# Scaling behaviour of non-hyperbolic coupled map lattices

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Coupled map lattices of non-hyperbolic local maps arise naturally in many physical situations described by discretised reaction diffusion equations or discretised scalar field theories. As a prototype for these types of lattice dynamical systems we study diffusively coupled Tchebyscheff maps of $N$-th order which exhibit strongest possible chaotic behaviour for small coupling constants $a$. We prove that the expectations of arbitrary observables scale with $\sqrt{a}$ in the low-coupling limit, contrasting the hyperbolic case which is known to scale with $a$. Moreover we prove that there are log-periodic oscillations of period $\log N^2$ modulating the $\sqrt{a}$-dependence of a given expectation value. We develop a general 1st order perturbation theory to analytically calculate the invariant 1-point density, show that the density exhibits log-periodic oscillations in phase space, and obtain excellent agreement with numerical results.

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Coupled map lattices (CMLs) as introduced by Kaneko and Kaplan [1, 2] are a paradigm of higher-dimensional dynamical systems exhibiting spatio-temporal chaotic behaviour. There is a variety of applications for CMLs to model hydrodynamical flows, turbulence, chemical reactions, biological systems, and quantum field theories (see e.g. reviews in [3, 4]). The analysis of chaotic CMLs is often restricted to numerical investigations and only a few analytical results are known. A notable exception is the case of hyperbolic maps (maps for which the absolute value of the slope of the local maps is always larger than 1) for small coupling $a$. In this case a variety of analytical results exists [5, 6, 7, 8] that guarantee the existence of a smooth invariant density and ergodic behaviour. The situation is much more complicated for non-hyperbolic maps which correspond to the generic case of physical interest. Here much less is known analytically, though some promising steps have been made [9, 10, 11, 12, 13].

In this paper, we present for the first time analytical results corresponding to the (generic) non-hyperbolic case and calculate invariant densities explicitely for the case of locally fully developed chaos that is diffusively coupled. We study local maps given by $N$-th order Tchebyscheff polynomials. In the uncoupled case these maps are conjugated to a Bernoulli shift of $N$ symbols. In the coupled case, this conjugacy is destroyed and the convential treatments for hyperbolic maps does not apply, since the Tchebyscheff maps have $N-1$ critical points where the slope vanishes, thus corresponding to a non-hyperbolic situation. From the physical point of view, the non-hyperbolic case is the most interesting one. For example, it has been shown that these types of non-hyperbolic CMLs naturally arise from stochastically quantised scalar field theories in the anti-integrable limit [4]. They can serve as useful models for vacuum fluctuations and dark energy in the universe [14]. Other applications include chemical kinetics as described by discretised reaction diffusion dynamics [3].

The case of two coupled Tchebyscheff maps was previously studied in [15] using periodic orbit theory. Here we consider infinitely many diffusively coupled Tschebyscheff maps on one-dimensional lattices with periodic boundary conditions, and apply Perron-Frobenius and convolution operator techniques. Our analytical techniques will yield explicit perturbative expressions for the invariant 1-point density for small couplings $a$. We will prove that the density exhibits log-periodic oscillations of period $\log N^2$ near the edges of the interval. Our explicit result for the invariant density will allow us to calculate expectations of arbitrary observables. We will prove that expectations of typical observables scale with $\sqrt{a}$ (rather than with $a$ as for hyperbolic coupled maps). We also show that there are log-periodic modulations of expectation values when the parameter $a$ is changed. As a by-product, our approach rigorously proves (and generalises) a conjecture of Ding et al. concerning the scaling behaviour of the largest Liapunov exponent for $a \to 0$ [10].

This paper is organised as follows: In Sec. I we introduce the relevant class of coupled map lattices and briefly explain their physical relevance. In Sec. II we give some numerical results for the scaling behaviour of these non-hyperbolic systems. In Sec. III we present our analytical results for the invariant density which we use to rigorously prove the scaling behaviour.

## THE CLASS OF SYSTEMS

Consider a scalar field $\varphi(\vec{x}, t)$ described by an equation of the form

$$\frac{\partial}{\partial t}\varphi = D \cdot \Delta\varphi + V'(\varphi). \tag{1}$$

Here $D$ is a diffusion constant and $V$ is some suitable potential. These types of equations occur in many different areas of physics. They describe reaction diffusion systems, Ginzburg-Landau type of models, nonlinear Schrödinger equations, stochastically quantised field theories, etc. In the one-dimensional case the Laplacian $\Delta$ is just given by $\partial^2/\partial x^2$. In many cases there is a fundamental length and time scale below which the above continuum theory is not valid anymore. For example, for stochastically quantised field theories this is the Planck length [4]. Writing $t = n\tau$, $x = i\delta$ where $n$ and $i$ are integers, and $\varphi(x,t) = p_{\max}\Phi_n^i$ where $\Phi_n^i$ is a dimensionless field and $p_{\max}$ is some constant with the same dimension as the field, the discretised Eq. (1) can be written in the form

$$\Phi_{n+1}^i = (1-a)T(\Phi_n^i) + \frac{a}{2}(\Phi_n^{i+1} + \Phi_n^{i-1}) \tag{2}$$

where the local map $T$ is given by

$$T(\Phi) = \Phi + \frac{\tau}{(1-a)p_{\max}} V'(p_{\max}\Phi), \tag{3}$$

and the dimensionless coupling $a$ is given by $a = 2D\tau/\delta^2$. For various applications of these types of coupled map lattices, see [3, 4]. The important point is that generically these types of models can lead to non-hyperbolic local maps $T$. For example, a $\varphi^4$-theory described by a double well potential $V(\varphi)$ with a sufficiently strong quartic term leads to cubic maps with two inflection points where the slope $T'(\Phi)$ vanishes. It is thus important to understand the generic behaviour of non-hyperbolic coupled map lattices.

We are particularly interested in cases where the local map exhibits strongest possible chaotic behaviour. The negative 3rd order Tchebyscheff map

$$\Phi_{n+1} = T_{-3}(\Phi_n) = -4\Phi_n^3 + 3\Phi_n \tag{4}$$

on the interval $\Phi \in [-1,1]$ is such an example. It is conjugated to a Bernoulli shift of 3 symbols and can be obtained in our context from the potential

$$V(\varphi) = \frac{1-a}{\tau}\left(\varphi^2 - \frac{\varphi^4}{p_{\max}^2}\right) + \text{const.} \tag{5}$$

In a similar way, one can construct potentials that lead to positive and negative Tchebyscheff maps of arbitrary order $N$ [4].

## OBSERVED SCALING BEHAVIOUR

Let us now study CMLs of type (2) where $T = T_N$ is a Tschebyscheff map of order $N$. One observes nontrivial scaling behaviour for small values of the coupling $a$ that is significantly different from that of hyperbolic systems.

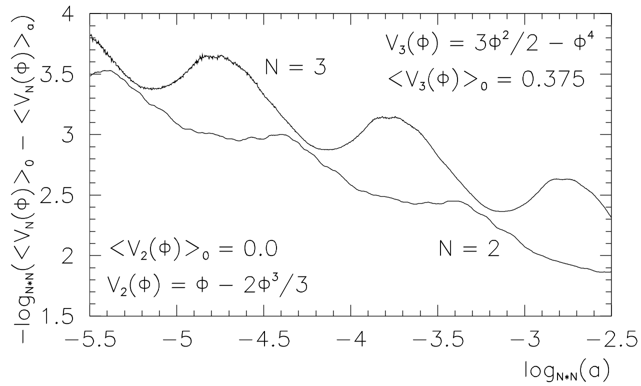


FIG. 1: The function $\sqrt{a}F^{(N)}(\log a)$ of Eq. (7) with $h(\Phi) = V_N(\Phi)$ as function of $a$ for $N = 2,3$ in a double-logarithmic plot with basis $N^2$.

Consider an arbitrary test function $h(\Phi)$ of the local iterates. Assuming ergodicity, expectations $\langle h(\Phi)\rangle_a$ for a given parameter $a$ are numerically calculable as time averages

$$\langle h(\Phi)\rangle_a = \lim_{M\to\infty, J\to\infty} \frac{1}{MJ}\sum_{n=1}^{M}\sum_{i=1}^{J} h(\Phi_n^i). \tag{6}$$

For $a \to 0$ one numerically observes the scaling behaviour

$$\langle h(\Phi)\rangle_a - \langle h(\Phi)\rangle_0 = \sqrt{a}\cdot F^{(N)}(\log a) \tag{7}$$

where $F^{(N)}$ is a periodic function of $\log a$ with period $\log N^2$. Examples are shown in FIG. 1. The choice $h(\Phi) = V_{\pm N}(\Phi) := \mp\int T_N(\Phi)d\Phi$ is important in the physical applications to estimate the vacuum energy generated by the chaotic field theory under consideration [4]. Generally the function $F = F[h]$ is a functional of the chosen test function $h$.

The above log-periodic scaling is observed for arbitrary test functions $h$ and hence is a general property of the invariant 1-point density $\rho_a(\Phi)$ of the CML for given small couplings $a$. In fact one observes that there is not only scaling in the parameter space $a$ but also in the phase space $\Phi$. This is shown in FIG. 2.

Near the left edge of the interval $[-1,1]$ one may write $\Phi = ay - 1$ and observe the scaling behaviour

$$\rho_a(ay-1) = a^{-1/2}g(y) \tag{8}$$

where the function $g$ is independent of $a$ for small $a$. At the right edge, writing $\Phi = 1 - ax$, one observes

$$\rho_a(1-ax) = \rho_0(1-ax) + \frac{1}{2}a^{-1/2}x^{-1}f(x) \tag{9}$$

where $f$ is independent of $a$ for small $a$. Moreover, $f$ exhibits log-periodic oscillations

$$f(N^2x) = f(x) \tag{10}$$

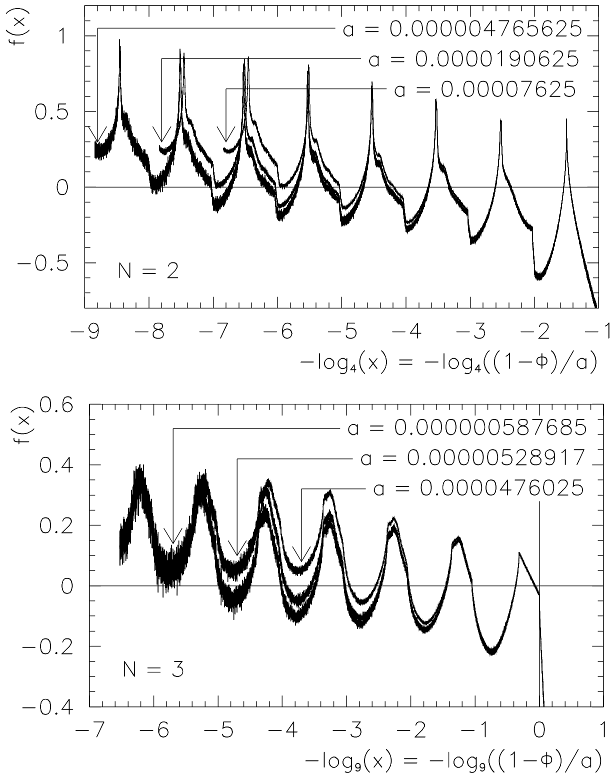


FIG. 2: Log-periodic oscillations of the rescaled invariant density $f(x)$ as observed for $N = 2$ (top) and $N = 3$ (bottom) and different values for $a$

over a large region of the phase space (see FIG. 2). The number of oscillations is approximately $-\log_{N^2}(a)$.

For $N$ odd the density is symmetric and the behaviour at the left and right edge is the same, whereas for $N$ even there is no left-right symmetry.

In the following section we will analytically prove the above numerical observations and provide explicit formulas for the functions $f$ and $g$ for a given local map $T_N$.

## PERTURBATIVE CALCULATION OF THE INVARIANT DENSITY

Our method is based on a perturbative treatment of the Perron-Frobenius operator of a perturbed local map and on convolution techniques. In a first approximation, the neighbouring lattice sites can be regarded as producing independent random noise with density $\rho_0(\Phi) = (1-\Phi^2)^{-1/2}/\pi$. While this picture works well in the middle of the interval $[-1,1]$, in the vicinity of the edges $\pm 1$ one has to take into account nontrivial nearest neighbour correlations, due to the non-hyperbolicity of the map. A first-order approximation of the density can then be further iterated to yield results of better precision. In each step we integrate over 2-point functions describing the joint probability of neighboured lattice sites to obtain the marginal distribution at a single lattice site. A detailed description of our calculations is out of the scope of a short letter, they will be published elsewhere [16]. Our final result is that for $N = 2$ one obtains in leading order of $\sqrt{a}$ at the left edge $\rho_a(ay-1) \approx \rho_a^{(0)}(ay-1)$ where

$$\rho_a^{(0)}(ay-1) = \frac{1}{\pi\sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{y-1+(\phi_+ + \phi_-)/2}}. \quad (11)$$

At the right edge one obtains by iterating our scheme $q$ times $\rho_a(1-ax) \approx \sum_{p=1}^{q} \rho_a^{(p)}(1-ax)$ where

$$\rho_a^{(p)}(1-ax) = \frac{1}{4^p \pi\sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/4^p + r_2^p(\phi_+) + r_2^p(\phi_-)}}. \quad (12)$$

Here the function $r_2^p(\phi)$ is defined as follows:

$$r_2^p(\phi) = \frac{1}{2} \sum_{q=0}^{p} \frac{T_{2^q}(\phi) - 1}{2^{2q}}. \quad (13)$$

The limits of the two integrations in Eqs. (11) and (12) are given by the condition that $|\phi_\pm| \leq 1$ and that the argument of the square root should always be positive.

A simple way to numerically evaluate our formulas is to replace the double integrals by ergodic averages of iterates of the uncoupled Tchebyscheff map. So for example we may evaluate the density at the left edge as

$$\rho_a^{(0)}(ay-1) = \frac{1}{\pi\sqrt{2a}} \frac{1}{M^2} \sum_{n_\pm=1}^{M} \frac{\theta(y-1+(\Phi_{n_+} + \Phi_{n_-})/2)}{\sqrt{y-1+(\Phi_{n_+} + \Phi_{n_-})/2}}, \quad (14)$$

and similar formulas apply to the right edge.

Eq. (11) can also be written as

$$\rho_a^{(0)}(ay-1) = \frac{1}{\pi\sqrt{2a}} \int_{1-y}^{1} \frac{\rho_{00}(z)dz}{\sqrt{y-1+z}} \quad (15)$$

$$\rho_{00}(z) = \frac{2}{\pi^2} K(\sqrt{1-z^2})\theta(1-z^2) \quad (16)$$

where $K(x)$ is the complete elliptic integral of the first kind. From the above formula it is obvious that the density is not differentiable at $y = 1$ and $y = 2$ for arbitrarily small $a$ (compare FIG. 3). By iteration of the Perron-Frobenius operator one can show that these two non-analytic points generate an entire cascade of such points at the right edge of the interval.

Note that sum over the terms in Eq. (12) converges rapidly with increasing $q$. In practice, a few terms are sufficient to obtain perfect agreement with the numerical histograms. FIG. 3 shows how well our analytical results (11) and (12) agree with the numerics. The agreement is so good that the analytic curves (given by dashed-dotted lines in FIG. 3) are not visible behind the data points.

Along similar lines, we obtain for $N = 3$

$$\rho_a^{(p)}(1-ax) = \frac{2}{9^p 3\pi\sqrt{2a}} \int \frac{\rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_-}{\sqrt{x/9^p + r_3^p(\phi_+) + r_3^p(\phi_-)}}, \quad (17)$$

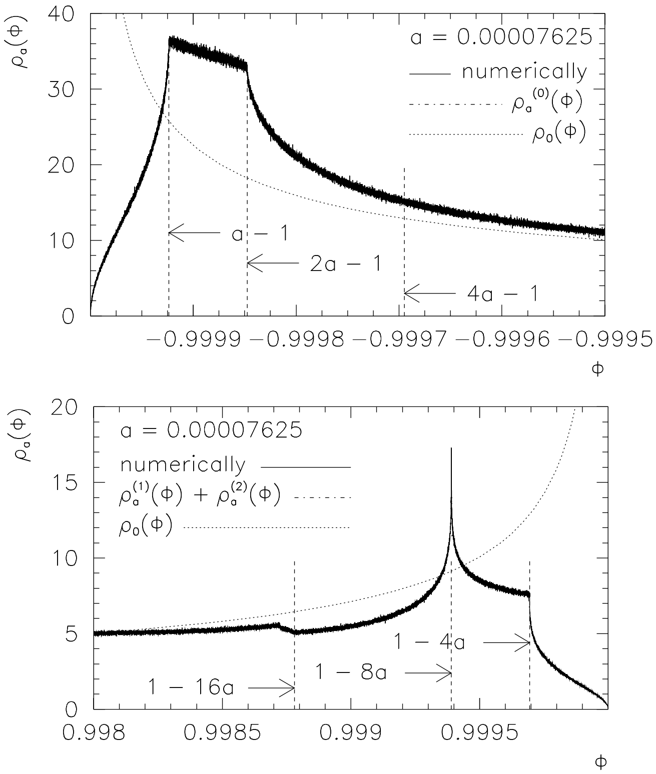


FIG. 3: Scaling behaviour of the invariant 1-point density near the left and right edge of the interval [-1,1] for $N = 2$ and $a = 0.00007625$. Shown is the density at the left (upper diagram) and right edge (lower diagram) in comparison with the exact results (dashed-dotted curve, covered by the numerical result) and the density $\rho_0(\phi)$ (dotted curve) together with several thresholds (dashed lines)

with

$$r_3^p(\phi) = \frac{1}{2}\sum_{q=0}^{p}\frac{T_{3^q}(\phi)-1}{9^q}. \tag{18}$$

The density is symmetric, i.e.

$$\rho_a^{(p)}(ax-1) = \rho_a^{(p)}(1-ax). \tag{19}$$

For general $N$ it is natural to conjecture that

$$\rho_a^{(p)}(1-ax) \sim \frac{1}{\sqrt{a}}\int\frac{\rho_0(\phi_+)d\phi_+\rho_0(\phi_-)d\phi_-}{\sqrt{x/N^{2p}+r_N^p(\phi_+)+r_N^p(\phi_-)}}, \tag{20}$$

with

$$r_N^p(\phi) = \frac{1}{2}\sum_{q=0}^{p}\frac{T_{N^q}(\phi)-1}{N^{2q}}. \tag{21}$$

With the above equations we have derived explicit formulas for the scaling functions $f$ and $g$ in Eqs. (8) and (9). It is easy to check from this integral representation that $f$ indeed satisfies property (10). Using Eqs. (8), (9) and (10) it is also possible to prove the general scaling relation (7) for arbitrary test functions $h$. Due to space restrictions we do not describe the details here but refer to a longer version [16]. The function $F^{(N)}$ is essentially given by a suitable integral of the observable $h$ folded with $f$.

## CONCLUSION AND OUTLOOK

We have derived in a perturbative way the 1-point density for coupled Tchebyscheff maps and rigorously proved the existence of several interesting scaling phenomena that are numerically confirmed. Although our formulas were worked out for the explicit example of Tchebyscheff maps, many of our techniques and results are expected to hold in a similar way for other non-hyperbolic systems exhibiting strongly chaotic behaviour. Scaling with $\sqrt{a}$ and log-periodic oscillations seem to be typical phenomena for these types of dynamical systems.

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