Lagrangian acceleration statistics in turbulent flows

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Abstract

We show that the probability densities of accelerations of Lagrangian test particles in turbulent flows as measured by Bodenschatz et al. [Nature 409, 1017 (2001)] are in excellent agreement with the predictions of a stochastic model introduced in [C. Beck, PRL 87, 180601 (2001)] if the fluctuating friction parameter is assumed to be log-normally distributed. In a generalized statistical mechanics setting, this corresponds to a superstatistics of log-normal type. We analytically evaluate all hyperflatness factors for this model and obtain a flatness prediction in good agreement with the experimental data. There is also good agreement with DNS data of Gotoh et al. We relate the model to a generalized Sawford model with fluctuating parameters, and discuss a possible universality of the small-scale statistics.

In the past two years there has been considerable experimental progress in mesasuring the statistics of Lagrangian test particles in turbulent flows [1, 2, 3]. While simple dynamical models like the Sawford model [4, 5] predict a Gaussian acceleration statistics, the measurements of Bodenschatz et al. [1, 3] have confirmed that the probability distribution of acceleration a is strongly non-Gaussian and has strongly pronounced tails. As shown in [6, 7], the measured acceleration distributions are well fitted by Tsallis distributions [8] for $|a| < 30$, in units of the standard deviation. However, very new recent measurements [9] and numerical simulations [10] indicate that there are deviations from Tsallis statistics for extremely large accelerations $|a| > 30$, i.e. for extremely rare events of very strong forces acting on the particle.

In this Letter we show that the simple dynamical model previously introduced in [7] can be generalized in such a way that it yields quite a perfect description of the observed Lagrangian acceleration statistics in turbulent flows. The coincidence is better than in previous models based on Tsallis statistics. In the language of statistical mechanics, the relevant new statistics that fits the data quite perfectly is 'superstatistics of log-normal type'. While it can be proved that log-normal superstatistics, as any superstatistics, is close to Tsallis statistics for not too large accelerations, there are significant differences in the tails.

The concept of superstatistics was introduced in [11]. A superstatistics is a 'statistics of a statistics', one given by ordinary Boltzmann factors $e^{-\beta E}$ and the other one by the distribution function of β, which is assumed to be fluctuating. Superstatistics are generally relevant for an effective description of driven nonequilibrium systems with stationary states which possess a fluctuating intensive parameter β. This can, for example, be the inverse temperature or, in the turbulence application, the energy dissipation rate. Tsallis statistics is a special superstatistics obtained for a χ^2-distribution of the intensive parameter. But as shown by Tsallis and Souza [12], more general versions of statistical mechanics can also be constructed for other superstatistics than Tsallis statistics, in particular for the superstatistics of log-normal type which turns out to be relevant here.

A superstatistics can be easily dynamically realized, by generalizing the approach of [7]. Consider a Langevin of the form

$$\dot{a} = \gamma F(a) + \sigma L(t), \tag{1}$$

where $L(t)$ is Gaussian white noise, $\gamma > 0$ is a friction constant, σ describes the strength of the noise, and $F(a) = -\frac{\partial}{\partial a}V(a)$ is a drift force. For most applications, it is sufficient to consider linear forces $F(a) = -a$ as generated by $V(a) = \frac{1}{2}a^2$. For an ordinary Brownian particle, a would be the velocity, but for the Lagrangian turbulence application, a actually stands for the acceleration of the Lagrangian test particle, i.e. a velocity difference on a very small time scale. We will consider linear Langevin equations of the form (1) for each component of the acceleration.

If γ and σ are constant then the stationary probability density of a is $p(a) \sim e^{-\beta V(a)}$, where $\beta := \frac{\gamma}{\sigma^2}$. However, since the energy dissipation fluctuates in a turbulent flow we now let the parameters γ and σ fluctuate such that $\beta := \frac{\gamma}{\sigma^2}$ has some probability density $f(\beta)$. These parameter fluctuations are assumed to be on a relatively long time scale (or a relatively large spatial scale) so that the system can temporarily reach local equilibrium. The Lagrangian test particle moves through these different regions with different β.

One obtains for the conditional probability $p(a|\beta)$ (i.e. the probability of a given some value of β)

$$p(a|\beta) = \frac{1}{Z(\beta)} \exp\left\{-\beta V(a)\right\}, \tag{2}$$

where $Z(\beta)$ is a normalization constant, and for the joint probability $p(a, \beta)$ (i.e. the probability to observe both a certain value of a and a certain value of β)

$$p(a, \beta) = p(a|\beta) f(\beta), \tag{3}$$

and finally for the marginal probability $p(a)$ (i.e. the probability to observe a certain value of a no matter what β is)

$$p(a) = \int p(a|\beta) f(\beta) d\beta. \tag{4}$$

In [7] a χ^2-distribution was chosen for $f(\beta)$, and it was shown that this generates Tsallis statistics for the marginal distribution, i.e. $p(a) \sim (1 + \tilde{\beta}(q-1)V(a))^{-1/(q-1)}$, where q is the entropic index of nonextensive statistical mechanics [8].

Let us here choose another distribution for $f(\beta)$, in fact one of the examples dealt with in [11], the log-normal distribution

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{\frac{-(\log\frac{\beta}{m})^2}{2s^2}\right\}, \tag{5}$$

where m and s are parameters. The average β_0 of the above log-normal distribution is given by $\beta_0 = m\sqrt{w}$ and the variance by $\sigma^2 = m^2 w(w-1)$, where $w := e^{s^2}$. For our Lagrangian turbulence model it is sufficient to consider linear drift forces $F(a) = -a$. The integral given by (4)

$$p(a) = \frac{1}{2\pi s}\int_0^\infty d\beta\, \beta^{-1/2} \exp\left\{ \frac{-(\log \frac{\beta}{m})^2}{2s^2} \right\} e^{-\frac{1}{2}\beta a^2}, \tag{6}$$

cannot be evaluated in closed form, but the equation is easily numerically integrated, and can be compared with experimentally measured densities $p(a)$. Similar distributions as in (6) have also been considered in [13], however without a dynamical realization in terms of a stochastic differential equation. The distribution $p(a)$ has variance 1 for the choice $m = \sqrt{w}$, hence only one fitting parameter s remains if one compares with experimental data sets that have variance 1.

Fig. 1 shows the histograms of accelerations of Lagrangian test particles as measured by Bodenschatz et al. [1, 3]. The distributions are rescaled to variance 1. As shown previously [6, 7], the experimental distributions are reasonably well approximated by Tsallis distributions for $|a| < 30$. But log-normal superstatistics yields a better fit, using only one fitting parameter s. The measured distributions for $R_\lambda = 690$ and $R_\lambda = 970$ are basically the same and very well fitted by eq. (6) with $s^2 = 3.0$. Note that m is not a free fitting parameter but fixed as $m = \sqrt{w} = e^{\frac{s^2}{2}}$ to give variance 1. Since Bodenschatz's data reach rather large accelerations a (in units of the standard deviation), the measured tails of the distributions allow for proper distinction between various superstatistics. The main difference between the χ^2-superstatistics studied in [7] and log-normal superstatistics studied here is the fact that $p(a)$ decays with a power law for the former ones, whereas it decays with a more complicated logarithmic decay for the latter ones. For alternative fitting attempts based on multifractal models, see [14].

All moments of the distribution (6) exist and can be easily evaluated as follows. The moments of a Gaussian distribution

$$p_G(a) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{1}{2}\beta a^2} \tag{7}$$

are given by

$$\langle a^r \rangle_G = \frac{1}{\beta^{r/2}}(r-1)!! \tag{8}$$

(r even). Moreover, the moments of the lognormal distribution (5) are given by

$$\langle \beta^r \rangle_{LN} = m^r w^{\frac{1}{2}r^2}. \tag{9}$$

Combining eq. (8) and (9) one obtains the moments of the distribution $p(a)$ given by eq. (6) as

$$\begin{aligned} \langle a^r \rangle &= \langle \langle a^r \rangle_G \rangle_{LN} && (10) \\ &= (r-1)!! \langle \beta^{-r/2} \rangle_{LN} && (11) \\ &= (r-1)!! m^{-\frac{r}{2}} w^{\frac{1}{8}r^2} && (12) \end{aligned}$$

In particular, the variance is given by

$$\langle a^2 \rangle = m^{-1}\sqrt{w}. \tag{13}$$

All hyperflatness factors F_r are independendent of m and given by

$$F_r := \frac{\langle a^{2r} \rangle}{\langle a^2 \rangle^r} = (2r-1)!! w^{\frac{1}{2}(r-1)}. \tag{14}$$

In particular, the flatness F_2 is given by

$$F_2 := \frac{\langle a^4 \rangle}{\langle a^2 \rangle} = 3w = 3e^{s^2}. \tag{15}$$

Measuring the flatness F_2 of some experimental data thus provides a very simple method to determine the fitting parameter s of lognormal superstatistics. Or, if s is fitted from the shape of the densities, then a prediction on the flatness factor can be given. Note that w is the analogue of q in the nonextensive approach [11].

Using the parameter $s^2 = 3.0$ that yields a good fit of the densities measured by Bodenschatz at $R_\lambda = 690$ and $R_\lambda = 970$, a flatness of $F_2 = 60.3$ is predicted from eq. (15), in agreement with the measured flatness values reported in [3, 9].

In direct numerical simulations (DNS) of the Navier-Stokes equations, even larger accelerations a can be reached. Fig. 2 shows Gotoh's results on the pressure distribution as obtained by DNS at $R_\lambda = 380$ [10]. In good approximation the pressure statistics coincides with the acceleration statistics of a Lagrangian test particle. Gotoh's histograms reach accelerations up to 150 (in units of the standard deviation), a much larger statistics than can be presently reached in Bodenschatz's experiment. Hence the tails of these distributions can very sensitively distinguish between various superstatistics models. Fig. 2 shows that log-normal superstatistics with $s^2 = 3.0$ again yields a good fit, keeping in mind that one compares data that vary over 12 orders of magnitude. The fit quality can be slightly further improved if one uses for $f(\beta)$ a log-normal distribution that has a lower and upper cutoff ($\beta_{min} \approx 0.0005$ and $\beta_{max} \approx 25$). The above truncation may effectively model finite Reynolds number and finite size effects, which are certainly present in any numerical simulation of the Navier-Stokes equation.

If the Reynolds number is sufficiently large ($R_\lambda > 300$), we notice that the same fitting parameter $s^2 = 3.0$ yields a good fit of all 3 different data sets ($R_\lambda = 380, 690, 970$). It seems that s^2 is essentially independent of the Reynolds number for large R_λ, in agreement with the measured approximate Reynolds number independence of the flatness reported in [3] for $R_\lambda > 300$. If this tendency is confirmed by further experiments then it seems natural to conjecture that the variance of the log-normal distribution converges to a finite universal value for $R_\lambda \to \infty$. This value might simply be given by $s^2 = 3$. The value 3 could stand in connection with the three spatial degrees of freedom: As it is apparent from eq. (5), the variable $Y := \log \frac{\beta}{m}$ is a Gaussian random variable with average 0 and variance s^2. The fluctuating β is related to the fluctuating energy dissipation rate. Since energy can flow independently into the three space directions, we may write

$$Y = \log \frac{\beta_x}{m} + \log \frac{\beta_y}{m} + \log \frac{\beta_z}{m}, \tag{16}$$

where each of the independent random variables $X_i := \log \frac{\beta_i}{m}$, $i = x, y, z$, is Gaussian with average 0. Our fitting observation $s^2 = 3$ is thus equivalent to the fact that each of the Gaussian random variables X_i, describing the energy flow in the i-th direction, has variance 1. Whether this is a random coincidence for finite Reynolds number or whether this is a deep fundamental principle underlying the small-scale statistics for $R_\lambda \to \infty$ is not clear at the moment.

What remains to be done is to precisely relate the parameter β of our superstatistics approach to the fluctuating energy dissipation ϵ in the turbulent flow. For this we may just compare with a previously studied Lagrangian model, the Sawford model [4, 5, 15]. The Sawford model assumes that the joint stochastic process $(a(t), u(t), x(t))$ of acceleration, velocity and position of a Lagrangian test particle obeys the stochastic differential equation

$$\begin{aligned} \dot{a} &= -(T_L^{-1} + t_\eta^{-1})a - T_L^{-1} t_\eta^{-1} u \\ &\quad + \sqrt{2\sigma_u^2 (T_L^{-1} + t_\eta^{-1}) T_L^{-1} t_\eta^{-1}}\, L(t) \qquad (17) \\ \dot{u} &= a \qquad (18) \\ \dot{x} &= u, \qquad (19) \end{aligned}$$

where $L(t)$ is again Gaussian white noise. T_L and t_η are two time scales, with $T_L >> t_\eta$. In this model one has

$$\begin{aligned} T_L &= \frac{2\sigma_u^2}{C_0 \bar{\epsilon}} \qquad (20) \\ t_\eta &= \frac{2a_0 \nu^{1/2}}{C_0 \bar{\epsilon}^{1/2}}, \qquad (21) \\ & \qquad (22) \end{aligned}$$

where $\bar{\epsilon}$ is the average energy dissipation, C_0, a_0 are Lagrangian structure function constants, and σ_u^2 is the variance of the velocity distribution. The Taylor scale Reynolds number is given by

$$R_\lambda = \frac{\sqrt{15}\sigma_u^2}{\sqrt{\nu \bar{\epsilon}}}. \qquad (23)$$

The Sawford model predicts Gaussian stationary distributions for a and u, and is thus at variance with the recent measurements. However, a straightforward idea is to generalize the Sawford model with constant parameters to a generalized Sawford model with fluctuating parameters, following a similar type of arguments as in [7]. This was recently worked out by Reynolds [15], who emphasized compatibility of the obtained stochastic model with Kolmogorov's refined similarity hypothesis.

We note that there are several possibilities to extend the Sawford model with constant parameters to an extended one with fluctuating ones, depending on which of the variables $\bar{\epsilon}$ in the above model equations are replaced by a fluctuating ϵ and which are not. For our purposes it is sufficient to consider the Sawford model in the limit $T_L \to \infty$, which is a reasonable approximation for large Reynolds numbers. In that limit the model with constant parameters becomes identical to eq. (1) with $F(a) = -a$ and with the identification

$$\gamma = \frac{C_0}{2a_0}\nu^{-1/2}\bar{\epsilon}^{1/2} \tag{24}$$

$$\sigma = \frac{C_0^{3/2}}{2a_0}\nu^{-1/2}\bar{\epsilon} \tag{25}$$

This clarifies the physical meaning of the average of our parameters γ and σ. To proceed to a model with fluctuting parameters, the simplest possibility is to replace in both of the above equations $\bar{\epsilon}$ by a fluctuating ϵ. This yields

$$\beta = \frac{\gamma}{\sigma^2} = \frac{2a_0}{C_0^2}\nu^{1/2}\epsilon^{-3/2}, \tag{26}$$

i.e. the fluctuating energy dissipation ϵ is proportional to $\beta^{-2/3}$. Based on this equation, one can derive a couple of interesting relations. For example, for the constant a_0 defined by the relation

$$\langle a^2 \rangle =: a_0 \langle \epsilon \rangle^{3/2} \nu^{-1/2} \tag{27}$$

one obtains after a short calculation the relation

$$a_0 = \frac{1}{\sqrt{2}} C_0 e^{\frac{1}{12}s^2} \tag{28}$$

Reynolds' DNS data suggest $C_0 \approx 7$ [15], hence $s^2 = 3$ yields $a_0 \approx 6$, in agreement with Bodenschatz's measurements [3]. Moreover, for the Gaussian random variable $Y = \log \frac{\beta}{m}$ studied in eq. (16) one obtains the relation

$$Y = -\frac{3}{2}\log\frac{\epsilon}{\langle \epsilon \rangle} - \frac{1}{3}s^2 \tag{29}$$

independent of the value of a_0 and C_0.

However, in general there are also other possible dynamical realizations of a Sawford model with fluctuating parameters than eq. (26). For example, we may keep γ constant in eq. (24) and let only σ in eq. (25) fluctuate. This leads to $\beta \sim \epsilon^{-2}$. Or, we may keep σ constant and let only γ fluctuate. This leads to $\beta \sim \epsilon^{1/2}$. All these cases have in common that $\beta \sim \epsilon^{\kappa}$, where κ is some power. Of course, for a log-normal distribution the power κ does not change the functional form of the log-normal distribution, which once again is a hint for the physical relevance of a log-normally distributed β. While all the above possibilities lead to the same marginal distribution $p(a)$ given by eq. (6), they can be dynamically distinguished by looking at the acceleration autocorrelation function, which exhibits different behaviour for the various cases.

To summarize, we have shown that the measured densities in Lagrangian turbulence experiments are very well described by log-normal superstatistics. This superstatistics is easily dynamically realized by considering a log-normal generalization of the model previously studied in [7]. Only one parameter s is fitted to obtain excellent agreement with the experimentally measured distributions and the DNS data. Log-normal superstatistics differs from χ^2-superstatistics, i.e. ordinary Tsallis statistics, but for moderately large accelerations Tsallis statistics is often a good approximation. A possible hypothesis is that the variance parameter s^2 converges for $R_\lambda \to \infty$ and is given by $s^2 = 3$. In that case no free parameters would be left, and a universal Lagrangian small scale statistics would arise.

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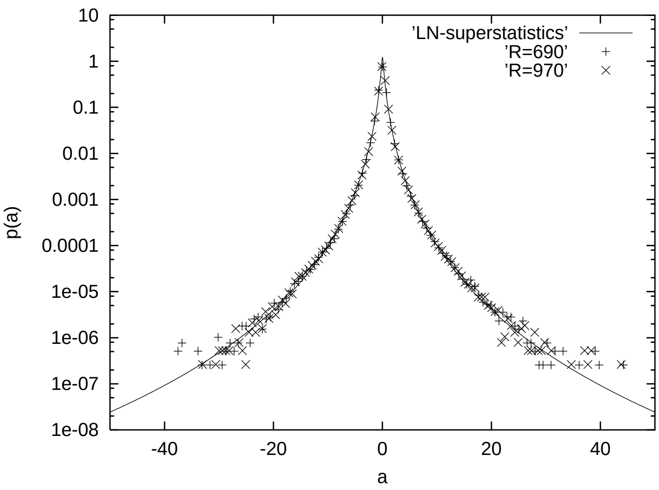

Fig.1 Acceleration distributions as measured by Bodenschatz et al. [1, 3] for Reynolds number $R_\lambda = 690$ and 970, and the log-normal superstatistics distribution $p(a)$ given by eq. (6) with $s^2 = 3.0$.

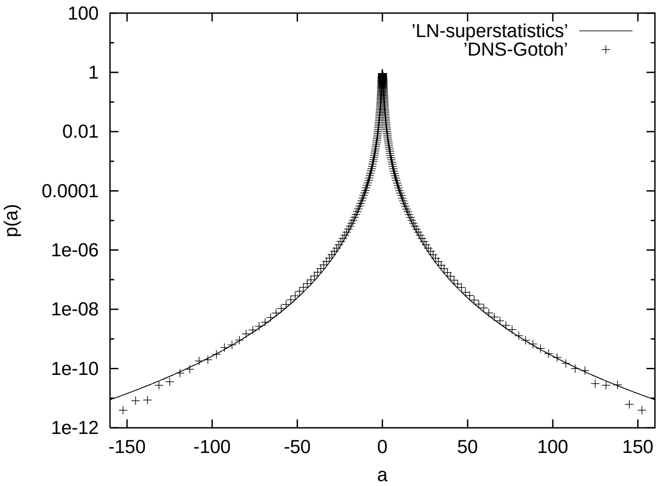

Fig.2 DNS data obtained by Gotoh et al. [10], and the log-normal superstatistics distribution eq. (6) with $s^2 = 3.0$.