D}$, by

$$\rho(z, w) := \sup\{\rho_{\mathbb{D}}(f(z), f(w)) : f \in H(D(\Omega), \mathbb{D})\} \quad (z, w \in D(\Omega))$$

which need not coincide with the integrated distance of the Carathéodory differential metric $\mathcal{C}$ on $V \oplus i\Omega$, defined in Example 2.3.

If the proper open cone Ω in V is normal, then the order-unit norms induced by elements in Ω are all equivalent to $\|\cdot\|$ by Lemma 3.3 and one can define a compatible tangent norm τ on Ω by

$$\tau(p,v) = \|v\|_p \quad ((p,v) \in \Omega \times V) \tag{4.1}$$

where $\|\cdot\|_p$ denotes the order-unit norm induced by the order unit $p \in \Omega$. To see that τ is continuous, let (p_n) converge to p in Ω and (v_n) converge to v in V. Given $1 > \varepsilon > 0$, $\|p_n - p\|_p \to 0$ implies $-\varepsilon p \le p_n - p \le \varepsilon p$ and $(1-\varepsilon)p \le p_n \le (1+\varepsilon)p$ from some n onwards, which gives

$$-(1+\varepsilon)\|v_n\|_{p_n} p \le -\|v_n\|_{p_n} p_n \le v_n \le \|v_n\|_{p_n} p_n \le (1+\varepsilon)\|v_n\|_{p_n} p$$

and hence $\|v_n\|_p \le (1+\varepsilon)\|v_n\|_{p_n}$. Likewise $p \le \frac{p_n}{1-\varepsilon}$ implies $\|v_n\|_{p_n} \le \frac{\|v_n\|_p}{1-\varepsilon}$ and therefore

$$1-\varepsilon \le \frac{\|v_n\|_p}{\|v_n\|_{p_n}} \le 1+\varepsilon.$$

Since $\|v_n\|_p \to \|v\|_p$ as $n \to \infty$, we conclude $\|v_n\|_{p_n} \to \|v\|_p$, proving continuity of τ. The above argument also implies that for each $a \in \mathcal{U} := \{v \in V : \|v - p\|_p < \varepsilon < 1\}$, we have

$$\frac{\|v\|_p}{1+\varepsilon} \le \|v\|_a \le \frac{\|v\|_p}{1-\varepsilon} \quad (v \in V).$$

Hence τ is a compatible tangent norm.

The tangent norm τ coincides with the tangent norm $b : T\Omega \to [0,\infty)$ in [32, 12.31, 22.37], which is defined as follows. Fix $e \in \Omega$. Then each $g \in G(\Omega)$ satisfying $g(e) = e$ is an isometry with respect to the order unit norm $\|\cdot\|_e$ and hence one can define

$$b(p,v) = \|h(v)\|_e \quad ((p,v) \in T\Omega) \tag{4.2}$$

for any $h \in G(\Omega)$ satisfying $h(p) = e$. In fact, τ is $G(\Omega)$-invariant, which implies $\tau = b$. For if $h \in G(\Omega)$, then we have

$$\tau(h(p), h'(p)(v)) = \tau(h(p), h(v)) = \|h(v)\|_{h(p)} = \|v\|_p = \tau(p,v)$$

for $v \in T_p\Omega = V$, where the third identity follows from the equivalent conditions

$$-\lambda h(p) \le h(v) \le \lambda h(p) \iff \lambda p \le v \le \lambda p \quad (\lambda > 0).$$

By [28, Lemma 1.3, Theorem 1.1], the integrated distance d_τ of τ on Ω coincides with Thompson's metric

$$d_\tau(x,y) = \max\left\{\log M\left(\frac{x}{y}\right), \log M\left(\frac{y}{x}\right)\right\} \quad (x,y \in \Omega),$$

where

$$M\left(\frac{a}{b}\right) := \inf\{\beta > 0 : \beta a \ge b\} \quad (a,b \in \Omega).$$

It has been shown in [33, (5.3), Theorem II] that the restriction of the Carathéodory distance ρ to $i\Omega$ can be expressed as

$$\rho(ix,iy) = \sup\left\{\left|\log\frac{f(x)}{f(y}\right| : f \in V^*,\ f(\Omega) \subset (0,\infty)\right\} \quad (x,y \in \Omega).$$

From this one can deduce that $d_\tau(x,y) = \rho(ix,iy)$, as shown in [14, Lemma 3.6.17].

Example 4.1. Let $\mathcal{A}$ be a JB-algebra with identity e, partially ordered by the closed cone $\mathcal{A}_+ = \{a^2 : a \in \mathcal{A}\}$. Let Ω be the interior of $\mathcal{A}_+$. Then $e \in \Omega$ is an order unit and the norm of $\mathcal{A}$ coincides with the order-unit norm $\|\cdot\|_e$. Hence Ω is a normal cone. Equip Ω with the tangent norm τ defined in (4.1). Each element $a \in \Omega$ is invertible and one can define a smooth map $\mu : \Omega \times \Omega \to \Omega$ in terms of the Jordan triple product by

$$\mu(x, y) = \{x, y^{-1}, x\} \quad (x, y \in \Omega).$$

It can be shown that (Ω, μ) is a Loos symmetric space (e.g. [25]) and moreover, each τ-isometry is a μ-homomorphism. By Lemma 2.7, a τ-symmetry $s_p : \Omega \to \Omega$ at $p \in \Omega$ must be unique since $s_p'(p) = -id : T_p\Omega \to T_p\Omega$.

Finally, we are ready to prove the main result.

Theorem 4.2. *Let Ω be a proper open cone in a real Banach space V, with closure $\overline{\Omega}$. The following conditions are equivalent:*

(i) *The Siegel domain $V \oplus i\Omega$ is biholomorphic to a bounded symmetric domain.*

(ii) *Ω is a normal linearly homogeneous Finsler symmetric cone.*

(iii) *V is a unital JB-algebra in an equivalent norm and $\overline{\Omega} = \{a^2 : a \in V\}$.*

Proof. (i) $\Leftrightarrow$ (iii) This has been proved in [8, 20].

(iii) $\Rightarrow$ (ii) This is essentially proved in [8,20], more details can be found in [32, 22.37]. It suffices to highlight the main arguments. First, Ω is a normal cone as noted in Example 4.1. Let $e \in V$ be the algebra identity. Then $e \in \Omega$ and each element in Ω is invertible. The linear automorphism group $G(\Omega)$ acts transitively on Ω and the tangent norm $b : T\Omega \to [0, \infty)$ defined in (4.2) is $G(\Omega)$-invariant. Equipped with this tangent norm, the inverse map $x \in \Omega \mapsto x^{-1} \in \Omega$ is a b-symmetry at e, which is unique, as noted in Example 4.1, and hence Ω is a symmetric Banach manifold by linear homogeneity.

(ii) $\Rightarrow$ (iii) Let Ω be a normal linearly homogeneous Finsler symmetric cone in a compatible $G(\Omega)$-invariant tangent norm ν. For each $p \in \Omega$, let $s_p : \Omega \to \Omega$ be the symmetry at p. By Example 2.6, (Ω, μ) is a Loos symmetric space, with the smooth map

$$\mu : (x, y) \in \Omega \times \Omega \mapsto x \cdot y = s_x(y) \in \Omega.$$

Denote by $\mathrm{Diff}(\Omega)$ the diffeomorphism group of Ω and let

$$\mathrm{Aut}\,\Omega = \{f \in \mathrm{Diff}(\Omega) : f \circ s_p = s_{f(p)} \circ f \text{ for all } p \in \Omega\}$$

be the subgroup of $\mathrm{Diff}(\Omega)$, consisting of μ-automorphisms of Ω.

By [21, Theorem 2.4, Theorem 5.12], $\mathrm{Aut}\,\Omega$ carries the structure of a real Banach Lie group, with Lie algebra

$$\mathrm{Kill}\,\Omega = \{X \in \mathcal{V}(\Omega) : \exp tX \in \mathrm{Aut}\,\Omega \text{ for all } t \in \mathbb{R}\}, \tag{4.3}$$

which is a Banach Lie algebra in some norm $|\cdot|$ and a subalgebra of the Lie algebra $\mathcal{V}(\Omega)$ of smooth vector fields on Ω. More details are given in the Appendix.

We note that the linear automorphism group $G(\Omega)$ is contained in $\mathrm{Aut}\,\Omega$. Indeed, given $p \in \Omega$ and $g \in G(\Omega)$, the composite map

$$g^{-1} \circ s_{g(p)} \circ g : \Omega \to \Omega$$

is a ν-isometry by $G(\Omega)$-invariance of ν, with isolated fixed-point p. Hence by uniqueness of the symmetry s_p, we have $g^{-1} \circ s_{g(p)} \circ g = s_p$ and $g \in \operatorname{Aut} \Omega$. It follows that $\mathfrak{g}(\Omega) \subset \operatorname{Kill} \Omega$ by (1.1) and (4.3).

Fix a point $e \in \Omega$, which induces an order-unit norm $\| \cdot \|_e$ on V, equivalent to the norm $\| \cdot \|$ of V, by Lemma 3.3.

The evaluation map

$$X \in \operatorname{Kill} \Omega \mapsto X(e) \in V$$

is surjective by [6, Proposition 5.9] (cf. [26, Theorem II.2.2]). In fact, the differential of the orbital map $\rho : g \in G(\Omega) \mapsto g(e) \in \Omega$ at the identity of $G(\Omega)$ is the evaluation map

$$X \in \mathfrak{g}(\Omega) \mapsto X(e) \in T_e\Omega = V \tag{4.4}$$

which is also surjective by linear homogeneity of Ω (see [13]; cf. [35, p. 110]).

Let $s_e : \Omega \to \Omega$ be the symmetry at e. Then $s_e \in \operatorname{Aut} \Omega$. Since s_e^2 is the identity map, the adjoint representation

$$\theta = Ad(s_e) : \operatorname{Kill} \Omega \to \operatorname{Kill} \Omega$$

is an involution and the Lie algebra $\operatorname{Kill} \Omega$ has an eigenspace decomposition

$$\operatorname{Kill} \Omega = \mathfrak{k} \oplus \mathfrak{p}$$

satisfying

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \tag{4.5}$$

where $\mathfrak{k}$ is the 1-eigenspace and $\mathfrak{p}$ the (-1)-eigenspace, both are $| \cdot |$-closed. Moreover, we have (cf. [12, Lemma 2.4.5])

$$\mathfrak{k} = \{X \in \operatorname{Kill} \Omega : X(e) = 0\} = \{X \in \operatorname{Kill} \Omega : \exp tX(e) = e \text{ for all } t \in \mathbb{R}\}.$$

Hence the linear map

$$X \in \mathfrak{p} \mapsto X(e) \in V \tag{4.6}$$

is bijective as $\mathfrak{k} \cap \mathfrak{p} = \{0\}$.

Let $I \in L(V)$ be the identity vector field, which belongs to the Lie algebra $\operatorname{Kill} \Omega$ since $\exp tI = \varepsilon^t I \in G(\Omega)$ for all $t \in \mathbb{R}$, where $\varepsilon = \log^{-1}(1)$ denotes Euler's number, to avoid confusion with the order unit $e \in \Omega$. Hence $[I, X] \in \operatorname{Kill} \Omega$ for all $X \in \operatorname{Kill} \Omega$. We show $\theta I = -I$.

We have

$$(\theta I)(\cdot) = \frac{d}{dt}\bigg|_{t=0} \exp t\theta I(\cdot) = \frac{d}{dt}\bigg|_{t=0} s_e(\exp tI)s_e(\cdot) = \frac{d}{dt}\bigg|_{t=0} s_e(\varepsilon^t s_e)(\cdot).$$

Since the symmetry s_e reverses the geodesic $\gamma(t) = \exp tI(e) = \varepsilon^t e$, we have

$$s_e(\varepsilon^t e) = s_e(\gamma(t)) = \gamma(-t) = \varepsilon^{-t} e.$$

By uniqueness of the symmetry, we have $\varepsilon^t s_e(\varepsilon^t \cdot) = s_e(\cdot)$, which gives

$$(\theta I)(\cdot) = \frac{d}{dt}\bigg|_{t=0} \varepsilon^{-t} I(\cdot) = -I(\cdot).$$

We show next that each $X = f\frac{\partial}{\partial x} \in \mathfrak{p}$ is a linear vector field. For this, we first note that $X = Z - \theta Z$ for some $Z \in \mathfrak{g}(\Omega) \subset \operatorname{Kill}\Omega$. Indeed, (4.4) implies the existence of $Y \in \mathfrak{g}(\Omega)$ such that $Y(e) = X(e)$, which gives

$$X(e) = Y(e) = \frac{1}{2}(Y + \theta Y)(e) + \frac{1}{2}(Y - \theta Y)(e) = \frac{1}{2}(Y - \theta Y)(e)$$

since $\frac{1}{2}(Y + \theta Y) \in \mathfrak{k}$. It follows that $X = \frac{1}{2}(Y - \theta Y) \in \mathfrak{p}$, where $Z = \frac{1}{2}Y \in \mathfrak{g}(\Omega)$. Since Z is a linear vector field by (1.1), linearity of $X = Z - \theta Z$ follows from that of θZ. By the remarks in Section 2, the latter is linear because

$$[I, \theta Z] = \theta[\theta I, Z] = -\theta[I, Z] = 0.$$

The linear isomorphism $X \in \mathfrak{p} \mapsto X(e) \in V$ in (4.6) is a continuous bijection and hence by the open mapping theorem, its inverse is also continuous and there is a constant $\kappa > 0$ such that

$$\kappa\|X(e)\| \geq |X|$$

for all $X \in \mathfrak{p}$. Let $L : V \to \mathfrak{p}$ be the inverse of the map in (4.6) so that

$$L(x)(e) = x \quad (x \in V)$$

and $|L(a)| \leq \kappa\|a\|$ for all $a \in V$.

On V, we can now define a product

$$xy := L(x)(y) \quad (x, y \in V), \tag{4.7}$$

where $L(x)$ is a linear vector field, identified as an element of $L(V)$.

We show that V is a Jordan algebra in this product, with identity e. First, we have

$$ae = L(a)(e) = a \quad (a \in V).$$

Given $a, b \in V$, we have

$$ab - ba = [L(a), L(b)](e) = 0,$$

where $L(a), L(b) \in \mathfrak{p}$ implies $[L(a), L(b)] \in \mathfrak{k}$, by (4.5).

Before deriving the Jordan identity, we need to establish some facts. By continuity of the evaluation map in (4.6), there is a constant $\rho > 0$ such that $\|Xe\| \leq \rho|X|$ for all $X \in \mathfrak{p}$. This implies $\|a\| = \|L(a)e\| \leq \rho|L(a)|$ and

$$\|ab\| = \|L(a)L(b)e\| \leq \kappa\|a\|\|L(b)e\| \leq \rho\kappa^2\|a\|\|b\| \quad (a, b \in V)$$

as well as

$$\|ab\|_e \leq \alpha\|a\|_e\|b\|_e \quad (a, b \in V) \tag{4.8}$$

for some $\alpha > 0$, since $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent.

We begin by showing that V is power associative. One can verify directly the identity

$$[[L(x), L(y)], L(z)](e) = L([L(x), L(y)]z)(e) \quad (x, y, z \in V)$$

where $[L(x), L(y)] \in \mathfrak{k}$ implies $[L(x), L(y)](e) = 0$. It follows that

$$[[L(x), L(y)], L(z)] = L([L(x), L(y)]z) \tag{4.9}$$

since both vector fields belong to $\mathfrak{p}$. By definition, $L(x)$ is the left multiplication by x on the commutative algebra V. By (4.9) and (3.1), $[L(x), L(y)]$ is a derivation on V for all $x, y \in V$. Hence Lemma 3.1 implies

$$[[L(x), L(x^2)], [[L(x), L(x^2)], L(x^2)]] = 0 \quad (x \in V). \tag{4.10}$$

Let $a \in V$ and consider the linear vector field $T = [L(a), L(a^2)] \in \mathfrak{k}$, identified as an element of $L(V)$. Since $\exp tT : \Omega \to \Omega$ satisfies $\exp tT(e) = e$ for all $t \in \mathbb{R}$, each $\exp tT$ is a positive linear map on $(V, \|\cdot\|_e)$ and $\|\exp tT\| = \|\exp tT(e)\| = \|e\| = 1$. Let $T_c \in L(V_c)$ be the complexification of $T \in L(V)$, as defined in Section 3. Then we have

$$\|\exp tT_c\| = \|(\exp tT)_c\| = \|\exp tT\| = 1 \quad (t \in \mathbb{R}).$$

Hence iT_c is a hermitian operator in $L(V_c)$ and it follows from (4.10) that

$$[iT_c, [iT_c, L(a^2)_c]] = -[T, [T, L(a^2)]]_c = 0.$$

The linear operator

$$\eta : S \in L(V_c) \mapsto [iT_c, S] = iT_cS - S(iT_c) \in L(V_c) \tag{4.11}$$

is hermitian, since both the left multiplication $S \in L(V_c) \mapsto iT_cS \in L(V_c)$ and right multiplication $S \in L(V_c) \mapsto S(iT_c) \in L(V_c)$ are hermitian. Hence Lemma 3.4 implies

$$\begin{aligned}\|[iT_c, L(a^2)_c]\|^2 &= \|\eta(L(a^2)_c)\|^2 \le 4\|L(a^2)_c\|\|\eta^2(L(a^2)_c)\| \\ &= 4\|L(a^2)_c\|\|[iT_c, [iT_c, L(a^2)_c]]\| = 0,\end{aligned}$$

which gives

$$[[L(a), L(a^2)], L(a^2)] = [T, L(a^2)] = 0. \tag{4.12}$$

In particular, we have

$$[L(a), L(a^2)](a^2) = [[L(a), L(a^2)], L(a^2)](e) = 0$$

since $[L(a), L(a^2)](e) = 0$. Further, by Lemma 3.1, we have

$$L(a)T = L(a)[L(a), L(a^2)] = \frac{1}{3}[L(a), L(a^3)] \in \mathfrak{k}$$

and hence

$$TL(a) = L(a)T - [L(a), T] = L(a)T - [L(a), [L(a), L(a^2)]] \in \operatorname{Kill}\Omega,$$

where $TL(a)$ is a linear vector field, identified as an element of $L(V)$.

By (3.3), we have $L(a)TL(a)(e) = a[L(a), L(a^2)](a) = 0$ and hence

$$(TL(a))^2(e) = TL(a)TL(a)(e) = 0$$

as well as

$$(TL(a))^{n+2}(e) = (TL(a))^n(TL(a))^2(e) = 0 \quad (n = 1, 2, \dots).$$

It follows that

$$\exp tTL(a)(e) = e + tTL(a)(e) + \frac{t^2(TL(a))^2(e)}{2!} + \cdots = e + tTL(a)(e)$$

for all $t \in \mathbb{R}$, where $\exp tTL(a) \in \operatorname{Aut}\Omega$ implies $e \pm tTL(a)(e) \in \Omega$ for all $t > 0$. In other words,

$$-\frac{1}{t}e \leq TL(a)(e) \leq \frac{1}{t}e \quad (t > 0)$$

and therefore $[L(a), L(a^2)](a) = TL(a)(e) = 0$. By (3.2), we have

$$[L(a), L(a^2)](a^n) = 0 \quad (n = 1, 2, \ldots).$$

That is, $a^{n+3} = a^{n+1}a^2$ for $n = 1, 2, \ldots$. It follows that

$$[L(a), L(a^m)](a) = a^{m+2} - a^m a^2 = 0 \quad (m = 2, 3, \ldots)$$

and again, (3.2) implies

$$[L(a), L(a^m)](a^n) = 0 \quad (n, m - 1 = 1, 2, \ldots),$$

which gives $a^m a^{n+1} = a(a^m a^n)$ for $m, n = 1, 2, \ldots$. From this we deduce

$$a^m a^n = a^{m+n} \quad (m, n = 1, 2, \ldots)$$

by induction, since $a^m a^n = a^{m+n}$ implies

$$a^m a^{n+1} = a(a^m a^n) = a a^{m+n} = a^{m+n+1}.$$

This proves power associativity of V and therefore the closed subalgebra $J(a, e)$ of V generated by e and any $a \in V$ is associative.

Since Ω is geodesically complete and the orbits of the one-parameter groups

$$t \in \mathbb{R} \mapsto \exp tX \quad (X \in \mathfrak{p})$$

are the geodesics through $e \in \Omega$ (cf. [27, Example 3.9]), we must have

$$\Omega = \{\exp X(e) : X \in \mathfrak{p}\}.$$

It follows that each $a \in \Omega$ can be written as $a = \exp X(e)$ for some $X \in \mathfrak{p}$, where X is a linear vector field, identified as an element of $L(V)$. For each $z \in V$, define

$$\operatorname{Exp} z = e + z + \frac{z^2}{2!} + \cdots.$$

Then we have

$$a = \exp X(e) = e + X(e) + \frac{X^2(e)}{2!} + \cdots = \operatorname{Exp} x,$$

where $x = X(e) \in V$. By power associativity, we have $a = (\operatorname{Exp} \frac{x}{2})^2$. This proves the first part of the following inclusions:

$$\Omega \subset \{x^2 : x \in V\} \subset \overline{\Omega}. \tag{4.13}$$

To prove the second inclusion in (4.13), let $v \in V$. We show $v^2 \in \overline{\Omega}$. By a remark before (3.4), there is some $\lambda_0 > 0$ and $a_0 \in \Omega$ such that $\lambda_0 v = e - a_0 \in J(a_0, e)$, where $J(a_0, e)$ is a commutative real Banach algebra in the order-unit norm by (4.8) (cf. [17]).

For each $x \in \Omega \cap J(a_0, e)$, we show $a_0 x \in \Omega$. Indeed, given $a_0 = \operatorname{Exp} z = \exp Z(e)$ for some $z = Z(e)$ and $Z \in \mathfrak{p}$, we have $x \in J(a_0, e) \subset J(z, e)$ and associativity of $J(z, e)$ implies

$$\begin{aligned} a_0 x &= x + zx + \frac{z^2 x}{2!} + \cdots = x + zx + \frac{z(zx)}{2!} + \cdots \\ &= x + Z(x) + \frac{Z^2(x)}{2!} + \cdots = \exp Z(x) \in \Omega. \end{aligned}$$

Further, for $y \in \overline{\Omega} \cap J(a_0, e)$, we show $a_0 y \in \overline{\Omega}$. Note that the cone $\Omega \cap J(a_0, e)$ is open in $J(a_0, e)$ and as before, we have

$$J(a_0, e) = \Omega \cap J(a_0, e) - \Omega \cap J(a_0, e)$$

and $e \in \Omega \cap J(a_0, e)$ is an order-unit in the induced ordering of $J(a_0, e)$ with respect to the cone $\overline{\Omega} \cap J(a_0, e)$. Repeating the remark before (3.4) for the cone $\Omega \cap J(a_0, e)$, one can find $\lambda > 0$ and $w \in \Omega \cap J(a_0, e)$ such that $\lambda y = e - w$, where $w = e - \lambda y \le e$ and $0 < f(w) \le 1$ for all states f in the state space S_e defined in (3.7). The latter implies

$$f\left(e - \left(1 - \frac{1}{n}\right) w\right) = 1 - \left(1 - \frac{1}{n}\right) f(w) > 0 \quad (n = 1, 2, \dots)$$

for all $f \in S_e$ and hence $e - (1 - \frac{1}{n}) w \in \Omega \cap J(a_0, e)$ by Lemma 3.2. Therefore the preceding argument yields $a_0(e - (1 - \frac{1}{n}) w) \in \Omega \cap J(a_0, e)$ and

$$\lambda a_0 y = \lim_n a_0 \left(e - \left(1 - \frac{1}{n}\right) w\right) \in \overline{\Omega} \cap J(a_0, e).$$

Let

$$S_{a_0} = \{\psi \in J(a_0, e)^* : \psi(e) = 1,\ \psi \text{ is positive on } J(a_0, e)\}$$

be the state space of $J(a_0, e)$. Let $\psi \in S_{a_0}$ be a pure state, that is, ψ is an extreme point of S_{a_0}. We show that $\psi(a_0^2) = \psi(a_0)^2$. Let

$$b = \frac{a_0}{2\|a_0\|_e} \in \Omega \cap J(a_0, e)$$

so that $\|b\|_e < 1$. Then we have $0 < \varphi(b) < 1$ for all $\varphi \in S_{a_0}$ and $e - b \in \Omega \cap J(a_0, e)$ by Lemma 3.2. One can define two states ψ_b and ψ_{e-b} in S_a by

$$\psi_b(x) = \frac{\psi(bx)}{\psi(b)}, \quad \psi_{e-b}(x) = \frac{\psi((e-b)x)}{1 - \psi(b)} \quad \text{for } x \in J(a_0, e).$$

This gives the convex combination

$$\psi = \psi(b)\psi_b + (1 - \psi(b))\psi_{e-b}$$

and therefore $\psi = \psi_b$, which gives $\psi(bx) = \psi(b)\psi(x)$ for all $x \in J(a_0, e)$ and in particular $\psi(a_0^2) = \psi(a_0)^2$.

It follows that $\psi((\lambda_0 v)^2) = \psi((e - a_0)^2) = \psi(e - 2a_0 + a_0^2) = (1 - \psi(a_0))^2 \ge 0$ for each pure state $\psi \in S_{a_0}$, and hence $\varphi(v^2) \ge 0$ for all states $\varphi \in S_{a_0}$, by the Krein–Milman theorem. As each state of V restricts to a state of $J(a_0, e)$, we have shown $f(v^2) \ge 0$ for all states f of V and hence $v^2 \in \overline{\Omega}$ by (3.8). This proves the second inclusion in (4.13).

The preceding arguments also reveal that $\|v\|_e^2 = \|v^2\|_e$ since $\psi(v^2) = \psi(v)^2$ for all pure states of $J(a_0, e)$ and $\|v\|_e$ is the supremum $\sup\{|\psi(x)|\}$, taken over all pure states ψ in S_{a_0}. Since $v \in V$ was arbitrary, we have shown $\|x^2\|_e = \|x\|_e^2$ for all $x \in V$.

In (4.8), we now actually have

$$\|xy\|_e \leq \|x\|_e \|y\|_e \quad (x, y \in V).$$

This follows from the fact that the map $(x, y) \in V^2 \mapsto f(xy) \in \mathbb{R}$ is a positive semi-definite symmetric bilinear form, for each state $f \in S_e$, and hence the Schwarz inequality gives

$$|f(xy)|^2 \leq f(x^2) f(y^2) \leq \|x^2\|_e \|y^2\|_e = \|x\|_e^2 \|y\|_e^2$$

and $\|xy\|_e = \sup\{|f(xy)| : f \in S_e\} \leq \|x\|_e \|y\|_e$.

Let $a \in V$. For all $x, y \in J(a, e)$, the inequality $0 \leq x^2 \leq x^2 + y^2$ implies

$$\|x^2\|_e \leq \|x^2 + y^2\|_e.$$

Therefore we have shown that $(J(a, e), \|\cdot\|_e)$ is an associative JB-algebra, which can be identified with the algebra $C(\mathcal{S}, \mathbb{R})$ of real continuous functions on a compact Hausdorff space $\mathcal{S}$ (see [16, Theorem 3.2.2]). Equipped with the injective tensor norm $\|\cdot\|_{\text{inj}}$, the complexification $J(a, e)_c = C(\mathcal{S}, \mathbb{R}) \otimes \mathbb{C}$ identifies with the C*-algebra $C(\mathcal{S}, \mathbb{C})$ of complex continuous functions on $\mathcal{S}$.

Equip the complexification $V_c = V \otimes \mathbb{C}$ of $(V, \|\cdot\|_e)$ with the injective tensor norm $\|\cdot\|_{\text{inj}}$. Then, for $a \in V$, the remarks at the end of Section 3 imply that the numerical range $\mathsf{V}(L_{a^2})$ of the left multiplication operator $L_{a^2} : V_c \to V_c$ is contained in

$$\mathsf{v}(a^2) = \{f(a^2) : f \in V_c^* \text{ satisfies } \|f\| = 1 = f(e)\},$$

where each f restricts to a state of the C*-algebra $J(a, e)_c = C(\mathcal{S}, \mathbb{C})$.

Since $a^2 \in J(a, e) \cap \overline{\Omega} \subset C(\mathcal{S}, \mathbb{R})$, we have $f(a^2) \geq 0$ and in particular

$$\mathsf{V}(L_{a^2}) \subset \mathsf{v}(a^2) \subset \mathbb{R}.$$

Hence the operator L_{a^2} is hermitian in $L(V_c)$ and as in (4.11), the linear operator

$$S \in L(V_c) \mapsto [L_{a^2}, S] = L_{a^2} S - S L_{a^2} \in L(V_c) \tag{4.14}$$

is hermitian.

We are now equipped to prove the Jordan identity. Indeed, we have

$$[L_{a^2}, [L_{a^2}, L_a]] = 0.$$

by (4.12) and as before, applying Lemma 3.4 to the hermitian operator in (4.14) yields

$$\|[L_{a^2}, L_a]\|^2 \leq 4\|L_a\| \|[L_{a^2}, [L_{a^2}, L_a]]\| = 0$$

and therefore $[L_{a^2}, L_a] = 0$, proving the Jordan identity in V.

It remains to show that $(V, \|\cdot\|_e)$ is a JB-algebra and $\overline{\Omega} = \{x^2 : x \in V\}$. To show the former, it suffices to prove

$$-e \leq a \leq e \implies 0 \leq a^2 \leq e \quad (a \in V)$$

by [16, Proposition 3.1.6].

Let $-e \leq a \leq e$. We have already shown $a^2 \in \overline{\Omega}$. Since $e \pm a \in \overline{\Omega} \cap J(a,e)$ and all pure states of $J(a,e) \approx C(\mathcal{S}, \mathbb{R})$ are multiplicative, we have

$$\psi(e - a^2) = \psi((e+a)(e-a)) = \psi(e+a)\psi(e-a) \geq 0$$

for all pure states ψ of $J(a,e)$, which implies $\varphi(e - a^2) \geq 0$ for all states φ of $J(a,e)$, by the Krein-Milman theorem. Hence $e - a^2 \in \overline{\Omega}$ since each state of V restricts to a state of $J(a,e)$. This proves that $(V, \|\cdot\|_e)$ is a JB-algebra. It follows that $\{x^2 : x \in V\}$ is closed and coincides with $\overline{\Omega}$, by (4.13). □

Remark. The proof of Theorem 4.2 reveals that condition (iii) in the theorem is equivalent to Ω being a normal linearly homogeneous Finsler symmetric cone in the tangent norm τ defined in (4.1). However, (iii) can also be equivalent to Ω being a normal linearly homogeneous Finsler symmetric cone in another $G(\Omega)$-invariant tangent norm. For instance, the other tangent norm can be the Riemannian metric given in Example 4.5 below.

Example 4.3. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. The Hilbert space direct sum $H \oplus \mathbb{R}$, with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, is a JH-algebra with identity $e = 0 \oplus 1$ and the Jordan product

$$(a \oplus \alpha)(b \oplus \beta) := (\beta a + \alpha b) \oplus (\langle a, b \rangle + \alpha\beta).$$

We have

$$\{x^2 : x \in H \oplus \mathbb{R}\} = \{a \oplus \alpha : \alpha \geq \|a\|\}.$$

Its interior Ω is linearly homogeneous [12, Lemma 2.3.17] and a Riemannian symmetric space in the metric

$$g_p(u, v) = \langle\!\langle \{p^{-1}, u, p^{-1}\}, v \rangle\!\rangle \quad (p \in \Omega,\ u, v \in H \oplus \mathbb{R})$$

(see [12, Theorem 2.3.19]), where $\{p^{-1}, u, p^{-1}\}$ denotes the Jordan triple product.

One can define an equivalent norm $\|\cdot\|_s$ on $H \oplus \mathbb{R}$ by

$$\|a \oplus \alpha\|_s = \|a\| + |\alpha|.$$

When $H \oplus \mathbb{R}$ is equipped with this norm, it becomes a JB-algebra and is called a *spin factor*, where $\|\cdot\|_s$ is the order-unit norm induced by e. In this setting, Ω is a linearly homogeneous Finsler symmetric cone with the tangent norm τ in (4.1), which differs from g. We have

$$\tau(e, a \oplus \alpha) = \|a \oplus \alpha\|_e = \|a \oplus \alpha\|_s = \|a\| + |\alpha|$$

whereas $g_e(a \oplus \alpha, a \oplus \alpha)^{\frac{1}{2}} = \sqrt{\|a\|^2 + |\alpha|^2}$.

The class of JB-algebras include the unital JH-algebras. Indeed, unital JH-algebras have been classified in [14, Section 3], they are of the form

$$A_1 \oplus \cdots \oplus A_n \quad (n \in \mathbb{N}) \tag{4.15}$$

where each summand A_j is either a finite-dimensional unital JH-algebra or of the form $H \oplus \mathbb{R}$, and the direct sum in (4.15) is equipped with coordinatewise Jordan product and the ℓ_2-norm

$$\|a_1 \oplus \cdots \oplus a_n\|_2 := (\|a_1\|^2 + \cdots + \|a_n\|^2)^{\frac{1}{2}}.$$

When the direct sum is equipped with the ℓ_∞-norm

$$\|a_1 \oplus \cdots \oplus a_n\|_\infty := \sup\{\|a_1\|, \dots, \|a_n\|\},$$

it becomes a JB-algebra. Finite-dimensional unital JH-algebras are exactly the class of finite-dimensional formally real Jordan algebras, which have been classified in [18].

Corollary 4.4. *Let Ω be a proper open cone in a real Hilbert space V, with closure $\overline{\Omega}$. The following conditions are equivalent:*

(i) *Ω is a normal linearly homogeneous Finsler symmetric cone.*

(ii) *Ω is a linearly homogeneous self-dual cone.*

(iii) *V is a unital JH-algebra in an equivalent norm and $\overline{\Omega} = \{a^2 : a \in V\}$.*

Proof. (ii) $\Rightarrow$ (iii) This has been proved in [13]. In fact, condition (ii) entails a decomposition $\mathfrak{g}(\Omega) = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ and the evaluation map $X \in \mathfrak{p}_1 \mapsto X(e) \in V$ induces an algebra product in V, as in (4.7). One can use the argument in the proof of Theorem 4.2 to derive the Jordan identity in place of the one given in [13].

(iii) $\Rightarrow$ (ii) This has been proved in [12, Lemma 2.3.17].

(iii) $\Rightarrow$ (i) This follows from Theorem 4.2 since V is a unital JB-algebra in an equivalent norm by the preceding remark.

(i) $\Rightarrow$ (iii) By Theorem 4.2, V is a unital JB-algebra in an equivalent norm and

$$\overline{\Omega} = \{a^2 : a \in V\}.$$

Since V is a Hilbert space, it is a reflexive JB-algebra and by [14, Corollary 3.3.6], V is an ℓ_∞-sum of a finite number of finite-dimensional formally real Jordan algebras or spin factors, or both. Hence V is a unital JH-algebra in an equivalent norm. □

Remark. It follows from the preceding corollary that one can view linearly homogeneous Finsler symmetric cones as a generalisation of linearly homogeneous self-dual cones to the setting of Banach spaces.

Example 4.5. A proper open cone Ω in a finite-dimensional Euclidean space $\mathbb{R}^n$, with inner product $\langle\cdot,\cdot\rangle$ and Euclidean measure dy, can be equipped with a canonical $G(\Omega)$-invariant Riemannian metric [34]

$$g = \frac{\partial^2 \log \varphi}{\partial x^i \partial x^j}\, dx^i\, dx^j,$$

where φ is the characteristic function of Ω defined by

$$\varphi(x) = \int_{\Omega^*} \exp -\langle x, y\rangle\, dy \quad (x \in \Omega).$$

The tangent norm ν defined by g is not the same as τ in (4.1). It has been shown in [31] and [30] that a linearly homogeneous cone Ω in $\mathbb{R}^n$ is self-dual if (Ω, g) is a symmetric space. We see that (i) $\Rightarrow$ (ii) in Corollary 4.4 provides an alternative proof of this result, as well as extends it to infinite dimension.

A. Appendix

In what follows, we provide some details of the fact that the automorphism group Aut Ω of a Finsler symmetric cone Ω in a Banach space V carries the structure of a real Banach Lie group, with Lie algebra Kill Ω.

This crucial result follows from [21, 22]. To begin, Ω is an open cone in V and a Banach manifold with analytic structure given by the identity map. As noted before, (Ω, μ) is a Loos symmetric space with the smooth map

$$\mu : (x, y) \in \Omega \times \Omega \mapsto x \cdot y = s_x(y) \in \Omega.$$

Further, Ω is equipped with an affine connection

$$\Gamma : T\Omega \oplus T\Omega \to TT\Omega$$

(denoted by B in [21]) which, by definition, is a morphism of vector bundles such that

$$(\pi_{T\Omega}, T\pi) \circ \Gamma = \mathrm{id}_{T\Omega \oplus T\Omega}$$

and

$$\Gamma_x : T_x\Omega \oplus T_x\Omega \to TT\Omega \quad (x \in \Omega)$$

is bilinear, where $\pi : T\Omega \to \Omega$ is the tangent bundle of Ω, and $\pi_{T\Omega} : TT\Omega \to T\Omega$ is that of $T\Omega$. The bilinear condition implies that, in the identity chart, Γ has a (local) representation

$$\Gamma(x, v, w) = (x, v, w, H_x(v, w)) \quad (x \in \Omega,\ v, w \in V),$$

where $H_x(v, w) = H(x)(v, w)$ and $H : \Omega \to L^2(V, V)$ is a smooth map into the Banach space $L^2(V, V)$ of continuous bilinear maps $V \times V \to V$.

Indeed, the affine connection Γ on the Finsler symmetric cone Ω is given by

$$H_x(v, w) = -\frac{1}{2} d^2\mu(x, x)(v, 0)(0, w),$$

which is geodesically complete, with the corresponding covariant derivative

$$\nabla : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathcal{V}(\Omega)$$

satisfying

$$\nabla_\xi \eta(x) = \eta'(x)\xi(x) - H_x(\eta(x), \xi(x)) \quad (x \in \Omega),$$

where, in the notation of [21], ξ and η denote smooth vector fields on Ω. The connection Γ is torsionfree and hence $H_x(v, w) = H_x(w, v)$.

It has been shown in [22, Theorem 3.15] that the automorphism group Aut (Ω, Γ), consisting of Γ-affine diffeomorphisms of Ω, carries the structure of a Banach Lie group, with Lie algebra Kill $(\Omega, \Gamma) = \{\xi \in \mathcal{V}(\Omega) : \exp t\xi \in \mathrm{Aut}(\Omega, \Gamma) \text{ for all } t \in \mathbb{R}\}$, which consists of complete vector fields $\xi \in \mathcal{V}(\Omega)$ satisfying

$$\text{(A.1)} \quad d^2\xi(x)(v)(w) + d\xi(x)(H_x(v, w)) = dH(x)(\xi(x))(v, w) + H_x(d\xi(x)(v), w) + H_x(v, d\xi(x)(w)) \quad (v, w \in V)$$

(cf. [22, Remark 3.10, Proposition 3.11]).

It follows that the automorphism group $\operatorname{Aut}\Omega$, consisting of μ-automorphisms of Ω, is a Banach Lie group since, by [21, Theorem 5.12], and also [27, Theorem 3.6], we have $\operatorname{Aut}\Omega = \operatorname{Aut}(\Omega, \Gamma)$ for a Loos symmetric space Ω.

To conclude, we show that the Lie algebra

$$\operatorname{Kill}\Omega = \{X \in \mathcal{V}(\Omega) : \exp tX \in \operatorname{Aut}\Omega \text{ for all } t \in \mathbb{R}\}$$

of *infinitesimal* μ*-automorphisms* is the Lie algebra $\operatorname{aut}\Omega$ of *derivations*, which are vector fields $\xi \in \mathcal{V}(\Omega)$ satisfying

$$\text{(A.2)} \qquad \xi(\mu(x, y)) = d\mu(x, y)(\xi(x), \xi(y)) \quad (x, y \in \Omega).$$

This has been proved in [26, p.84] for finite-dimensional connected Loos symmetric spaces.

Lemma. *We have* $\operatorname{Kill}(\Omega, \Gamma) = \operatorname{aut}\Omega$.

Proof. We follow the arguments in [21, Proposition 5.17]. Let $\xi \in \operatorname{Kill}(\Omega, \Gamma)$. Then we have $\exp t\xi \in \operatorname{Aut}(\Omega, \Gamma) = \operatorname{Aut}\Omega$ for $t \in \mathbb{R}$. Hence

$$\begin{aligned}
\xi(\mu(x, y)) &= \frac{d}{dt}\Big|_{t=0} \exp t\xi(\mu(x, y)) = \frac{d}{dt}\Big|_{t=0} \exp t\xi(x \cdot y) \\
&= \frac{d}{dt}\Big|_{t=0} \exp t\xi(x) \cdot \exp t\xi(y) = \frac{d}{dt}\Big|_{t=0} \mu(\exp t\xi(x), \exp t\xi(y)) \\
&= d\mu(x, y)\left(\frac{d}{dt}\Big|_{t=0} \exp t\xi(x), \frac{d}{dt}\Big|_{t=0} \exp t\xi(y)\right) \\
&= d\mu(x, y)(\xi(x), \xi(y)).
\end{aligned}$$

Therefore $\xi \in \operatorname{aut}\Omega$.

Conversely, let $\xi \in \operatorname{aut}\Omega$. We show that ξ satisfies (A.1). Differentiating (A.2), we get

$$d\xi(\mu(x, y))d\mu(x, y)(v, 0) = d^2\mu(x, y)(\xi(x), \xi(y))(v, 0) + d\mu(x, y)(d\xi(x)v, 0).$$

Differentiating the above with respect to y in the direction $w \in V$ gives

$$\begin{aligned}
\text{(A.3)} \quad & d^2\xi(\mu(x, y))(d\mu(x, y)(v, 0), d\mu(x, y)(0, w)) \\
&\quad + d\xi(\mu(x, y))d^2\mu(x, y)(v, 0)(0, w) \\
&= d^3\mu(x, y)(\xi(x), \xi(y))((v, 0), (0, w)) + d^2\mu(x, y)(0, d\xi(y)(w))(v, 0) \\
&\quad + d^2\mu(x, y)(d\xi(x)(v), 0)(0, w).
\end{aligned}$$

We note that, by [27, Proposition 3.3] (cf. [26, p.74]), the tangent bundle $T\Omega$ is a Loos symmetric space $(T\Omega, T\mu)$, where

$$T\mu : T\Omega \times T\Omega \to T\Omega$$

is the tangent map of $\mu : \Omega \times \Omega \to \Omega$, and we have

$$T\mu(v, w) = 2v - w \quad (v, w \in T_p\Omega, p \in \Omega).$$

Finally, putting $y = x$ in (A.3), where $\mu(x, x) = x$, and making use of the preceding equation, we arrive at

$$\begin{aligned}
& d^2\xi(x)(2v, -w) + d\xi(x)(-2H_x(v, w)) \\
&\quad = -2dH(x)(\xi(x))(v, w) - 2H_x(v, d\xi(x)(w)) - 2H_x(d\xi(x)(v), w),
\end{aligned}$$

which gives (A.1). □

Acknowledgement. The author wishes to thank the referee for valuable comments and suggesting the inclusion of an appendix with an exposition of the Lie structure of the automorphism group Aut Ω of a Finsler symmetric cone Ω.

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Eingegangen 16. Oktober 2020, in revidierter Fassung 28. April 2021