Superstatistics, escort distributions, and applications

Christian Beck

School of Mathematical Sciences
Queen Mary, University of London
Mile End Road, London E1 4NS, England

Abstract

Superpositions of different statistics on different time or spatial scales (in short, superstatistics) can naturally lead to an effective description by nonextensive statistical mechanics. We first discuss the role of escort distributions within the superstatistical framework, and then briefly describe recent physical applications of this concept to turbulent and patttern forming systems.

Key words: nonextensive statistical mechanics, superstatistics, escort distributions
PACS: 05.40.-a, 05.70.-a

1 Introduction

In this paper we will briefly review the so-called superstatistics concept[1]. It yields a simple and plausible argument why nonextensive methods, as developed by Tsallis and many others [2,3,4], are relevant for nonequilibrium systems with large-scale spatio-temporal fluctuations of an intensive parameter.

Superstatistics is a superposition of two (or even more) different statistics: One given by ordinary Boltzmann factors, and another one given by large-scale fluctuations of one (or several) intensive parameters (e.g. the inverse temperature). The corresponding stationary probability distributions arise as a convolution of the various statistics. Tsallis statistics is a special case that arises naturally in this approach for a χ^2-distributed parameter, if the marginal distributions are formed. But other generalized statistics are possible as well. Recently it has

Email address: c.beck@qmul.ac.uk (Christian Beck).

become clear that the concept is not only a theoretical construct but of practical physical relevance. For example, it is possible to develop superstatistical models that well describe the statistical properties of measured accelerations in Lagrangian turbulence experiments [5,6], or the statistics of defect velocities in pattern forming systems [7], or the measured energy distribution of cosmic rays [8,9].

2 Basic idea of superstatistics

Consider a driven nonequilibrium systems with spatio-temporal fluctuations of an intensive parameter β. These fluctuations are externally produced, by constantly putting energy into the system which is dissipated. The intensive parameter may be the inverse temperature, or an effective chemical potential, or a function of the fluctuating energy dissipation in the flow (for the turbulence application). Locally, i.e. in spatial regions (cells) where β is approximately constant, the system may be described by ordinary statistical mechanics, i.e. ordinary Boltzmann factors $e^{-\beta E}$, where E is an effective energy in each cell. In the long-term run, the system is described by a spatio-temporal average over the fluctuating β.

Suppose the probability density of β is $f(\beta)$. By integrating over β one obtains marginal distributions of the form

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} \rho(E) e^{-\beta E} d\beta, \tag{1}$$

where $\rho(E)$ is the density of states and $Z(\beta)$ is the normalization constant of $\rho(E)e^{-\beta E}$ for a given β. Eq. (1) describes the long-term behaviour of the system (we have chosen type-B superstatistics [1]).

Let us consider three important examples (many more are possible).

1. β is distributed according to the χ^2-distribution

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\}. \tag{2}$$

n is a parameter. The average of the fluctuating β is given by

$$\langle\beta\rangle = \int_0^\infty \beta f(\beta) d\beta = \beta_0 \tag{3}$$

and the second moment of β is given by

$$\frac{\langle \beta^2 \rangle}{\beta_0^2} = 1 + \frac{2}{n}. \tag{4}$$

Let $E = \frac{1}{2}u^2$ be a kinetic energy. The marginal distribution (1), obtained by integration over all β, yields the generalized canonical distributions of nonextensive statistical mechanics [2,3,4]

$$p(u) = \frac{1}{Z_q} \frac{1}{\left(1 + \frac{1}{2}\tilde{\beta}(q-1)u^2\right)^{\frac{1}{q-1}}} \tag{5}$$

where Z_q is a normalization constant and

$$q = 1 + \frac{2}{n+1} \tag{6}$$

$$\tilde{\beta} = \frac{2}{3-q}\beta_0. \tag{7}$$

The distributions (5) maximize the Tsallis entropies S_q subject to suitable constraints [2,3,4].

2. β is log-normally distributed:

$$f(\beta) = \frac{1}{\beta s \sqrt{2\pi}} \exp\left\{ \frac{-(\log \frac{\beta}{m})^2}{2s^2} \right\} \tag{8}$$

The average β_0 of the above log-normal distribution is given by $\beta_0 = m\sqrt{w}$ and the variance by $\sigma^2 = m^2 w(w-1)$, where $w := e^{s^2}$. One obtains the superstatistics distribution

$$p(u) = \frac{1}{2\pi s} \int_0^\infty d\beta \; \beta^{-1/2} \exp\left\{ \frac{-(\log \frac{\beta}{m})^2}{2s^2} \right\} e^{-\frac{1}{2}\beta u^2}. \tag{9}$$

The integral cannot be evaluated in closed form, but it can be easily numerically evaluated and compared with experimental data [5,6,10].

3. Yet another interesting example of a superstatistics is obtained if we assume that the temperature $T = \beta^{-1}$ (rather than β itself) is χ^2-distributed. In this case the corresponding marginal distribution $p(u)$ has exponential tails [11].

3 Escort distributions in superstatistical systems

For a given probability distribution $\{p_i\}$ of microstates i and a given parameter q the escort distribution $\{P_i\}$ is defined as [12]

$$P_i = \frac{p_i^q}{\sum_i p_i^q}. \tag{10}$$

Both types of distributions, $\{p_i\}$ and $\{P_i\}$, notoriously occur in the formulation of nonextensive statistical mechanics [3,4,13], like two spin-degrees of freedom which are not there in classical statistical mechanics ($q = 1$). We will now show that within the superstatistical approach these two degrees of freedom can be easily understood.

Generally, it makes sense to β-average typical thermodynamic relations if the local cells are sufficiently large and if they are in local equilibrium. In this case one has for each local cell

$$S = \beta(U - F), \tag{11}$$

where S is the ordinary Boltzmann-Gibbs-Shannon entropy, U is the internal energy and F the free energy associated with each cell. The averaged version reads

$$\int_0^\infty f(\beta) S d\beta = \int_0^\infty \beta f(\beta) U d\beta - \int_0^\infty \beta f(\beta) F d\beta. \tag{12}$$

Note that on the right-hand side we obtain averages formed with $\beta f(\beta)$. For χ^2-distributed β this is like transforming $n \to n + 2$ in the distribution (2). Since

$$\frac{1}{q-1} = \frac{n+1}{2} \tag{13}$$

and

$$\frac{q}{q-1} = \frac{n+3}{2} \tag{14}$$

this means one is forming averages with respect to the escort distributions on the right-hand side of eq. (12), as required in the modern versions of nonextensive statistical mechanics [3].

Our argument is valid for arbitrary energies E. We see that within the superstatistical β-averaging approach entropic quantities most naturally correspond to the original distribution p_i, energetic quantities such as U or F most naturally to the escort distribution P_i, since there is an additional factor β on the right-hand side of eq. (12).

In general, the averaged Shannon entropy on the left-hand side does not coincide with the Tsallis entropy S_q, which satisfies [2,3,4]

$$S_q = \tilde{\beta}(U_q - F_q). \tag{15}$$

However, it might be interesting to construct special situations where eq. (12) and (15) are equivalent.

4 Applications

Langevin equations where the parameters fluctuate on a large spatio-temporal scale yield a concrete dynamical realization of superstatistics [14]. For example, to model the motion of a single test particle in a turbulent flow one considers a superstatistical extension of the Sawford model [15]. The Sawford model describes the joint stochastic process $(a(t), u(t), x(t))$ of acceleration, velocity and position of a Lagrangian test particle in a turbulent flow by the stochastic differential equation

$$\begin{aligned} \dot{a} &= -(T_L^{-1} + t_\eta^{-1})a - T_L^{-1}t_\eta^{-1}u \\ &\quad + \sqrt{2\sigma_u^2(T_L^{-1} + t_\eta^{-1})T_L^{-1}t_\eta^{-1}}\ L(t) && (16) \\ \dot{u} &= a && (17) \\ \dot{x} &= u, && (18) \end{aligned}$$

where $L(t)$ is Gaussian white noise. Note that in this model the acceleration a and not the velocity u is driven by white noise, so the meaning of the variables is different as compared to an ordinary Brownian particle. T_L and t_η are two time scales, with $T_L >> t_\eta$ and

$$\begin{aligned} T_L &= \frac{2\sigma_u^2}{C_0 \epsilon} && (19) \\ t_\eta &= \frac{2a_0 \nu^{1/2}}{C_0 \epsilon^{1/2}}. && (20) \\ & && (21) \end{aligned}$$

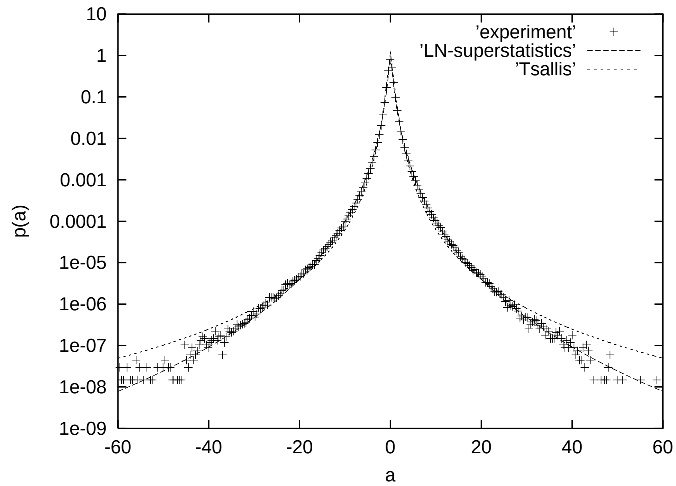

Fig. 1 Histogram of accelerations a as measured in [16] and the log-normal superstatistics prediction eq. (9) with $s^2 = 3.0$. For comparison, the figure also shows Tsallis statistics (eq. (5) with $q = 1.5$).

ϵ is the energy dissipation, ν the kinematic viscosity, C_0, a_0 are Lagrangian structure function constants, and σ_u^2 is the variance of the velocity distribution. The Sawford model with constant coefficients predicts Gaussian stationary distributions for a and u, and is thus at variance with recent measurements [16]. However, a superstatistical generalization of the Sawford model [5,6] well fits the data.

For $T_L \to \infty$ one can derive [6] that the fluctuating parameter β is given by

$$\beta = \frac{2a_0}{C_0^2}\nu^{1/2}\epsilon^{-3/2}, \tag{22}$$

i.e. the fluctuating energy dissipation ϵ is proportional to $\beta^{-2/3}$. Fig. 1 shows the measured probability density of the acceleration of a Lagrangian test particle in a turbulent flow as obtained in the experiment of Bodenschatz et al. [16]. Log-normal superstatistics with $s^2 = 3.0$ yields an excellent fit [6].

Similar superstatistical models can be formulated for pattern forming systems, just that the meaning of the variables in the Langevin equation is different. The probability density of defect velocities in inclined layer convection experiments can be quite precisely measured. As shown in Fig. 2, it quite precisely coincides with a Tsallis distribution with $q \approx 1.46$ [7]. A superstatistical model well describes various stochastic properties of these defects.

Superstatistical models also have applications in high energy physics, for example as simple models to explain the measured statistics of cosmic rays [8,9].

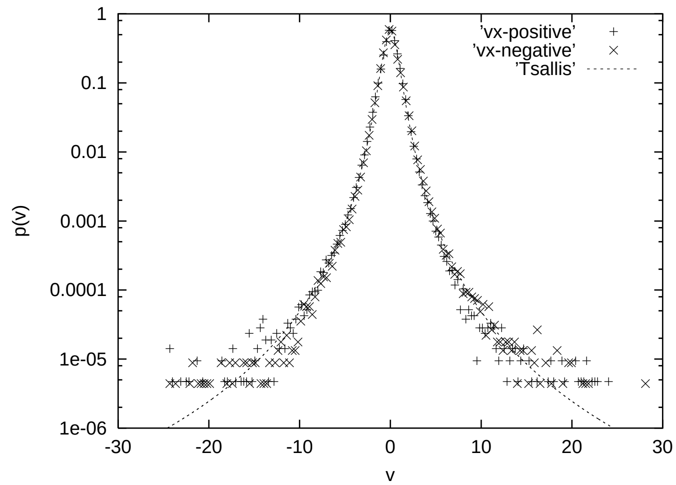

Fig. 2 Measured distribution of positive and negative defect velocities in inclined layer convection and comparison with a nonextensive canonical distribution of type (5) with $q = 1.46$ (more details in [7]).

References

[1] C. Beck and E.G.D. Cohen, Physica **322A**, 267 (2003)

[2] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)

[3] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica **261A**, 534 (1998)

[4] S. Abe, Y. Okamoto (eds.), *Nonextensive Statistical Mechanics and Its Applications*, Springer, Berlin (2001)

[5] A.M. Reynolds, Phys. Rev. Lett. **91**, 084503 (2003)

[6] C. Beck, Europhys. Lett. **64**, 151 (2003)

[7] K.E. Daniels, C. Beck, and E. Bodenschatz, Physica D, in press (2003) (cond-mat/0302623)

[8] C. Tsallis, J.C. Anjos, E.P. Borges, Phys. Lett. **310A**, 372 (2003)

[9] C. Beck, Physica **331A**, 173 (2004)

[10] B. Castaing, Y Gagne, E.J. Hopfinger, Physica **46D**, 177 (1990)

[11] F. Sattin and L. Salasnich, Phys. Rev. **65E**, 035106(R) (2002)

[12] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems*, Cambridge University Press, Cambridge (1993)

[13] S. Abe, Phys. Rev. **68E**, 031101 (2003)

[14] C. Beck, Phys. Rev. Lett. **87**, 180601 (2001)

[15] B.L. Sawford, Phys. Fluids **A3**, 1577 (1991)

[16] G.A. Voth et al., J. Fluid Mech. **469**, 121 (2002)