Nonextensive scalar field theories and dark energy models

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Abstract

Current astronomical measurements indicate that approximately 73% of the universe is made up of dark energy. Stochastically quantized self-interacting scalar fields can serve as suitable models to generate dark energy. We study a particular model where the scalar field theory underlying dark energy exhibits strongest possible chaotic behaviour. It is shown that the fluctuating chaotic field momenta obey a generalized (nonextensive) statistical mechanics, with an entropic index q given by $q = 3$, respectively $q = -1$ if the escort formalism is used.

1 Introduction

The formalism of nonextensive statistical mechanics has been developed as a useful tool to statistically describe complex systems [1,2,3]. The concept is based on generalizing the Shannon entropy S_1 to more general information measures, such as the Tsallis entropies S_q[1], or other generalized information measures as well[4,5,6], and doing statistical mechanics based on these more general information measures. Often, the formalism is relevant to nonequilibrium systems with spatio-temporal parameter fluctuations, which exhibit a superposition of different statistics on different time scales (in short, a 'superstatistics') [6,7,8,9]. In recent years, several physical applications have been pointed out for the formalism with $q \neq 1$, among them being turbulent flows[8,9], pattern forming systems[10], heavy ion collisions[11], cosmic rays[12,13,4], cosmology[14], econophysics[15], and biological systems [16].

In this paper we want to point out yet another potential application of the nonextensive formalism, namely to stochastically quantized dark energy models. There is growing observational evidence that the universe is presently dominated by some unknown form of vacuum energy with equation of state close to $w = -1$, so-called dark energy. Current astronomical data indicate that there is approximately 73 % dark energy, 23 % dark matter, and 4% ordinary matter in our universe[17,18]. The nature and origin of the dominating dark energy component is not understood, and many different models are currently being discussed [19,20,21].

The simplest way to generate dark energy is via a self-interacting scalar field. The dark energy density is then essentially given by the potential of the field, in either a static way as for a true cosmological constant or in a dynamical way as in quintessence models[19,20]. In this paper we consider a model for dark energy which associates dark energy with self-interacting scalar fields corresponding to a φ^4-theory, which is second quantized [21,22,23]. The difference to other scalar field approaches is that our fields are very strongly (rather than weakly) self-interacting, and that they are second-quantized. We use as the relevant method to quantize the scalar fields the stochastic quantization method introduced by Parisi and Wu [24,25]. In the fictitious time variable of this approach, the fields behave in a deterministic chaotic way. Our physical interpretation is to associate the chaotic behaviour of the scalar fields with vacuum fluctuations [22,23]. When quantum mechanical averages are formed, the fields generate quite smoothly distributed dark energy [21].

We will show that the probability density of the fluctuating chaotic momenta in our model is given by the generalized canonical distributions in the formalism of nonextensive statistical mechanics. The relevant entropic index is $q = 3$ (see also [14]), or $q = -1$ if the escort formalism is used. This means the dark energy component behaves quite different from a system of particles described by ordinary statistical mechanics. The expectation of the scalar field potential plays the role of a generalized thermodynamic potential in this setting. We will show that a theory formulated in terms of escort distributions, i.e. $q = -1$, has many advantages as compared to $q = 3$. We will explicitly calculate quantities like the entropy S_q, the generalized internal energy U_q and the generalized free energy F_q for our model.

If our model is indeed the correct description of dark energy in the universe, then this means that Boltzmann-Gibbs type of statistical mechanics ($q = 1$) is only relevant for a minority of the contents of the universe (ordinary and dark matter, together 27 %), whereas the dominating dark energy contents of the universe (73 %) is described by a different type of statistical mechanics, with $q = -1$.

This paper is organized as follows. In section 2 we first recall how to second-quantized a scalar field dynamics using stochastic quantization. We then show that a chaotic dynamics can be obtained if the potential is very strong. Our main example in section 3 is a φ^4-theory leading to an effective dynamics given by weakly coupled 3rd order Tchebyscheff maps. This dark energy model exhibits strongest possible chaotic behavior. In section 4 we show how the parameters have to be chosen in order to reproduce the currently measured dark energy. Finally, in section 5 we embed the chaotic field model into the formalism of nonextensive statistical mechanics and calculate various thermodynamic quantities.

2 Stochastic quantization of strongly self-interacting scalar fields

Let us consider a homogeneous self-interacting scalar field φ in Robertson-Walker metric. To second-quantize it, one can use the Parisi-Wu approach of stochastic quantization [24,25]. This means one considers the following stochastic differential equation:

$$\frac{\partial}{\partial s}\varphi = \ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) + L(t,s) \tag{1}$$

Here H is the Hubble parameter, V is the potential under consideration, t is physical time, and s is an artificial coordinate (called fictitious time) which is just introduced for quantization purposes. $L(t,s)$ is Gaussian white noise, δ-correlated both in t and s. The expectations with respect to the stochastic process generated by this stochastic differential equation for $s \to \infty$ are identical to the quantum mechanical expectations of the field theory under consideration.

We may discretize eq. (1) using $s = n\tau$ and $t = i\delta$, where n and i are integers and τ is a fictitious time lattice constant, δ is a physical time lattice constant. The continuum limit requires $\tau \to 0$, $\delta \to 0$, but we will later see that it makes physical sense to keep small but finite lattice constants of the order of the Planck length. We obtain

$$\frac{\varphi^i_{n+1} - \varphi^i_n}{\tau} = \frac{1}{\delta^2}(\varphi^{i+1}_n - 2\varphi^i_n + \varphi^{i-1}_n) + 3\frac{H}{\delta}(\varphi^i_n - \varphi^{i-1}_n) + V'(\varphi^i_n) + noise \tag{2}$$

equivalent to

$$\begin{aligned}\varphi^i_{n+1} = &(1-\alpha)\left\{\varphi^i_n + \frac{\tau}{1-\alpha}V'(\varphi^i_n)\right\} + 3\frac{H\tau}{\delta}(\varphi^i_n - \varphi^{i-1}_n) + \frac{\alpha}{2}(\varphi^{i+1}_n + \varphi^{i-1}_n) \\ &+ \tau \cdot noise,\end{aligned} \tag{3}$$

where a dimensionless coupling constant α is introduced as $\alpha := \frac{2\tau}{\delta^2}$. We also introduce a dimensionless field variable Φ^i_n by writing $\varphi^i_n = \Phi^i_n p_{max}$, where p_{max} is some (so far) arbitrary energy scale. The discretized stochastically quantized system is equivalent to a coupled map lattice of the form

$$\Phi_{n+1}^{i} = (1-\alpha)T(\Phi_n^i) + \frac{3}{2}H\delta\alpha(\Phi_n^i - \Phi_n^{i-1}) + \frac{\alpha}{2}(\Phi_n^{i+1} + \Phi_n^{i-1}) + \tau \cdot noise, \quad (4)$$

where the local map T is given by

$$T(\Phi) = \Phi + \frac{\tau}{p_{max}(1-\alpha)} V'(p_{max}\Phi). \quad (5)$$

Here the prime means $' = \frac{\partial}{\partial\varphi} = \frac{1}{p_{max}}\frac{\partial}{\partial\Phi}$. For large t (old universes) the term proportional to H can be neglected and one obtains a symmetric diffusively coupled map lattice of the form

$$\Phi_{n+1}^{i} = (1-\alpha)T(\Phi_n^i) + \frac{\alpha}{2}(\Phi_n^{i+1} + \Phi_n^{i-1}) + \tau\ noise. \quad (6)$$

Apparently the iteration of a coupled map lattice of the form (6) with a given map T has physical meaning: It means that one is considering the second-quantized dynamics of a self-interacting real scalar field φ with a force V' given by

$$V'(\varphi) = \frac{1-\alpha}{\tau}\left\{-\varphi + p_{max}T\left(\frac{\varphi}{p_{max}}\right)\right\}, \quad (7)$$

equaivalent to

$$V(\varphi) = \frac{1-\alpha}{\tau}\left\{-\frac{1}{2}\varphi^2 + p_{max}\int d\varphi\, T\left(\frac{\varphi}{p_{max}}\right)\right\} + const. \quad (8)$$

3 Chaotic φ^4-theory

It is interesting to see that our model can exhibit chaotic behaviour for very strong forces $V' \sim \tau^{-1}$. As a distinguished example, let us consider the map

$$\Phi_{n+1} = T_{-3}(\Phi_n) = -4\Phi_n^3 + 3\Phi_n \quad (9)$$

on the interval $\Phi \in [-1,1]$. T_{-3} is the negative third-order Tchebyscheff map, a strongly chaotic map conjugated to a Bernoulli shift of 3 symbols. This map arises in the scalar field dynamics (6) if the scalar field potential is given by

$$V_{-3}(\varphi) = \frac{1-\alpha}{\tau}\left\{\varphi^2 - \frac{1}{p_{max}^2}\varphi^4\right\} + const, \quad (10)$$

or, in terms of the dimensionless field Φ,

$$V_{-3}(\varphi) = \frac{1-\alpha}{\tau} p_{max}^2 (\Phi^2 - \Phi^4) + const. \tag{11}$$

Apparently, this potential describes a field φ that rapidly fluctuates in fictitious time on some finite interval . The small noise term in eq. (6) can be neglected as compared to the deterministic chaotic fluctuations of the field.

Of physical relevance are the expectations of suitable observables with respect to the ergodic chaotic dynamics. For example, one can calculate the expectation of the potential V_{-3}, which describes the vacuum energy generated by this chaotic field theory:

$$\langle V_{-3}(\varphi)\rangle = \frac{1-\alpha}{\tau} p_{max}^2 (\langle \Phi^2\rangle - \langle \Phi^4\rangle) + const \tag{12}$$

For uncoupled Tchebyscheff maps ($\alpha = 0$), expectations of any observable A can be evaluated as the ergodic average

$$\langle A\rangle = \int_{-1}^{+1} A(\Phi) p(\Phi) d\Phi, \tag{13}$$

with the natural invariant density $p(\Phi)$ being given by

$$p(\Phi) = \frac{1}{\pi\sqrt{1-\Phi^2}}. \tag{14}$$

From eq. (14) one obtains $\langle \Phi^2\rangle = \frac{1}{2}$ and $\langle \Phi^4\rangle = \frac{3}{8}$, thus

$$\langle V_{-3}(\varphi)\rangle = \frac{1}{8}\frac{p_{max}^2}{\tau} + const. \tag{15}$$

Alternatively, we may consider the positive Tchebyscheff map $T_3(\Phi) = 4\Phi^3 - 3\Phi$. This basically exhibits the same dynamics as T_{-3}, up to a sign. Repeating the same calculation we obtain

$$V_3(\varphi) = \frac{1-\alpha}{\tau} p_{max}^2 (-2\Phi^2 + \Phi^4). \tag{16}$$

For the expectation of the vacuum energy one gets

$$\langle V_3(\varphi)\rangle = \frac{1-\alpha}{\tau} p_{max}^2 (-2\langle \Phi^2\rangle + \langle \Phi^4\rangle) + const, \tag{17}$$

which for $\alpha = 0$ reduces to

$$\langle V_3(\varphi)\rangle = -\frac{5}{8}\frac{p_{max}^2}{\tau} + const. \tag{18}$$

Symmetry considerations between T_{-3} and T_3 suggest to take the additive constant $const$ as $const = +\frac{1-\alpha}{\tau}p_{max}^2\frac{1}{2}\langle\Phi^2\rangle$. In this case one obtains the fully symmetric equation

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{1-\alpha}{\tau}p_{max}^2\left\{-\frac{3}{2}\langle\Phi^2\rangle + \langle\Phi^4\rangle\right\}, \tag{19}$$

which for $\alpha \to 0$ reduces to

$$\langle V_{\pm 3}(\varphi)\rangle = \pm\frac{p_{max}^2}{\tau}\left(-\frac{3}{8}\right). \tag{20}$$

The above vacuum energy changes in a nontrivial way with the coupling α. Extensive numerical simulations were performed in [22,23], and it was found that as a function of α it has several local minima that numerically coincide with running standard model coupling constants, evaluated at the mass scales of known fermions and bosons. This emphasizes the physical relevance of the model. The role of chaotic fields in the universe could be to fix and stabilize fundamental constants as local minima in the dark energy landscape.

4 Constructing a dark energy model

Let us now fix the free parameters in our chaotic field model in such a way that the currently experimentally measured properties of dark energy come out correctly.

Firstly, the measured dark energy density is positive, not negative. This means we have to choose the negative Tchebyscheff map T_{-3}.

Next, consider the parameter τ. It is the lattice constant of fictitious time s and has dimension GeV^{-2}. Ordinary stochastic quantization based on Gaussian white noise requires the continuum limit $\tau \to 0$. But since quantum field theory runs into difficulties at the Planck scale m_{Pl} and is expected to be replaced by a more advanced theory at this scale, it is most reasonable to take the small but finite value

$$\tau \sim m_{Pl}^{-2} \sim 10^{-38} GeV^{-2}. \tag{21}$$

Next, the classical equation of state of dark energy should be close to $w = -1$, as experimentally measured [17,18]. The classical equation of state is defined by $w = \langle p \rangle / \langle \rho \rangle$, where p is the pressure and ρ the energy density. As shown in [21], this condition requires that the Tchebyscheff maps are weakly coupled. $\alpha = 0$ generates an equation of state exactly given by $w = -1$, and $\alpha << 1$ generates an equation of state $w \approx -1$.

Finally, the parameter p_{max} can be determined from the currently measured dark energy density [17,18]

$$\rho_\varphi^{Obs} = (2.9 \pm 0.3) \cdot 10^{-47} GeV^4. \tag{22}$$

For $\alpha \approx 0$, eq. (20), (21) and (22) fix the parameter p_{max} as

$$p_{max} \sim 10^{-42} GeV. \tag{23}$$

In static models, where the vacuum energy density does not change with the expansion of the universe, the smallness of this parameter is the famous cosmological constant problem [19,20]. In models with dynamically evolving dark energy, the (current) smallness of this parameter is related to the fact that the universe is old[21].

5 Generalized statistical mechanics

The type of chaotic vacuum fluctuations considered here have rather interesting properties, which can be described in the language of statistical mechanics. For $\alpha = 0$ the natural invariant measure describing the probability distribution of the fluctuating variables Φ_n^i is given by eq. (14), and the measure factorizes at all lattice sites. The relevant probability density can be regarded as a generalized canonical distribution in non-extensive statistical mechanics [1]. As it is well known, one defines for a dimensionless continuous random variable X with probability density $p(x)$ the Tsallis entropies as

$$S_q = \frac{1}{q-1}(1 - \int p(x)^q dx). \tag{24}$$

Here q is the entropic index. The Tsallis entropies contain the Shannon entropy S_1 as a special case for $q \to 1$. Extremizing S_q subject to the constraint

$$\int p(x)E(x)dx = U \tag{25}$$

one ends up with q-generalized canonical distributions. These are given by

$$p(x) \sim (1 + (q-1)\beta E(x))^{-\frac{1}{q-1}}, \tag{26}$$

where E is the energy associated with microstate x, and β is the inverse temperature. Of course, for $q \to 1$ one obtains the usual Boltzmann factor $e^{-\beta E}$.

Alternatively, one can work with the escort distributions, defined by [26]

$$P(x) = \frac{p(x)^q}{\int p(x)^q dx}. \tag{27}$$

If the energy constraint (25) is implemented using the escort distributions $P(x)$ rather than the original distribution $p(x)$, one obtains generalized canonical distributions of the form

$$P(x) \sim (1 + (q-1)\beta E(x))^{-\frac{q}{q-1}}. \tag{28}$$

Again the limit $q \to 1$ yields ordinary Boltzmann factors $e^{-\beta E}$.

Let us now apply this formalism to the chaotic fields $X = \Phi_n^i$. We might identify $E = \frac{1}{2}m\Phi^2$ as a formal kinetic energy associated with the chaotic fields. We then get by comparing eq. (14) and (26)

$$q = 3 \tag{29}$$
$$\beta^{-1} = -m \tag{30}$$

Two problems arise: Firstly, the temperature that formally comes out of this approach is negative. Secondly, and more seriously, the Tsallis entropy of the distribution (14) as defined by the integral eq. (24) does not exist, since the integral $\int_{-1}^{1}(1-x^2)^{-3/2}dx$ diverges.

However, it is remarkable that these problems do not occur if the escort formalism [2] is used. By comparing eq. (14) and (28) we obtain

$$q = -1 \tag{31}$$
$$\beta^{-1} = m. \tag{32}$$

We obtain the result that our dark energy model behaves similar to an ideal gas in the nonextensive formalism but with entropic index $q = -1$, rather than $q = 1$ as in ordinary statistical mechanics. The 'velocity' v is given by the dimensionless variable $v = \Phi_n^i$, the 'energy' by the non-relativistic formula

$E = \frac{1}{2}mv^2$, and the inverse temperature is $\beta = m^{-1}$. This nonextensive gas has the special property that the temperature coincides with the mass of the 'particles' considered. For nonzero α a similar formalism remains valid, just that the effective energies E become more complicated [22].

We can now evaluate all interesting thermodynamic properties of the system using the escort formalism. Regarding the invariant density (14) as an escort distribution $P(\Phi)$, the original distribution is given by

$$p(\Phi) = \frac{2}{\pi}\sqrt{1-\Phi^2} \sim P(\Phi)^{1/q}. \tag{33}$$

For the Tsallis entropy of the chaotic fields we obtain from eq. (24) and (33)

$$S_q[p] = \frac{1}{4}(\pi^2 - 2) = S_q[P] \quad (q=-1) \tag{34}$$

It is invariant under the transformation $p \to P$ (a distinguished property of the entropic index $q = \pm 1$).

For the generalized internal energy we obtain

$$U_q[P] = \int_{-1}^{1} P(\phi)\frac{1}{2}m\Phi^2 = \frac{1}{4}m \quad (q=-1) \tag{35}$$

and for the generalized free energy

$$F_q = U_q - TS_q = \frac{m}{4}(3-\pi^2) \quad (q=-1). \tag{36}$$

All expectations formed with the invariant measure can be regarded as corresponding to escort expectations within the more general type of thermodynamics that is relevant for our chaotically evolving scalar fields. Whereas ordinary matter is described by a statistical mechanics with $q=1$, the chaotic fields generating dark energy are described by $q=-1$. The role of dark energy is similar to that of a suitable thermodynamic potential in this more general type of statistical mechanics. In general, one can easily verify that the entropic indices $q=+1$ and $q=-1$ are very distingushed cases: Only for these two cases the Tsallis entropy of the escort distribution is equal to the Tsallis entropy of the original distribution, for arbitrary distributions $p(x)$. If our model of dark energy is correct, then both types of statistical mechanics are realized in the universe.

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