Scaling exponents in fully developed turbulence from nonextensive statistical mechanics

Christian Beck

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK

Abstract

We combine Tsallis statistics with extended selfsimilarity (ESS) to calculate scaling exponents of moments in a nonextensive statistical model of fully developed turbulence. One obtains numerical values of scaling exponents that agree very well with experimental data. Our approach is based on a scale-dependent nonextensitivity parameter $q > 1$ and can be embedded into previously successful models for probability densities of velocity differences.

The scaling exponents ζ_m of moments of radial velocity differences $u := v(x+r) - v(x)$ measured at distance r in fully developed turbulent flows have a long history in turbulence theory. In the inertial range one observes scaling behaviour of the form

$$\langle |u|^m \rangle \sim r^{\zeta_m}, \tag{1}$$

and up to now there is a controversy on the exact numerical values of the ζ_m. The earliest model, Kolmogorov's K41 theory [1] predicted linear behaviour $\zeta_m = \frac{1}{3}m$, being soon replaced by the lognormal model (K62)[2] and numerous other models with a nonlinear dependence of ζ_m on m (see [3, 4] for a survey). The need for these new models came from the experimental measurements, which (although being plagued by large error bars) provided evidence for a nontrivial dependence of ζ_m on m. Presently, the formula that seems to fit the experimentally measured ζ_m best is the She-Leveque formula [5]

$$\zeta_m = \frac{m}{9} + 2\left(1 - \left(\frac{2}{3}\right)^{m/3}\right) \tag{2}$$

though the theoretical basis underlying this formula is somewhat unclear.

Recently, it has been suggested to use methods from nonextensive statistical mechanics [6, 7, 8] for a statistical description of turbulence [9, 10, 11]. Two different approaches have been suggested, one with a constant nonextensitivity parameter $q < 1$ [9] and another one with a scale dependent $q(r) > 1$ [10, 11]. Here we will work within the second approach as described in [11], which has been shown to reproduce experimentally measured probability densities very precisely [11, 12].

Generally, to derive values for the scaling exponents in turbulence, Tsallis statistics alone is not sufficient — one also has to use additional assumptions. For example, in [11] the assumption of a logarithmic dependence of $(q-1)^{-1}$ on r was used to derive a formula for the scaling exponents that is similar to that of Kolmogorov's lognormal model. Or, in [9] the assumption of a certain scale independent distribution of local scaling exponents was used to derive another formula for the ζ_m within the multifractal model.

Here we want to describe yet another nonextensive approach to the scaling exponents, which is new but can be consistently embedded into the approach of [11, 12]. This new approach is very closely related to experiments, in the sense that we only use assumptions that are experimentally verified. We will combine Tsallis statistics with a concept well known from turbulence theory, so called extended self similarity (ESS) [13]. The ESS method is in fact used for most experimental precision measurements of the scaling exponents and is thus a reasonable approach to use in a theoretical context as well. Using Tsallis statistics together with the exact validity of ESS for the second and third moment, one can eliminate the unknown scale dependent variance parameter $\beta(q)$ in the Tsallis distributions. One ends up with concrete numerical values for the scaling exponents ζ_m. This new theoretical approach, as we will see, predicts very good numerical values for the scaling exponents, possibly in even better agreement with experiments than the celebrated She-Leveque formula. The advantage compared to other theoretical models is that not only the scaling exponents but also the scale dependent probability densities of the velocity differences come out correctly at the same time.

Our approach is as follows. Neglecting skewness corrections, probability densities of velocity differences u in fully developed turbulent flows have been observed to be very well approximated by Tsallis distributions of the form

$$p(u) = \frac{1}{Z_q} \frac{1}{(1 + \frac{1}{2}(q-1)\beta |u|^{2\alpha})^{\frac{1}{q-1}}} \tag{3}$$

(see [12] for experimental evidence). The exponent α is close to one. In [12] best fits to the experimental data were obtained for the choice $\alpha = 2-q$. The non-extensitivity parameter q depends on the spatial scale r in a non-trivial way. Very detailed measurements of $q(r)$ have been presented in [12]. The skewness of the experimentally observed distributions can also be understood on a theoretical basis, yielding a generalization of eq. (3) that is slightly asymmetric [11, 12]. However, in the following we will neglect the asymmetric corrections, which are small.

The normalization constant Z_q of the above probability distributions can be calculated as

$$Z_q^{-1} = \left(\frac{1}{2}(q-1)\beta \right)^{\frac{1}{2\alpha}} \cdot \frac{\alpha}{B\left(\frac{1}{2\alpha}, \frac{1}{q-1} - \frac{1}{2\alpha} \right)}, \tag{4}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} \tag{5}$$

denotes the beta function. With this Z_q, all moments of the distribution (3) can be analytically evaluated. One obtains after a short calculation

$$\langle |u|^m \rangle = \int_{-\infty}^{+\infty} |u|^m p(u) du = \left\{ \frac{2}{(q-1)\beta} \right\}^{\frac{m}{2\alpha}} \frac{B\left(\frac{m+1}{2\alpha}, \frac{1}{q-1} - \frac{m+1}{2\alpha}\right)}{B\left(\frac{1}{2\alpha}, \frac{1}{q-1} - \frac{1}{2\alpha}\right)}. \tag{6}$$

Let us now consider ESS [13]. The basic idea of this commonly used method in turbulence theory is that one just plots the logarithm of the m-th moment versus the logarithm of the 3rd moment. The slope in this plot defines the scaling exponents ζ_m. The advantage of this method is that the explicit r-dependence of the structure functions is irrelevant. Nonuniversal features of the structure functions are averaged out, and usually one observes a much larger scaling region for experimental data if ESS is used.

We can now eliminate the unknown scale dependent parameter β in eq. (6) by postulating the exact validity of the ESS condition

$$\log\langle |u|^2 \rangle = \zeta_2 \log\langle |u|^3 \rangle + const, \tag{7}$$

or

$$\langle |u|^2 \rangle = c \cdot \langle |u|^3 \rangle^{\zeta_2}. \tag{8}$$

Combining eq. (8) with eq. (6) one can solve for β, obtaining the formula

$$\beta(q) = c^{\frac{2\alpha}{3\zeta_2 - 2}} \cdot \frac{2}{q-1} \frac{B\left(\frac{2}{\alpha}, \frac{1}{q-1} - \frac{2}{\alpha}\right)^{\frac{2\zeta_2 \alpha}{3\zeta_2 - 2}} B\left(\frac{1}{2\alpha}, \frac{1}{q-1} - \frac{1}{2\alpha}\right)^{\frac{2\alpha(1-\zeta_2)}{3\zeta_2 - 2}}}{B\left(\frac{3}{2\alpha}, \frac{1}{q-1} - \frac{3}{2\alpha}\right)^{\frac{2\alpha}{3\zeta_2 - 2}}}. \tag{9}$$

With this $\beta(q)$ and $\alpha(q) = 2 - q$ one can now exactly evaluate the moments using eq. (6). In particular, one can plot $\log\langle |u|^m \rangle$ versus $\log\langle |u|^3 \rangle$ for various values of q (remember that a change of q describes a change of the spatial distance r in our nonextensive model of fully developed turbulence [11]). One obtains straight lines with a slope given by the scaling exponents ζ_m of the moments (Fig. 1). The slope is independent of the constant c. However, it weakly depends on ζ_2, which is an input parameter. In fact, almost all relevant models of turbulent scaling exponents need an input parameter, which is often expressed in terms of the intermittency parameter μ. Our input parameter is the exponent ζ_2 of the 2nd moment. Given ζ_2, all other exponents ζ_m follow. In the following, we choose for ζ_2 the value 0.695937 that is given by the She-Leveque formula (2) for $m = 2$. This allows for a direct comparison between the She-Leveque model and our new model based on Tsallis statistics.

It is possible to determine the slopes s in Fig. 1 with very high precision, using the integral representation (5) of the beta function and our analytical formulas. Actually, it turns out that for $m > 3$ the ESS method yields straight lines that are curved by a tiny little amount (just similar to experimentally observed ESS plots). These tiny changes of slope are not visible to the blank eye but can of course be precisely determined in our model, since we have the exact formulas. We observe that for each moment $m > 3$ there is a distinguished q value, call it q^*, where the (very small) change of slope with q has a local minimum, i.e. $\frac{\partial^2}{\partial q^2} s(q^*) = 0$. The slope $s(q^*)$ at this point can be used to precisely *define* the scaling exponents. If only a precision of 2-3 digits is wished for (a realistic precision for turbulence data), then the curves of the ESS plot are sufficiently straight and linear regression will do a sufficiently good job.

Our results for the scaling exponents are listed in Tab. 1 and plotted in Fig. 2. There is good agreement between our theoretical values and the values measured in true turbulence experiments. In fact, if one performs the average over the data of the 5 experimental groups that have measured the exponents [13] – [17], one obtains values that very well coincide with the prediction of our nonextensive model (Fig. 3). The coincidence seems to be even better than that of the celebrated She-Leveque formula.

We have only evaluated moments up to $m = 10$, since the reliability of the experimental data rapidly decreases for larger m. Also, as can be seen from eq. (3) the theoretically predicted probability densities asymptotically decay as $p(u) \sim |u|^{-w}$ with a rather large exponent $w = 2\alpha/(q-1) = (4-2q)/(q-1)$, and hence only moments with $m < w-1$ exist. In the precision measurements of the Swinney group [12], w was estimated to be ~ 9 for very small spatial scales, increasing to ~ 15 at medium scales.

Summarizing, it appears that extremizing the Tsallis entropies is a very useful theoretical approach to turbulence. To the best of our knowledge, no other theoretical model yields fits of probability densities of similar quality as the non-extensive model studied here does (see [12] for a precision comparison with experimental data). Moreover, the scaling exponents of the moments come out in the same way as measured in experiments. Hence there is a lot of evidence that the Tsallis formalism works very well in turbulence. The most interesting open question, of course, is *why* it works so well. Presumably, a fully developed turbulent flow (as a driven system) cannot extremize the

Boltzmann Gibbs entropy and then just chooses the second best information measures that are available, the Tsallis entropies. These have nice convexity properties and satisfy a generalized version of the Shannon-Khinchin axioms [18].

References

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Tables

m	ζ_m
1	0.363
2	0.696
3	1
4	1.277
5	1.527
6	1.754
7	1.95
8	2.13
9	2.28
10	2.42

Tab. 1: Scaling exponents as obtained from our theoretical model.

Figure captions

Fig. 1 m-th moment ($m = 1, 2, \ldots, 10$) versus 3rd moment for the probability densities (3). q varies in the interval [1.005,1.100] and $\beta(q)$ is given by eq. (9) with $c = \frac{2}{3}$ and $\alpha = 2 - q$. The slope in the double logarithmic plot yields the scaling exponents.

Fig. 2 Scaling exponents as obtained from our theoretical model (solid line) and experimental data from various turbulence experiments [13] – [17].

Fig. 3 Same as Fig. 2 but the experimental data are averaged and the She-Leveque formula is shown for comparison.

Fig.1

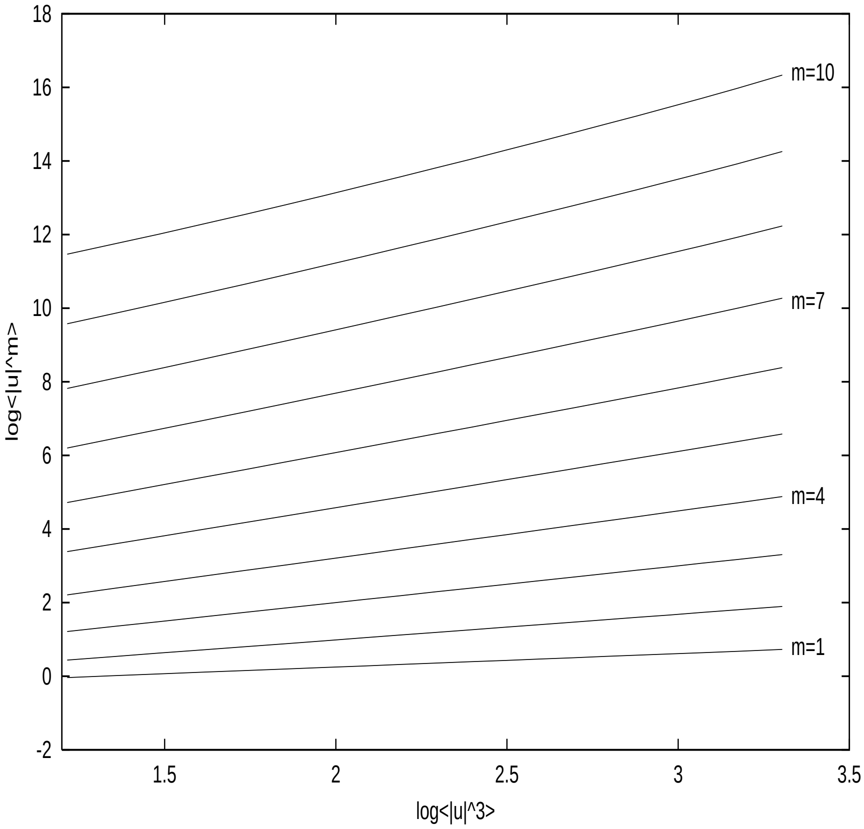
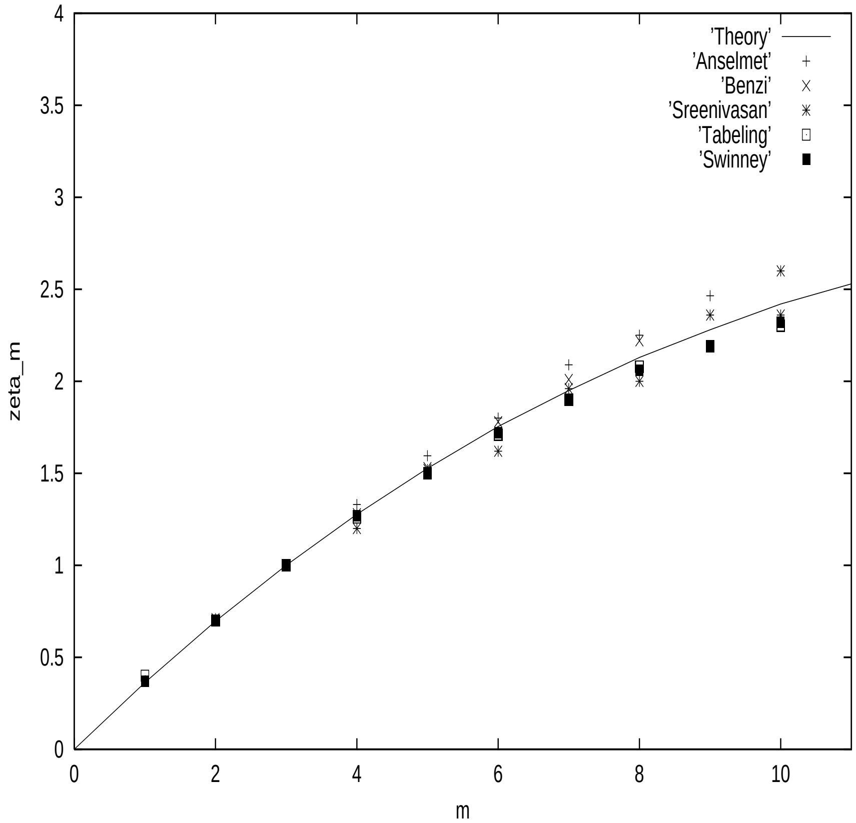


Fig.2


'Theory'
'Anselmet'
'Benzi'
'Sreenivasan'
'Tabeling'
'Swinney'
zeta_m
m
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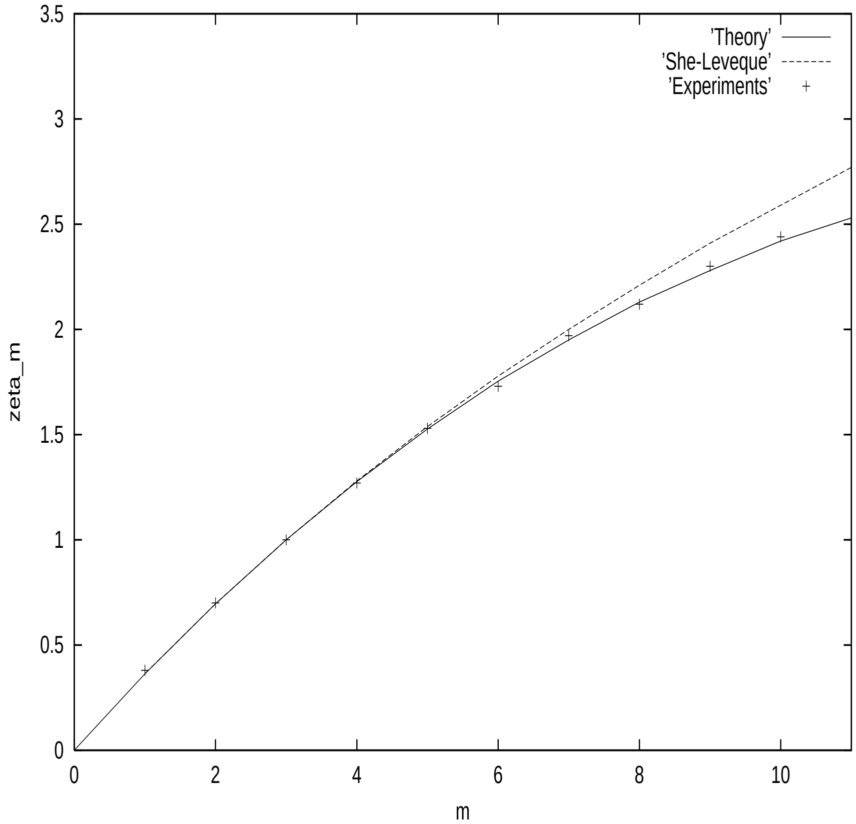
1.5
2
2.5
3
3.5
4
0
2
4
6
8
10

Fig.3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'She-Leveque'
'Experiments'
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0
2
4
6
8
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