Nonextensive methods in turbulence and particle physics

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Abstract

We describe some recent applications of Tsallis statistics in fully developed hydrodynamic turbulence and in high energy physics. For many of these applications nonextensive properties arise from spatial fluctuations of the temperature or the energy dissipation rate. The entropic index q is related to the relative magnitude of these fluctuations. We concentrate on a recently derived formula for the energy dependence of q that is experimentally verified by fits of cross sections in e^+e^- annihilation experiments. Evaluating this formula for much smaller energies E of the order of the recombination temperature, one obtains the correct order of magnitude of the fluctuations of the cosmic microwave background. Evaluating it for $E \to \infty$, one obtains (under certain assumptions) possible evidence for the existence of 6 compactified dimensions, as predicted by superstring theory.

1 Introduction

Nonextensive statistical mechanics as introduced by Tsallis in 1988 [1] and further developed by many others [2]–[8] is a useful generalization of ordinary statistical mechanics, with potential applications not only for equilibrium systems but also for nonequilibrium systems with a stationary state. It is based on the extremization of the Tsallis entropies

$$S_q = \frac{1}{q-1}\left(1-\sum_i p_i^q\right) \tag{1}$$

rather than the Shannon entropy S_1, which is contained as a special case in this more general setting. The p_i are the probabilities of the physical microstates, and q is the entropic index.

Given two independent subsystems I and II with probabilities p_i^I and p_i^{II}, respectively, the Tsallis entropy of the composed system I+II (with probabilities $p_{ij}^{I+II} = p_i^I p_j^{II}$) satisfies

$$S_q^{I+II} = S_q^I + S_q^{II} - (q-1)S_q^I S_q^{II}. \tag{2}$$

Hence the system is nonextensive, i.e. there is additivity of the entropy for $q=1$ only.

Broadly speaking, the formalism with $q \neq 1$ has so far been observed to be relevant for three different classes of systems. The first class is systems with long-range interactions (e.g. [9]–[13]), the second one is multifractal systems (e.g. [14]–[15]), and the third one is systems with fluctuations of temperature or energy dissipation (e.g. [16]–[20]).

In this paper we describe some interesting new physical applications of the nonextensive approach. Our examples mainly seem to be lying in the 3rd class. One example is fully developed hydrodynamic turbulence [18]–[28], another one the statistical behaviour of particles produced in high energy e^+e^- collider experiments [29]–[31]. We will also deal with possible aspects of Tsallis statistics in the early universe [32]–[37], and in particular point out a connection with the fluctuations of the cosmic microwave background. In the last section we will show that (under certain assumptions on the orign of the temperature fluctuations) the e^+e^- scattering data may provide experimental evidence for the existence of compactified dimensions, as predicted by superstring theory.

2 Fluctuations of temperature and Tsallis statistics

Let us here sketch the way in which fluctuations can lead to nonextensive behaviour. This is a relatively new approach to nonextensitivity [16]–[20], which is in particular relevant for nonequilibrium systems with a stationary state. The fluctuations may be either local temperature fluctuations or fluctuations of an effective friction coefficient (see [38, 39] for some related nonequilibrium systems).

Consider a system of ordinary statistical mechanics with Hamiltonian H. Tsallis statistics with $q > 1$ can arise from this ordinary Hamiltonian if the inverse temperature β is fluctuating. For example, we may think of these fluctuations as spatial fluctuations. From the integral representation of the gamma function one can easily derive the formula [16, 18]

$$(1 + (q-1)\beta_0 H)^{-\frac{1}{q-1}} = \int_0^\infty e^{-\beta H} f(\beta) d\beta, \tag{3}$$

where

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left\{ \frac{1}{(q-1)\beta_0} \right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{ -\frac{\beta}{(q-1)\beta_0} \right\} \tag{4}$$

is the probability density of the χ^2 (or gamma) distribution. The above formula is valid for arbitrary Hamiltonians H und thus quite important. The left-hand side of eq. (3) is just the generalized Boltzmann factor emerging out of nonextensive statistical mechanics. It can directly be obtained by extremizing S_q. The right-hand side is a weighted average over Boltzmann factors of ordinary statistical mechanics. If we consider a nonequilibrim system with fluctuating β variables, then the generalized distribution functions of nonextensive statistical mechanics are a consequence of integrating over all possible fluctuating inverse temperatures β, provided β is χ^2 distributed.

Wilk et al. [17] have also suggested an extension of this approach which is valid for $q < 1$. However, in that case the distribution function $f(\beta)$ is more complicated and different from the simple form (4). In particular, for $q < 1$ one has $f(\beta) = f(\beta, H)$, i.e. the probability density of the fluctuating temperature also depends on the Hamiltonian H. This case seems physically less straightforward, hence we restrict ourselves to $q > 1$ in the following.

For $q > 1$, the χ^2 distribution obtained in eq. (4) is a universal distribution that occurs in many very common circumstances [40]. For example, it arises if β is the sum of squares of n independent Gaussian random variables X_j with average 0

$$\beta = \sum_{j=1}^{n} X_j^2, \tag{5}$$

with

$$n = \frac{2}{q-1}. \tag{6}$$

Hence one expects Tsallis statistics with $q > 1$ to be relevant in many applications where n (approximately) independent and (approximately) Gaussian variables contribute to the variable β. Of course, other distribution functions $f(\beta)$ can also be considered which may lead to other generalized statistics. Tsallis statistics is just one possibility, though a very relevant one. It is distinguished by generalized Khinchin axioms and formal invariance of most thermodynamic relations [2, 3].

The constant β_0 in eq. (4) is the average of the fluctuating β,

$$\langle \beta \rangle := \int_0^\infty \beta f(\beta) d\beta = \beta_0 \tag{7}$$

The deviation of q from 1 can be related to the relative variance of β. One can easily check that

$$q - 1 = \frac{\langle \beta^2 \rangle - \langle \beta \rangle^2}{\langle \beta \rangle^2}. \tag{8}$$

3 Application to turbulent flows

A lot of recent papers [18]–[28] showed that extremizing Tsallis entropies is a very useful tool in fully developed turbulence, leading to predictions for scaling exponents and probability density functions that coincide quite precisely with experimental data. Different models were developed, some of them using Tsallis statistics with a constant $q < 1$ within the multifractal model of turbulence [21]–[23], others doing Tsallis statistics with a scale-dependent $q > 1$ directly in velocity space [24]–[28]. Apparently, the way Tsallis statistics can be used in turbulence is not unique, but often the predictions given are quite similar.

Here, let us follow the approach introduced in [24] and experimentally verified in [25]. Consider turbulent velocity differences $u := v(x+r) - v(x)$ at a certain scale r of the flow. Consider a nonlinear Langevin equation of the form

$$\dot{u} = -\gamma F(u) + \sigma L(t) \tag{9}$$

where $F(u) = -\frac{\partial}{\partial u} V(u)$ is a nonlinear forcing and $L(t)$ is Gaussian white noise. To be specific, let us assume that $V(u) = C|u|^{2\alpha}$ is a power-law potential. The parameter β is defined as

$$\beta = \frac{\gamma}{\sigma^2}. \tag{10}$$

For $\alpha = 1$ we just obtain the ordinary Ornstein-Uhlenbeck process, for which $\beta = \gamma/\sigma^2$ is related to the temperature of the surrounding heat bath by $\beta = 1/kT$. Given some β and α, the stochastic differential equation (9) generates the stationary probability density

$$p(u|\beta) = \frac{\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)} (C\beta)^{\frac{1}{2\alpha}} \exp\left\{-\beta C |u|^{2\alpha}\right\}. \tag{11}$$

So far this invariant density deviates from what is measured in turbulence experiments, so our model is yet too simple for turbulence. But now comes the important point. Let us assume that not only $L(t)$ is a fluctuating quantity but also $\beta = \gamma/\sigma^2$. We assume that β changes spatially on a relatively large scale, i.e. the test particle first moves in a certain region with a given β, then moves to another region with another β and so on. Within each region there is local equilibrium. If β is a χ^2 distributed random variable in the various regions, with an n given by eq. (5), then for the marginal distributions $p(u) = \int p(u|\beta) f(\beta) d\beta$ we obtain after a short calculation

$$p(u) = \frac{1}{Z_q} \frac{1}{(1+(q-1)\tilde{\beta} C |u|^{2\alpha})^{\frac{1}{q-1}}}, \tag{12}$$

where

$$Z_q^{-1} = \alpha \left(C(q-1)\tilde{\beta} \right)^{\frac{1}{2\alpha}} \cdot \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\frac{1}{q-1} - \frac{1}{2\alpha}\right)} \tag{13}$$

and

$$q = 1 + \frac{2\alpha}{\alpha n + 1} \tag{14}$$

$$\tilde{\beta} = \frac{2\alpha}{1 + 2\alpha - q} \beta_0. \tag{15}$$

The densities (12) extremize the Tsallis entropies. Hence Tsallis statistics is rigorously proved for this simple model with fluctuating friction forces [20]. Moreover, for a test particle moving in a turbulent flow the dynamics is physically plausible, since one knows that the dissipation fluctuates in turbulent flows.

The densities $p(u)$ as given by eq. (12) were just those that were recently successfully used as fits of experimentally measured turbulent densities [25]. They perfectly coincide with the experimental data (Fig. 1 in [25], Fig. 1 in [18]), thus making this nonextensive model a very good model of turbulent statistics. The generalized Langevin equation yields a dynamical reason for Tsallis statistics. If one replaces the Gaussian white noise in the above stochastic differential equation by deterministic chaotic noise, one can also understand the (very small) asymmetry (skewness) of the observed distributions [41, 42, 24, 25].

4 Application to e^+e^- annihilation

Not only velocity distributions in hydrodynamics, but also momentum distributions of particles produced in high-energy scattering experiments can be successfully described using nonextensive methods. The differential cross section $\frac{1}{\sigma}\frac{d\sigma}{dp}$ measured in a collider experiment is proportional to the probability density to observe particles with a certain momentum p. Bediaga et al. [29] looked at experimentally measured differential cross sections of transverse momenta p_T in e^+e^- annihilation experiments and showed that the formula

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = c \cdot u \int_0^\infty dx \ \left(1 + (q-1)\sqrt{x^2 + u^2 + m_\beta^2}\right)^{-\frac{q}{q-1}} \tag{16}$$

yields an extremely good fit of the experimentally measured cross sections. Here $x = p_L/kT_0$, $u = p_T/kT_0$ and $m_\beta := m_0/kT_0$ are the longitudinal momentum, transverse momentum and mass of the particles in units of the Hagedorn temperature T_0, respectively. c is a constant related to the multiplicity M, i.e. the number of charged particles produced.

The Hagedorn temperature T_0 is a kind of phase transition temperature of 'boiling' nuclear matter. This fundamental concept was introduced by Hagedorn in a seminal paper in 1965 [31, 43, 44]. These days the Hagedorn phase transition is seen in connection with the QCD phase transition from confined to non-confined states.

Bediaga et al. just wrote down eq. (16) without deriving it. They neglected m_β and used three free fitting parameters for their fits, namely c, T_0, q. The question was if one can derive suitable equations for these fitting parameters, thus proceeding from a mere fit to a theory. Although the complete theory taking into account all possible interactions within the hadronization cascade is out of reach, a first step was performed in [30], where eq. (16) was derived from first principles by q-generalizing Hagedorn's original approach. The tool for this are suitable factorizing nonextensive partition functions for fermions and bosons, actually of similar form as previously suggested in [34].

One can then go a step further by considering suitable approximations and by making plausible assumptions on the energy dependences of various parameters, being guided by the existence of moments. The final result of [30] was that in good approximation one has

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = \frac{1}{T_0} M \cdot p(u) \tag{17}$$

where

$$\begin{aligned}
p(u) &= \frac{1}{Z_q} u^{3/2} \left(1 + (q-1)u\right)^{-\frac{q}{q-1}+\frac{1}{2}} && (18) \\
Z_q &= (q-1)^{-5/2} B\left(\frac{5}{2}, \frac{q}{q-1} - 3\right) && (19) \\
T_0(q) &= \left(1 - \frac{q}{3}\right) p_T^* && (20) \\
p_T^* &\approx 180\ MeV && (21) \\
q(E) &= \frac{11 - e^{-E/E_0}}{9 + e^{-E/E_0}} && (22) \\
E_0 &\approx \frac{1}{2} m_Z = 45.6\ GeV && (23) \\
M(E) &\approx \left(\frac{E}{T_0(1)}\right)^{5/11}. && (24)
\end{aligned}$$

Here E is the center of mass energy of the beam, and the function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ denotes the beta function.

Formula (17) with $p(u)$ given by eq. (18), $q(E)$ given by eq. (22), $T_0(q)$ given by eq. (20) and multiplicity $M(E)$ given by eq. (24) turns out to very well reproduce the experimental results of cross sections for all energies

E in the experimentally accessible region 14...161 GeV (see Fig. 3 in [30], which compares with experimental cross sections measured by the TASSO and DELPHI collaborations [45, 46]).

The theoretical progress is that eqs. (17)–(24) provide concrete equations for the energy dependence of all parameters T_0, M, q used in the fits, and only two fundamental constants are left: One is p_T^*, the energy scale of the QCD phase transition, and the other one E_0, the energy scale of the electroweak phase transition.

The agreement of eq. (17) with the experimentally measured cross sections is remarkably good. Nonextensitivity seems to be the right approach to tackle this high energy problem. q is not constant but depends on the spatial scale $r = \hbar/E$, just similar as it does in turbulence. Of course, an alternative for particle physicists is to do Monte-Carlo simulations. The simulated Monte-Carlo cross sections look very similar to our theoretical curves [47]. Thus the complexity inherent in the Monte-Carlo simulation effectively reduces to some model with nonextensive properties.

5 Fluctuations of the cosmic microwave background

Many authors have already dealt with possible applications of nonextensive statistical mechanics in high energy physics. Work so far has concentrated on the solar neutrino problem [48]–[51], quark gluon plasmas [52], heavy ion collisions [53], cosmic rays [54], e^+e^- annihilation [29, 30], and the cosmic microwave background [32, 34]. Useful bounds on $|q-1|$ in the early universe were derived [32]–[37]. Here we want to deal with the cosmic microwave background from a new point of view. We will estimate the order of magnitude of the temperature fluctuations using our formula for $q(E)$ from the previous section.

Although formula (22) is only experimentally confirmed for energies in the region 14...161 GeV, we can certainly also evaluate it for other energies as well. For $E << E_0$ we may write $exp(-E/E_0) \approx 1 - E/E_0$, which yields

$$q(E) = 1 + \frac{1}{5}\frac{E}{E_0} \qquad (E << E_0). \tag{25}$$

In section 2 we saw that any $q > 1$ can be interpreted in terms of spatial fluctuations in $\beta = 1/kT$. It is now interesting to see what the typical size of these fluctuations is if we evaluate our formulas at the temperature $E \approx 1$ eV where the universe becomes transparent (the recombination temperature). One obtains from formulas (8), (23), (25)

$$\frac{\Delta\beta}{\beta_0} = \frac{\Delta T}{T} = \sqrt{q-1} = \sqrt{\frac{E}{5E_0}} \approx 2 \cdot 10^{-6}. \tag{26}$$

Here $\Delta\beta$ denotes the standard deviation of β. Remarkably, we obtain a value that has the same order of magnitude as the experimentally measured value of the fluctuations of the cosmic microwave background [55]:

$$1.5 \cdot 10^{-6} \leq \frac{\Delta T}{T} \leq 10 \cdot 10^{-6} \tag{27}$$

Of course, the correct order of magnitude may just be a random coincidence, but it may also not be! If true, then nonextensive statistical mechanics is probably relevant for the thermodynamic description of the early universe, due to fluctuations that were neglected so far. We expect that q is energy dependent and given by eq. (22) or something of similar order of magnitude. Our value of $q - 1 \approx 4 \cdot 10^{-12}$ at the recombination temperature is certainly consistent with previous bounds derived, which were of the order $|q-1| \leq 10^{-4}$ [32, 33, 35, 37].

6 Compactified dimensions

Another striking observation is related to the $E \to \infty$ limit, where our formula (22) yields

$$q(\infty) = \frac{11}{9}. \tag{28}$$

The largest possible energy E corresponds to the smallest possible scale $r \sim \hbar/E$. In a quantum gravity setting this is the Planck scale. It is now reasonable to assume that at the smallest possible scale r (described by the largest possible energy E) the number n of *independent* contributions X_j to the fluctuating β is given by the number d of spatial dimensions of the space. The reason is that the momenta whose kinetic energy define β can fluctuate in each space direction independently. At the smallest scales, there is (by definition of the smallest scale) no other structure and degree of freedom except for the various directions in d-dimensional space. Using this (somewhat speculative) argument we obtain from eq. (6)

$$q - 1 = \frac{2}{n} = \frac{2}{d} = \frac{2}{9} \qquad \text{(at smallest scales)}, \tag{29}$$

hence $d = 9$ at the smallest scales. 9 spatial dimensions mean that space-time is 10-dimensional. Remarkably, we get the number of space-time dimensions that are necessary for superstring theory to be formulated in a consistent way [56].

The e^+e^- annihilation data actually already show a crossover from $q \approx 1$ to $q \approx 11/9$ at the electroweak energy scale of about 100 GeV [29, 30]. Should the saturation of q near the value 11/9 be confirmed by future experiments with larger center of mass energies $E > 161$ GeV, this could be interpreted as possible experimental evidence that the compactified extra dimensions of superstring theory start already becoming visible at the electroweak scale. This means, their diameter would be of the order of $r \sim \hbar c (100 GeV)^{-1}$, much larger than the Planck length. This is compatible with other theoretical arguments, which also suggest relatively large sizes of the extra dimensions [57]. If this picture is correct, then the ultimate reason for the validity of Tsallis statistics in the early universe could be fluctuations originating from the Planck era, where $q = 11/9$.

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