From time series to superstatistics

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Complex nonequilibrium systems are often effectively described by a ‘statistics of a statistics’, in short, a ‘superstatistics’. We describe how to proceed from a given experimental time series to a superstatistical description. We argue that many experimental data fall into three different universality classes: χ^2-superstatistics (Tsallis statistics), inverse χ^2-superstatistics, and log-normal superstatistics. We discuss how to extract the two relevant well separated superstatistical time scales τ and T, the probability density of the superstatistical parameter β, and the correlation function for β from the experimental data. We illustrate our approach by applying it to velocity time series measured in turbulent Taylor-Couette flow, which is well described by log-normal superstatistics and exhibits clear time scale separation.

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I. INTRODUCTION

Driven nonequilibrium systems of sufficient complexity are often effectively described by a superposition of different dynamics on different time scales. As a simple example consider a Brownian particle moving through a changing environment. A relatively fast dynamics is given by the velocity of the Brownian particle, and a slow dynamics is given, e.g., by temperature changes of the environment. The two effects are associated with two well separated time scales, which result in a superposition of two statistics, or in a short, a superstatistics (SS) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The stationary distributions of superstatistical systems typically exhibit non-Gaussian behavior with fat tails, which can decay with a power law, or as a stretched exponential, or in an even more complicated way. An essential ingredient of SS models is the existence of an intensive parameter β that fluctuates on a large spatio-temporal scale T. For the above example of a Brownian particle, β is the fluctuating inverse temperature of the environment, but in general β can also be an effective friction constant, a changing mass parameter, a changing amplitude of Gaussian white noise, the fluctuating energy dissipation in turbulent flows, or simply a local variance parameter extracted from a signal. The SS concept is quite general and has recently been applied to a variety of physical systems, including Lagrangian [10, 11, 12] and Eulerian [13, 14] turbulence, defect turbulence [15], atmospheric turbulence [16, 17], cosmic ray statistics [18], solar flares [19], random networks [20, 21] and mathematical finance [22, 23].

In this paper we address a problem that is of great interest in experimental applications. Given an experimentally measured time series or signal, how can we check if this time series is well described by a superstatistical model, i.e., does it contain two separate time scales, and how can we extract the relevant superstatistical parameters from the time series? Further, since there are infinitely many SS [1], which ones are the most relevant for typical experimental situations?

We argue that many experimental data (see, e.g., [11, 12, 13, 16, 24, 25, 26]) are well described by three major universality classes, namely χ^2, inverse χ^2, and log-normal SS. These SS represent a universal limit statistics for large classes of dynamical systems. We then show how to extract the superstatistical parameters from a given experimental signal. Our example is a time series of longitudinal velocity differences measured in turbulent Taylor-Couette flow [27]. We will extract the two relevant time scales τ and T from the data and show that there is clear time scale separation for our example. We will also investigate the probability density of β and the β-correlation function. While our turbulent time series appears to fall into the universality class of log-normal superstatistics, our concepts are general and can in principle be applied to any time series.

II. SUPERSTATISTICAL UNIVERSALITY CLASSES

Consider a driven nonequilibrium system which is inhomogeneous and consists of many spatial cells with different values of some intensive parameter β (e.g., the inverse temperature). The cell size can be determined by the correlation length of the continuously varying β-field. Each cell is assumed to reach local equilibrium very fast, i.e., the associated relaxation time τ is short. The parameter β in each cell is approximately constant during the time scale T, then it changes to a new value. In the long-term run ($t >> T$), the stationary distributions of this inhomogeneous system arise as a superposition of Boltzmann factors $e^{-\beta E}$ weighted with the probability density $f(\beta)$ to observe some value β in a randomly chosen cell:

$$p(E) = \int_0^\infty f(\beta)\frac{1}{Z(\beta)}\rho(E)e^{-\beta E}d\beta \tag{1}$$

Here E is an effective energy for each cell, $\rho(E)$ is the density of states, and $Z(\beta)$ is the normalization constant of $\rho(E)e^{-\beta E}$ for a given β. The simplest example is a Brownian particle of mass m moving through a changing environment in d dimensions. For its velocity $\vec{v}$ one has the local Langevin equation

$$\dot{\vec{v}} = -\gamma\vec{v} + \sigma\vec{L}(t) \tag{2}$$

($\vec{L}(t)$: d-dimensional Gaussian white noise) which becomes superstatistical because for a fluctuating environment the parameter $\beta := \frac{2}{m}\frac{\gamma}{\sigma^2}$ becomes a random variable as well: it varies from cell to cell on the large spatio-temporal scale T. Of course, for this example $E = \frac{1}{2}m\vec{v}^2$, and while on the time scale T the local stationary distribution in each cell is Gaussian,

$$p(\vec{v}|\beta) = \left(\frac{\beta}{2\pi}\right)^{d/2} e^{-\frac{1}{2}\beta m\vec{v}^2}, \tag{3}$$

the marginal distribution describing the long-time behavior of the particle for $t >> T$,

$$p(\vec{v}) = \int_0^\infty f(\beta)p(\vec{v}|\beta)d\beta \tag{4}$$

exhibits non-trivial behavior. The large-$|v|$ tails of the distribution (4) depend on the behavior of $f(\beta)$ for $\beta \to 0$ [3]. For example, if $f(\beta)$ is a χ^2-distribution of degree n, then eq. (4) generates Tsallis statistics [28, 29], with entropic index q given by $q = 1 + \frac{2}{n+d}$ [5]. Of course, a necessary condition for a superstatistical description to make sense is the condition $\tau = \gamma^{-1} << T$, because otherwise the system is not able to reach local equilibrium before the next change of β takes place. In superstatistical turbulence models [5, 10, 11, 12, 14], one formally replaces the variable $\vec{v}$ in eq. (2) by the velocity difference $\vec{u}$ (or acceleration $\vec{a}$ on smallest scales), and β is related to energy dissipation ϵ.

The distribution $f(\beta)$ is determined by the spatio-temporal dynamics of the entire system under consideration. By construction, β is positive, so $f(\beta)$ cannot be Gaussian. Let us here consider three examples of what to expect in typical experimental situations.

(a) There may be many (nearly) independent microscopic random variables ξ_j, $j = 1, \ldots, J$, contributing to β in an additive way. For large J their rescaled sum $\frac{1}{\sqrt{J}}\sum_{j=1}^J \xi_j$ will approach a Gaussian random variable X_1 due to the Central Limit Theorem. In total, there can be many different random variables consisting of microscopic random variables, i.e., we have n Gaussian random variables $X_1, \ldots, X_n$ due to various relevant degrees of freedom in the system. As mentioned before, β needs to be positive; a positive β is obtained by squaring these Gaussian random variables. The resulting $\beta = \sum_{i=1}^n X_i^2$ is χ^2-distributed with degree n,

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{n}{2\beta_0}\right)^{n/2}\beta^{n/2-1}e^{-\frac{n\beta}{2\beta_0}}, \tag{5}$$

where β_0 is the average of β. As shown in [5, 30], the SS resulting from (4) and (5) is Tsallis statistics [28]. It exhibits power-law tails for large $|\vec{v}|$. Our above argument shows that Tsallis statistics arises as a universal limit dynamics, i.e., the details of the microscopic random variables ξ_j (e.g., their probability densities) are irrelevant.

(b) The same considerations as above can be applied if the 'temperature' β^{-1} rather than β itself is the sum of several squared Gaussian random variables arising out of many microscopic degrees of freedom ξ_j. The resulting $f(\beta)$ is the inverse χ^2-distribution given by

$$f(\beta) = \frac{\beta_0}{\Gamma(\frac{n}{2})}\left(\frac{n\beta_0}{2}\right)^{n/2}\beta^{-n/2-2}e^{-\frac{n\beta_0}{2\beta}}. \tag{6}$$

It generates superstatistical distributions (4) that have exponential decays in $|\vec{v}|$ [3, 24]. Again this superstatistics is universal: details of the ξ_j are irrelevant.

(c) Instead of β being a sum of many contributions, for other systems (in particular, turbulent ones) the random variable β may be generated by multiplicative random processes. We may have a local cascade random variable $X_1 = \prod_{j=1}^J \xi_j$, where J is the number of cascade steps and the ξ_j are positive microscopic random variables. By the Central Limit Theorem, for large J the random variable $\frac{1}{\sqrt{J}}\log X_1 = \frac{1}{\sqrt{J}}\sum_{j=1}^J \log\xi_j$ becomes Gaussian for large J. Hence X_1 is log-normally distributed. In general there may be n such product contributions to β, i.e., $\beta = \prod_{i=1}^n X_i$. Then $\log\beta = \sum_{i=1}^n \log X_i$ is a sum of Gaussian random variables; hence it is Gaussian as well. Thus β is log-normally distributed, i.e.,

$$f(\beta) = \frac{1}{\sqrt{2\pi}s\beta}\exp\left\{\frac{-(\ln\frac{\beta}{\mu})^2}{2s^2}\right\}, \tag{7}$$

where μ and s^2 are suitable mean and variance parameters [1]. For related turbulence models, see, e.g., [10, 11, 25, 31]. Again this log-normal result is independent of the details of the microscopic cascade random variables ξ_j; hence log-normal SS is universal as well.

Although more complicated cases can be constructed, we believe that most experimentally relevant cases fall into one of these three universality classes, or simple combinations of them. χ^2 superstatistics has been found for wind velocity fluctuations [16, 17], and log-normal superstatistics has been found for Lagrangian [10, 11, 12] and Eulerian [13, 14] turbulence. Candidate systems for inverse χ^2 superstatistics are systems exhibiting velocity distributions with exponential tails [3, 24].

III. APPLICATION TO EXPERIMENTAL TIME SERIES

Suppose some experimental time series $u(t)$ is given [32]. Our goal is to test the hypothesis that it is due to a superstatistics and if so, to extract the two basic time scales τ and T as well as $f(\beta)$. First let us determine the large time scale T. For this we divide the time series into N equal time intervals of size Δt. The total length of the signal is $t_{max} = N\Delta t$. We then define a function $\kappa(\Delta t)$ by

$$\kappa(\Delta t) = \int_0^{t_{max}-\Delta t} dt_0 \frac{\langle (u-\bar{u})^4\rangle_{t_0,\Delta t}}{\langle (u-\bar{u})^2\rangle^2_{t_0,\Delta t}} \qquad (8)$$

Here $\langle\cdots\rangle_{t_0,\Delta t} = \int_{t_0}^{t_0+\Delta t}\cdots dt$ denotes an integration over an interval of length Δt starting at t_0, and $\bar{u}$ is the average of $u(t)$ (we may either choose $\bar{u}$ to be a local average in each cell or a global average over the entire time series — our results do not depend on this choice in a significant way). Equation (8) simply means that the local flatness is evaluated in each interval of length Δt, and the result is then averaged over all t_0. We now define the superstatistical time scale T by the condition

$$\kappa(T) = 3. \qquad (9)$$

Clearly this condition simply implies that we are looking for the simplest SS, a superposition of local Gaussians, which have local flatness 3 (see [13] for similar ideas). If Δt is so small that only one constant value of u is observed in each interval, then of course $\kappa(\Delta t) = 1$. On the other hand, if Δt is so large that it includes the entire time series, then we obtain the flatness of the distribution of the entire signal, which will be larger than 3, since superstatistical distributions are fat-tailed. Hence there exists a time scale T satisfying (9).

The function $\kappa(\Delta t)$ is shown in Fig. 1 for longitudinal velocity differences, $u(t) = v(t+\delta) - v(t)$, measured in Taylor-Couette flow at Reynolds number $Re = 540000$ [27]. The total number of measurement points was 2×10^7, and in the present analysis $\Delta t \leq 1000$, so $N \geq 2 \times 10^4$, which means there is sufficient statistics to obtain precise values for the time scales T and τ. For details of the experiment, see [27]. For each time difference δ (measured in units of the sampling frequency, which was 2500 times the inner cylinder rotation frequency), the relevant superstatistical time scale T leading to locally Gaussian behavior is extracted in Fig. 1. The time scales T have to be compared with the relaxation times $\tau = \gamma^{-1}$ of the dynamics, which can be estimated from the short-time exponential decay of the correlation function $C_u(t-t') = \langle u(t)u(t')\rangle$ of the velocity difference u

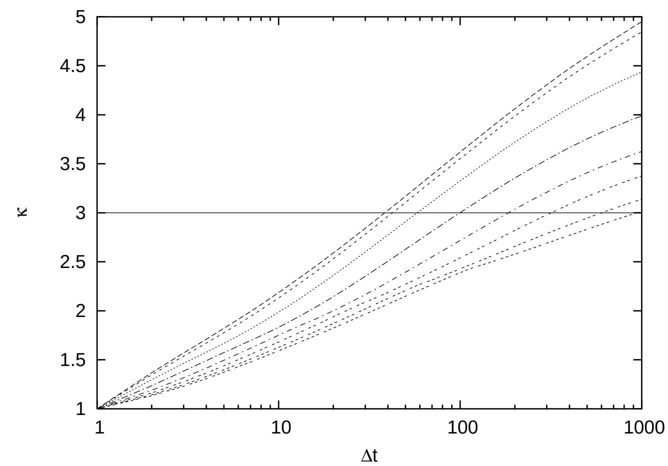

FIG. 1: Determination of the superstatistics long time scale T from the flatness function $\kappa(\Delta t)$ given in eq. (8), for $\delta = 2^j, j = 0, 1, 2, \ldots, 7$ (from top to bottom). The intersections with the line $\kappa = 3$ yield $T = 39, 42, 58, 100, 184, 320, 600, 948$, respectively. Time is nondimensional (see text).

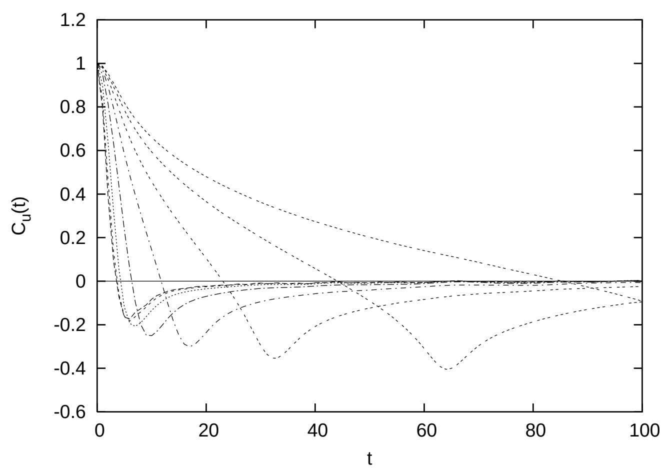

FIG. 2: Determination of the superstatistics short time scale τ from the decay of the correlation function $C_u(t)$ of the velocity difference. Defining τ by $C_u(\tau) = e^{-1}C_u(0)$, we obtain for $\delta = 2^j, j = 0, 1, 2, \cdots, 7$, $\tau = 2.1, 2.3, 2.8, 4.3, 7.2, 12.1, 19.9, 29.5$, respectively.

(Fig. 2). We find that the ratio $T/\tau \approx 17...34$ is large compared to unity, and the ratio has only a weak (logarithmic) dependence on δ (Fig. 3). Thus there are indeed two well separated time scales in the time series for turbulent Couette-Taylor flow.

Laboratory data were obtained for a wide range of Reynolds numbers, so we can also examine how the time scale ratio changes with the Reynolds number Re. Fig. 4 shows that T/τ increases with increasing Re, meaning that the superstatistics approach becomes more and more exact for $Re \to \infty$.

Next, we are interested in the analysis of the slowly varying stochastic process $\beta(t)$. Since the variance of

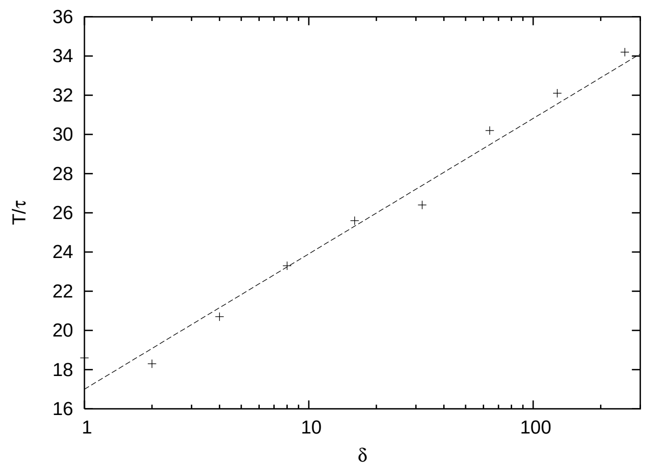

FIG. 3: The time scale ratio T/τ, given as a function of δ for turbulence data at $Re = 540000$, is large compared to unity. Thus the long and short time scales are well-separated, as required for superstatistics. The dashed line is a fit given by $T/\tau = 17 + 3 \ln \delta$.

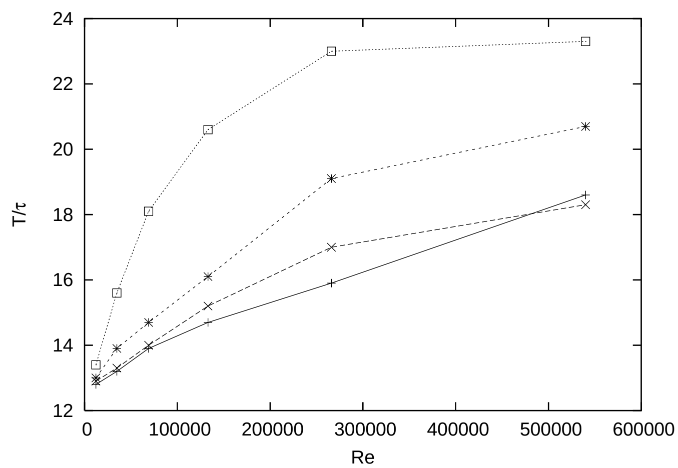

FIG. 4: Time scale ratio T/τ as a function of Reynolds number Re for $\delta = 8, 4, 2, 1$ (from top to bottom).

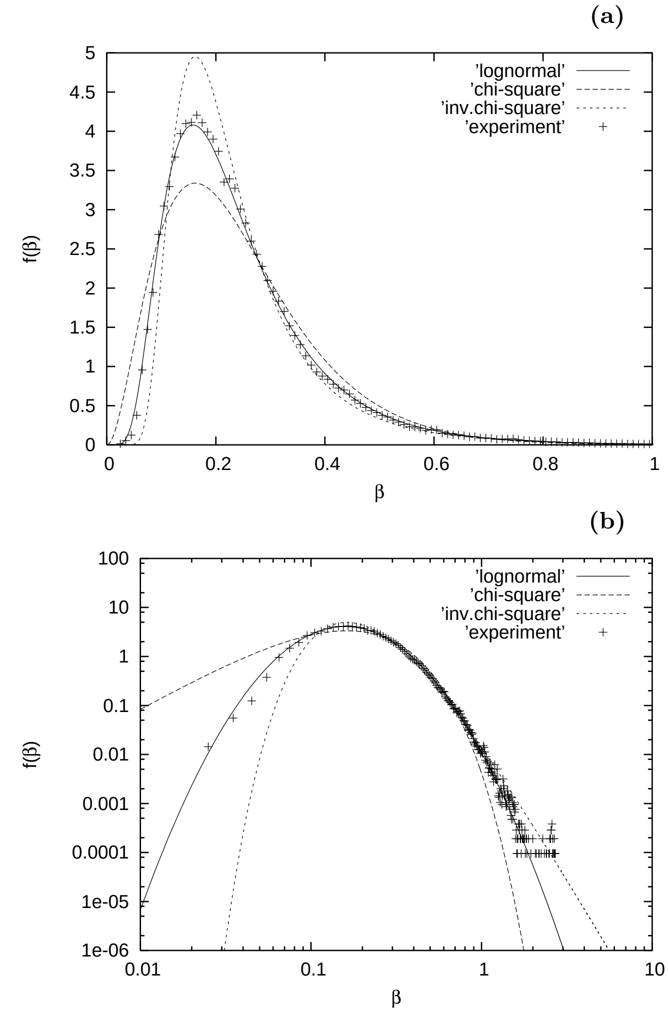

FIG. 5: Probability density $f(\beta)$ extracted from the turbulent time series ($\delta = 16$), and compared with log-normal, χ^2, and inverse χ^2 distributions, on (a) linear-linear and (b) log-log plots. All distributions have the same mean and variance as the experimental data.

the local Gaussians $\sqrt{\frac{\beta}{2\pi}} e^{-\frac{1}{2}\beta u^2}$ is given by β^{-1}, we can determine the process $\beta(t)$ from the time series as

$$\beta(t_0) = \frac{1}{\langle u^2 \rangle_{t_0,T} - \langle u \rangle_{t_0,T}^2}. \tag{10}$$

We obtain the probability density $f(\beta)$ as a histogram of $\beta(t_0)$ for all values of t_0, as shown in Fig. 5. We compare the experimental data with log-normal, χ^2 and inverse χ^2 distributions with the same mean $\langle \beta \rangle$ and variance $\langle \beta^2 \rangle - \langle \beta \rangle^2$ as the experimental data. The fit of the data to a log-normal distribution is significantly better than to a χ^2 or inverse χ^2. Indeed, the cascade picture of energy dissipation in turbulent flows suggests that our time series should belong to the log-normal universality class of superstatistics (see section II (c)). A power-law relation between the energy dissipation rate ϵ and β was found for the Couette-Taylor turbulence data in an analysis in [13]. Note that if such a power-law relation is valid, then a log-normally distributed ϵ implies a log-normally distributed β, and vice versa.

For superstatistics to make physical sense, the variable β must change slowly compared to u. This is indeed the case for our turbulence data, as Fig. 6 illustrates for a sample time series. A slow β-dynamics also implies a slow correlation decay of the β-correlation function $C_\beta(t - t') = \langle \beta(t)\beta(t') \rangle$. For our data we observe a power-law decay with a small exponent, $C_\beta(t) \sim t^{-0.9}$ (Fig. 7). This means that $\beta(t)$ indeed has a long memory and changes slowly, a necessary consistency condition for the superstatistics approach. For comparison, the correlation function of the longitudinal velocity difference $u(t)$ first decays exponentially fast and only finally, for very large times, approaches a power-law decay of the

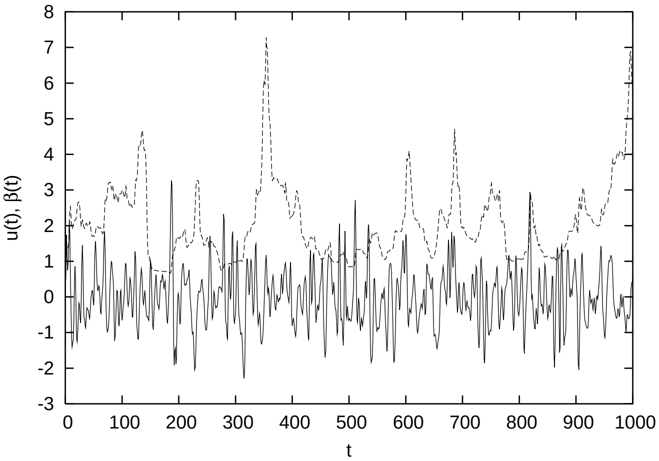

FIG. 6: Time series of $\beta(t)$ (top) and $u(t)$ (bottom). For this example $\delta = 2$ and $T = 42$.

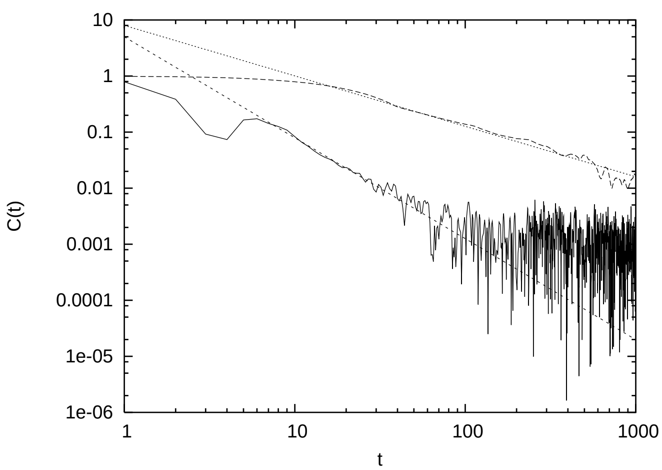

FIG. 7: Correlation functions $C_\beta(t)$ (top) and $|C_u(t)|$ (bottom) for $\delta = 1$. The straight lines represent power laws with exponents -0.9 and -1.8.

form $C_u(t) \sim t^{-1.8}$, as shown in Fig. 7. Note that for $t > 6$ the β-correlation is larger than the u-correlation by a factor 10...100.

Finally, we may check the validity of the general SS formula

$$p(u) = \int_0^\infty f(\beta)p(u|\beta)d\beta, \tag{11}$$

where $p(u|\beta)$ is the conditional distribution of the signal $u(t)$ in cells of size T, and $p(u)$ is the marginal stationary distribution of the entire signal. For log-normal superstatistics this means

$$p(u) = \frac{1}{2\pi s}\int_0^\infty d\beta\beta^{-1/2}\exp\left\{\frac{-(\ln(\beta/\mu))^2}{2s^2}\right\}e^{-\frac{1}{2}\beta u^2}. \tag{12}$$

As shown in Fig. 8, there is excellent agreement between the experimental histogram and the superstatistical model prediction, both in the tails and in the region around the peak of the distribution.

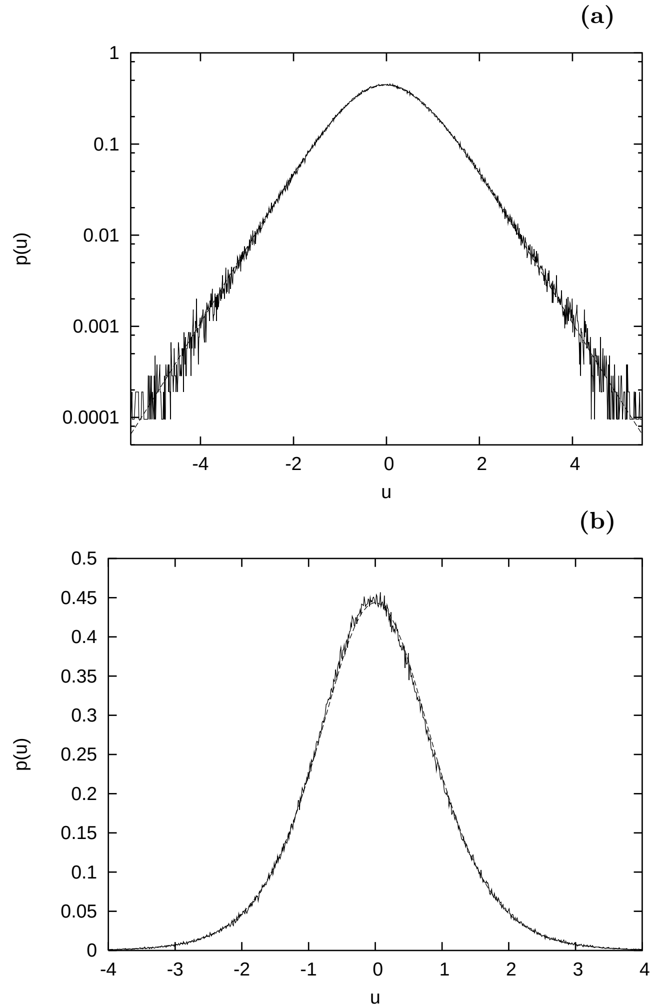

FIG. 8: Comparison of the measured (fluctuating lines) and predicted (dashed lines) probability distribution $p(u)$ for velocity differences on (a) semi-log plot, which emphasizes the tails, and (b) linear-linear plot, which emphasizes the peak. The predicted $p(u)$ was obtained from eq. (12) with $f(\beta)$ being a log-normal distribution with the same parameters as in Fig. 5 ($\delta = 16$).

For any superstatistics one can formally define a parameter q by [1]

$$q := \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2}. \tag{13}$$

q measures in a quantitative way the deviation from Gaussian statistics. No fluctuations in β at all correspond to $f(\beta) = \delta(\beta - \beta_0)$ and $q = 1$, i.e., ordinary equilibrium statistical mechanics. For χ^2-superstatistics, the above q is given by $q = 1 + 2/n$ [1, 30] and is nearly the same as the entropic index q_T introduced by Tsallis [5, 28, 29] ($q_T = 1 + 2/(n + 1)$). For log-normal (LN) superstatistics, one can relate q to the flatness $F = \frac{\langle u^4\rangle}{\langle u^2\rangle^2}$

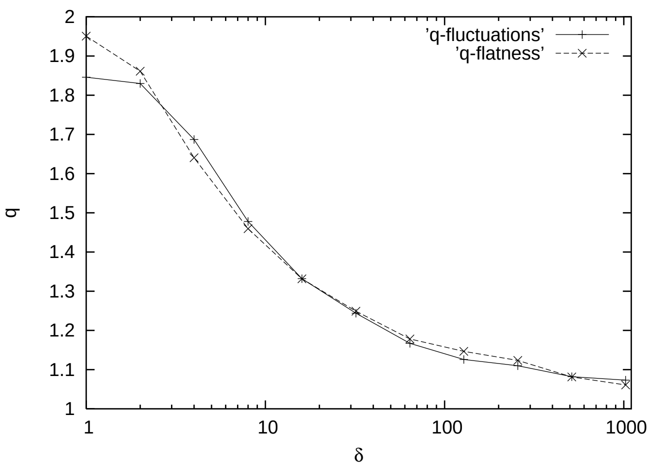

FIG. 9: The parameter q (eq. (13)) as a function of δ, as evaluated from eq. (13) (q-fluctuations) and eq. (18) (q-flatness).

of the distribution $p(u)$. Since for log-normal superstatistics [14]

$$\langle\beta\rangle_{LN} = \mu e^{\frac{1}{2}s^2} \tag{14}$$

$$\langle\beta^2\rangle_{LN} = \mu^2 e^{2s^2} \tag{15}$$

$$\langle u^2\rangle = \mu^{-1} e^{\frac{1}{2}s^2} \tag{16}$$

$$\langle u^4\rangle = 3\mu^{-2} e^{2s^2}, \tag{17}$$

where μ and s^2 are the mean and variance parameters of the distribution (7), we arrive at the following simple relation,

$$q = e^{s^2} = \frac{1}{3}F, \tag{18}$$

using eq. (13). We thus have two different equations to evaluate q for our time series. The first one, based on eq. (13), is always valid (i.e., for any $f(\beta)$), whereas the second one, based on eq. (18), should coincide with the first one provided the system is described by log-normal superstatistics. Figure 9 shows the q-values that we extract from the experimental data for $Re = 540000$ using both methods. As expected, q decreases monotonically with scale δ. Both curves agree well for $\delta \geq 2$. This indicates that log-normal superstatistics is a good model for our data, and that our extraction of the time scale T of the process $\beta(t)$ is consistent. Significant deviations between the two q-values occur only on the smallest scale $\delta = 1$, where the experiment reaches its resolution limits.

IV. CONCLUSIONS

In this paper we have advocated the view that the non-Gaussian behavior of many complex driven nonequilibrium systems can often be understood as a superposition of two different statistics on different time scales, in short, a superstatistics. We have argued that typical experimental situations are described by three relevant universality classes, namely χ^2, inverse χ^2, and log-normal superstatistics. Our example, turbulent Taylor Couette flow, falls into the universality class of log-normal superstatistics. This means the time series is essentially described by local Boltzmann factors $e^{-\frac{1}{2}\beta u^2}$ whose variance parameter β varies slowly according to a log-normal distribution function. We have developed a general method to extract from data the process $\beta(t)$, its probability density $f(\beta)$, and its correlation function. Our approach is applicable to any experimental time series. We have extracted the two relevant time scales τ (the relaxation time to local equilibrium) and T (the large time scale on which the intensive parameter β fluctuates). Our main result is that for turbulent Taylor-Couette flow there is clear time scale separation, which is a necessary condition for a superstatistical description to make physical sense. The ratio T/τ grows logarithmically with the scale separation δ on which longitudinal velocity differences are investigated. Moreover, T/τ also increases with increasing Reynolds number, making the superstatistical approach more and more exact for increasing Reynolds number. The experimentally measured distributions of β and u agree very well with the superstatistical model predictions.

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