Non-additivity of Tsallis entropies and fluctuations of temperature

Christian Beck[1]

Isaac Newton Institute for Mathematical Sciences, University of Cambridge, 20 Clarkson Road, Cambridge CB3 0EH, UK

Abstract

We show that the non-additivity relation of the Tsallis entropies in nonextensive statistical mechanics has a simple physical interpretation for systems with fluctuating temperature or fluctuating energy dissipation rate. We also show that there is a distinguished dependence of the entropic index q on the spatial scale that makes the Tsallis entropies quasi-additive. Quasi-additivity implies that q is a strictly monotonously decreasing function of the spatial scale r, as indeed observed in various experiments.

[1] permanent address: School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS.

The formalism of nonextensive statistical mechanics has been developed over the past 13 years as a powerful and beautiful generalization of ordinary statistical mechanics [1]–[4]. It is based on the extremization of the Tsallis entropies

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right) \tag{1}$$

subject to suitable constraints. Here the p_i are the probabilities of the physical microstates, and q is the entropic index. The Tsallis entropies reduce to the Boltzmann Gibbs (or Shannon) entropy $S_1 = -\sum_i p_i \log p_i$ for $q \to 1$, hence ordinary statistical mechanics is contained as a special case in the generalized formalism. There is growing experimental evidence that $q \neq 1$ yields a correct decription of many complex physical phenomena, including hydrodynamic turbulence [5]–[8], scattering processes in particle physics [9, 10], and self-gravitating systems in astrophysics [11, 12], to name just a few. A complete list of references can be found in [4].

The Tsallis entropies are non-extensive. Given two independent subsystems I and II with probabilities p_i^I and p_i^{II}, respectively, the entropy of the composed system I+II (with probabilities $p_{ij}^{I+II} = p_i^I p_j^{II}$) satisfies

$$S_q^{I+II} = S_q^I + S_q^{II} - (q-1) S_q^I S_q^{II}. \tag{2}$$

Hence there is additivity for $q = 1$ only. This property has occasionally lead to unjustified prejudices, using arguments of the type 'entropy must be extensive'. In the following, however, we will see that the non-additivity property is not at all a 'bad' property, but rather a natural and physically consistent property for those types of systems where nonextensive statistical mechanics is expected to work.

Broadly speaking, the formalism with $q \neq 1$ has so far been observed to be relevant for two different classes of systems. One class is systems with long-range interactions (e.g. [11, 12, 13]), the other one is systems with fluctuations of temperature or of energy dissipation (e.g. [6, 9, 10, 14]). For the first class of systems, the concept of *independent* subsystems does not make any sense, since all subsystems are interacting. Hence there is no contradiction with the non-additivity of entropy for independent subsystems since independent subsystems do no exist for these systems (by definition of the long-range interaction).

Let us thus concentrate on the other class, systems with fluctuations. Consider a system of ordinary statistical mechanics with Hamiltonian H.

Tsallis statistics with $q > 1$ can arise from this ordinary Hamiltonian if one assumes that the temperature β^{-1} is locally fluctuating. From the integral representation of the gamma function one can easily derive the formula [7, 14]

$$(1+(q-1)\beta_0 H)^{-\frac{1}{q-1}} = \int_0^\infty e^{-\beta H} f(\beta) d\beta, \tag{3}$$

where

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left\{\frac{1}{(q-1)\beta_0}\right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{\beta}{(q-1)\beta_0}\right\} \tag{4}$$

is the probability density of the χ^2 distribution. The above formula is valid for arbitrary Hamiltonians H und thus of great significance. The left-hand side of eq. (3) is just the generalized Boltzmann factor emerging out of nonextensive statistical mechanics. It can directly be obtained by extremizing S_q. The right-hand side is a weighted average over Boltzmann factors of ordinary statistical mechanics. In other words, if we consider a nonequilibrim system (formally described by a fluctuating β), then the generalized distribution functions of nonextensive statistical mechanics are a consequence of integrating over all possible fluctuating inverse temperatures β, provided β is χ^2 distributed.

The χ^2 distribution is a universal distribution that occurs in many very common circumstances (see any statistics handbook on this, e.g. [15]). For example, it arises if β is the sum of squares of n Gaussian random variables, with $n = 2/(q-1)$. Hence one expects Tsallis statistics to be relevant in many applications. For fully developed turbulent hydrodynamic flows, where Tsallis statistics has been observed to work very well [6], β^{-1} is not the physical temperature of the flow but a formal temperature defined by the fluctuating energy dissipation rate times a time scale of the order of the Kolmogorov time [7]. In the application to scattering processes in collider experiments [9, 10], β^{-1} is a fluctuating inverse temperature near the Hagedorn phase transition.

The constant β_0 in eq. (4) is the average of the fluctuating β,

$$E(\beta) := \int_0^\infty \beta f(\beta) d\beta = \beta_0 \tag{5}$$

(E denotes the expectation with respect to $f(\beta)$). The deviation of q from 1 can be related to the relative variance of β. One has

$$q-1=\frac{E(\beta^2)-E(\beta)^2}{E(\beta)^2}. \tag{6}$$

We can now give physical meaning to eq. (2) for $q>1$. Consider two independent subsystems I and II that are composed into one system I+II. In system I (as well as II) the fluctuations of temperature T are expected to be larger than in I+II, due to the smaller size of I (or II) as compared to I+II. Remember that generally entropy is a measure of missing information on the system [16]. The entropy $S_q^I+S_q^{II}$ is larger than S_q^{I+II}, since in the single systems the probability distribultion of T is broader, thus our missing information on these systems is larger. Hence there must be a negative correction term to $S_q^I+S_q^{II}$ in eq. (2). It is physically plausible that this correction term is proportional a) to the relative strength of temperature fluctuations as given by $q-1$ and b) to the entropies S_q^I and S_q^{II} in the single systems. Hence we end up with eq. (2).

For most physical applications the fluctuations of β (or the relevant value of q) are observed to be dependent on the spatial scale r. For example, in the turbulence application detailed measurements of $q(r)$ have been presented in [6]. q turns out to be a strictly monotonously decreasing function of the distance r on which the velocity differences are measured. Similarly, for the application to e^+e^- annihilation [10], $q(r)$ turns out to be again a strictly monotonously descreasing function of the scale r, which in this case is given by $r\sim\hbar/E_{cm}$, where E_{cm} is the center of mass energy of the beam. Let us now present a theoretical argument why physical systems may like to choose a scale-dependent monotonously decreasing q.

The observation is that it is possible to make the Tsallis entropies quasi-additive by choosing different entropic indices at different scales. I.e., given a certain q for two small independent subsystems I and II we may choose another q' for the larger, composed system I+II such that

$$S_q^I+S_q^{II}=S_{q'}^{I+II}. \tag{7}$$

We may call this property *quasi-additivity.* For practical applications, q is often close to 1, so that a perturbative expansion in $q-1$ makes sense. One obtains

$$\sum_i p_i^q \;=\; \sum_i p_i e^{(q-1)\log p_i}$$

$$= \quad 1 + (q-1)\sum p_i \log p_i + \frac{1}{2}(q-1)^2 \sum p_i (\log p_i)^2 + \ldots, \quad (8)$$

where the dots denote higher-order terms in $q-1$. This yields for the sum of entropies of the two identical subsystems

$$\begin{aligned} S_q^I + S_q^{II} = 2S_q^I &= \frac{2}{q-1}(1 - \sum_i p_i^q) \\ &= -2\sum p_i \log p_i - (q-1)\sum p_i(\log p_i)^2 - \ldots \end{aligned} \quad (9)$$

On the other hand, by squaring eq. (8) one obtains

$$\begin{aligned} \left(\sum p_i^q\right)^2 &= 1 + 2(q-1)\sum p_i \log p_i \\ &+ (q-1)^2(\sum p_i \log p_i)^2 + (q-1)^2 \sum p_i (\log p_i)^2 + \ldots \end{aligned} \quad (10)$$

and hence the entropy of the composed system satisfies

$$\begin{aligned} S_{q'}^{I+II} &= \frac{1}{q'-1}\left(1 - \sum_{ij} p_{ij}^{q'}\right) \\ &= \frac{1}{q'-1}\left(1 - (\sum_i p_i^{q'})^2\right) \\ &= -2\sum p_i \log p_i - (q'-1)\left((\sum p_i \log p_i)^2 + \sum p_i (\log p_i)^2\right) \\ &\quad + \ldots \end{aligned} \quad (11)$$

Quasi-additivity thus implies a relation between q' and q, namely

$$\frac{q'-1}{q-1} = \frac{\sum p_i (\log p_i)^2}{\sum p_i(\log p_i)^2 + (\sum p_i \log p_i)^2}, \quad (12)$$

which can be written in the simple form

$$\frac{q'-1}{q-1} = \left(1 + \frac{\langle B_i \rangle^2}{\langle B_i^2 \rangle}\right)^{-1}. \quad (13)$$

Here $B_i := \log p_i$ is the so-called 'bit number' [16]. The negative expectation $-\langle B_i \rangle$ of the bit number is just the ordinary Boltzmann-Gibbs (or Shannon) entropy. We thus see that quasi-additive behaviour necessarily implies a change of q with system size. For q close to 1, this change is determined by eq. (13), which involves the average and variance of the bit number, i.e. quantities related to fluctuations of the Shannon entropy. It is now clear from eq. (13) that $q'-1$ (corresponding to the larger system) is smaller than $q-1$ (corresponding to the smaller system) for arbitrary probabilities p_i. In other words, $q(r)$ is a strictly monotonously decreasing function of the scale r, just as observed in the experiments.

References

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)

[2] S. Abe, Phys. Lett. **271A**, 74 (2000)

[3] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica **261A**, 534 (1998)

[4] http://tsallis.cat.cbpf.br/biblio.htm

[5] C. Beck, Physica **277A**, 115 (2000)

[6] C. Beck, G.S. Lewis and H.L. Swinney, Phys. Rev. **63E**, 035303(R) (2001)

[7] C. Beck, *On the small-scale statistics of Lagrangian turbulence*, preprint (2001)

[8] T. Arimitsu and N. Arimitsu, J. Phys. **33A**, L235 (2000)

[9] I. Bediaga, E.M.F. Curado, and J. Miranda, Physica **286A**, 156 (2000)

[10] C. Beck, Physica **286A**, 164 (2000)

[11] A.R. Plastino and A. Plastino, Phys. Lett. **174A**, 384 (1993)

[12] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, Astrophys. Lett. and Comm. **35**, 449 (1998)

[13] M. Antoni and S. Ruffo, Phys. Rev. **52E**, 2361 (1995)

[14] G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. **84**, 2770 (2000)

[15] N.A.J. Hastings and J.B. Peacock, *Statistical Distributions*, Butterworth, London (1974)

[16] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems*, Camdridge University Press, Cambridge (1993)