A QUICK INTRODUCTION TO FIBERED CATEGORIES AND TOPOLOGICAL STACKS

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Abstract. This is a quick introduction to fibered categories and topological stacks.

Contents

1. Categories fibered in groupoids

The formalism of categories fibered in groupoids provides a convenient framework for dealing with lax groupoid-valued functors. In this section, we recall some basic facts about categories fibered in groupoids.

We recall that a *groupoid* is a (small) category in which all morphisms are isomorphisms. A set, viewed as a category with only identity morphisms, is a groupoid. By abuse of terminology, we sometimes use the term *set* for any groupoid that is equivalent to a groupoid of this form (the correct terminology is *equivalence relation*).

Let T be a fixed category. An example to keep in mind is $\mathsf{T} = \mathsf{Top}$, the category of topological spaces. A **category fibered in groupoids** over T is a category $\mathcal{X}$ together with a functor $\pi\colon \mathcal{X} \to \mathsf{T}$ satisfying the following properties:

- **(i)** (Lifting arrows.) For every arrow $f\colon V \to U$ in T, and for every object X in $\mathcal{X}$ such that $\pi(X) = U$, there is an arrow $F\colon Y \to X$ in $\mathcal{X}$ such that $\pi(F) = f$.

(**ii**) (Lifting triangles.) Given a commutative triangle in T, and a partial lift for it to $\mathcal{X}$ as in the diagram

$$\begin{array}{ccccccc} Y & & & & V & & \\ & \overset{F}{\searrow} & & & & \overset{f}{\searrow} & \\ & & X & \xrightarrow{\pi} & h\downarrow & & U \\ & \underset{G}{\nearrow} & & & & \underset{g}{\nearrow} & \\ Z & & & & W & & \end{array}$$

there is a unique arrow $H: Y \to Z$ such that the left triangle commutes and $\pi(H) = h$.

We often drop the base functor π from the notation and denote a fibered category $\pi: \mathcal{X} \to \mathsf{T}$ by $\mathcal{X}$.

For a fixed object $T \in \mathsf{T}$, let $\mathcal{X}(T)$ denote the category of objects $X \in \mathcal{X}$ such that $\pi(X) = T$. Morphisms in $\mathcal{X}(T)$ are morphisms $F: X \to Y$ in $\mathcal{X}$ such that $\pi(F) = \mathrm{id}_T$. It is easy to see that $\mathcal{X}(T)$ is a groupoid. It is called the *fiber of* $\mathcal{X}$ *over* T. We sometimes abuse terminology and call $\mathcal{X}(T)$ the groupoids of T*-points* of $\mathcal{X}$. An object (resp., a morphism) in $\mathcal{X}(T)$ is called an object (resp., a morphism) over T.

Remark 1.1.

1. Conditions (**i**) and (**ii**) imply that, for every morphism $f: T' \to T$ in T, every object $X \in \mathcal{X}(T)$ has a "pull-back" $f^*(X)$ in $\mathcal{X}(T')$. The pull-back is unique up to a unique isomorphism. We sometimes denote $f^*(X)$ by $X|_{T'}$.
2. The pull-back functors f^* (whose definition involves making some choices) give rise to a lax groupoid-valued functor $T \mapsto \mathcal{X}(T)$. Conversely, given a lax groupoid-valued functor on T, it is possible to construct a category fibered in groupoids over T via the so-called Grothendieck construction.
3. A fibered category $\mathcal{X} \to \mathsf{T}$ whose fibers $\mathcal{X}(T)$ are sets gives rise to a presheaf of sets $T \mapsto \mathcal{X}(T)$. Conversely, every presheaf of sets over T gives rise, via the Grothendieck construction, to a category fibered in sets over T. The conclusion is that, categories fibered in sets over T are the same as presheaves of sets over T; see Example 1.2.**2** below.

Example 1.2.

1. Let $\mathsf{T} = \mathsf{Top}$, and let G be a topological group. Let $\mathcal{B}G$ be the category of principal G-bundles $P \to T$. A morphism in $\mathcal{B}G$ is, by definition, a G-equivariant cartesian diagram

$$\begin{array}{ccc} P' & \to & P \\ \downarrow & & \downarrow \\ T' & \to & T \end{array}$$

The base functor $\mathcal{B}G \to \mathsf{Top}$ is the forgetful functor that sends $P \to T$ to T. This makes $\mathcal{B}G$ a category fibered in groupoids over Top. The fiber $\mathcal{B}G(T)$ of $\mathcal{B}G$ over T is the groupoid of principal G-bundles over T.

2. Let $\mathsf{T} = \mathsf{Top}$, and let X be a topological space. Let $\mathcal{X}$ be the category of continuous maps $T \to X$. A morphism in $\mathcal{X}$ is a commutative triangle

$$\begin{array}{ccc} T' & \longrightarrow & T \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The forgetful functor that sends $T \to X$ to T makes $\mathcal{X}$ a category fibered in groupoids over Top.

The groupoid $\mathcal{X}(T)$ is in fact a set, namely, the set of continuous maps $T \to X$ (i.e., the set of T-points of X). The functor $T \mapsto \mathcal{X}(T)$ is a presheaf (in fact, a sheaf) of sets on Top.

Remark 1.3. There are two ways of thinking of a fibered category $\mathcal{X} \to \mathsf{T}$. One is to think of it as a device for cataloguing the objects parameterized by a *moduli problem* over T. In this case, an object $X \in \mathcal{X}(T)$ is viewed as a "family parameterized by T."

The second point of view is to think of $\mathcal{X}$ as some kind of a *space*. In this case, an object in $\mathcal{X}(T)$ is simply thought of as a T-valued point of $\mathcal{X}$, that is, a map from T to $\mathcal{X}$.

The Yoneda Lemma 1.4 clarifies the relation between the two points of view.

1.1. **The 2-category of fibered categories.** Categories fibered in groupoids over T form a 2-category. Let us explain how.

A *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ of fibered categories is a functor $f\colon \mathcal{X} \to \mathcal{Y}$ between the underlying categories such that $\pi_{\mathcal{Y}} \circ f = \pi_{\mathcal{X}}$. Given two such morphisms $f, g\colon \mathcal{X} \to \mathcal{Y}$, a *2-morphism* $\varphi\colon f \Rightarrow g$ between them is a natural transformation of functors φ from f to g such that the composition $\pi_{\mathcal{Y}} \circ \varphi$ is the identity transformation from $\pi_{\mathcal{X}}$ to itself.

With morphisms and 2-morphisms as above, categories fibered in groupoids over T form a 2-category $\mathfrak{Fib}_{\mathsf{T}}$. The 2-morphisms in $\mathfrak{Fib}_{\mathsf{T}}$ are automatically invertible. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called an *equivalence* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ such that $f \circ g$ and $g \circ f$ are 2-isomorphic to the corresponding identity morphisms. It is not uncommon in the literature to call two equivalent fibered categories isomorphic.

The construction in Example 1.2.2 can be performed in any category T and it gives rise to a functor $\mathsf{T} \to \mathfrak{Fib}_{\mathsf{T}}$. From now on, we will use the same notation for an object T in T and for its corresponding category fibered in groupoids. This is justified by the following Yoneda-type lemma (also see the ensuing paragraph).

Lemma 1.4 (Yoneda lemma)**.** *Let $\mathcal{X}$ be a category fibered in groupoids over T, and let T be an object in T. Then, the natural functor*

$$\mathrm{Hom}_{\mathfrak{Fib}_{\mathsf{T}}}(T, \mathcal{X}) \to \mathcal{X}(T)$$

is an equivalence of groupoids.

The functor in the above lemma is defined by sending $f\colon T \to \mathcal{X}$ to the image of the identity map $\mathrm{id}\colon T \to T$, viewed as an element in $T(T)$, under the map $f(T)\colon T(T) \to \mathcal{X}(T)$.

This lemma implies that the functor $\mathsf{T} \to \mathfrak{Fib}_{\mathsf{T}}$ is fully faithful. That is, we can think of the category T as a full subcategory of $\mathfrak{Fib}_{\mathsf{T}}$. For this reason, in the sequel we quite often do not distinguish between an object T and the fibered category associated to it.

1.2. **Fiber products and inner-homs in $\mathfrak{Fib}_{\mathsf{T}}$.** The 2-category $\mathfrak{Fib}_{\mathsf{T}}$ is closed under 2-fiber products and inner-homs. The 2-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is defined by the rule

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(T) := \mathcal{X}(T) \times_{\mathcal{Z}(T)} \mathcal{Y}(T).$$

The latter is 2-fiber product (or homotopty fiber product) in the 2-category of groupoids. Sometimes instead of saying 2-fiber product we simply say fiber product.

Given categories $\mathcal{X}$ and $\mathcal{Y}$ fibered over T, the inner-hom between them, denoted $\mathbf{Hom}(\mathcal{Y}, \mathcal{X})$, is defined by the rule

$$\mathbf{Hom}(\mathcal{Y}, \mathcal{X})(T) := \mathrm{Hom}(T \times \mathcal{Y}, \mathcal{X}).$$

The inner-hom has the expected exponential property. That is, given $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, we have a natural equivalence of fibered categories

$$\mathbf{Hom}(\mathcal{Z} \times \mathcal{Y}, \mathcal{X}) \cong \mathbf{Hom}(\mathcal{Z}, \mathbf{Hom}(\mathcal{Y}, \mathcal{X})).$$

1.3. **Descent condition.** To simplify the exposition, and to avoid the discussion of Grothendieck topologies, we will assume from now on that $\mathsf{T} = \mathsf{Top}$, with the Grothendieck topology being the usual open-cover topology. There is no subtlety in generalizing the discussion to arbitrary Grothendieck topologies.

We say that a category $\mathcal{X}$ fibered in groupoids over T is a **stack**, if the following two conditions are satisfied:

(i) (*Gluing morphisms.*) Given two objects X and Y in $\mathcal{X}$ over a fixed topological spaces T, morphisms between them form a sheaf. That is, the presheaf of sets on T defined by

$$U \mapsto \mathrm{Hom}_{\mathcal{X}(U)}(X|_U, Y|_U)$$

is a sheaf.

(ii) (*Gluing objects.*) Let T be a topological space, and let $\{U_i\}$ be an open cover of T. Assume that we are given objects $X_i \in \mathcal{X}(U_i)$, together with isomorphisms $\varphi_{ij} \colon X_j|_{U_i \cap U_j} \to X_i|_{U_i \cap U_j}$ in $\mathcal{X}(U_i \cap U_j)$ which satisfy the cocycle condition

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$, for every triple of indices i, j and k. Then, there is an object X over T, together with isomorphisms $\varphi_i \colon X|_{U_i} \to X_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$, for all i, j.

The data given in (**ii**) is usually called a *gluing data* or a *descent data*. It follows from (**i**) that the object X in (**ii**) is unique up to a unique isomorphism (respecting φ_i).

Stacks over T form a full sub 2-category $\mathfrak{St}_{\mathsf{T}}$ of $\mathfrak{Fib}_{\mathsf{T}}$ which is closed under fiber products and inner-homs. Any fibered category over T which is equivalent to a stack is itself a stack.

Example 1.5.

1. The fibered category $\mathcal{B}G$ of Example 1.2.1 is a stack. This is because one can glue principal G-bundles over a fixed space T using a gluing data (and the same thing is true for morphisms of principal G-bundles as well). The stack $\mathcal{B}G$ is called the *classifying stack of G*.

2. The fibered category $\mathcal{X}$ of Example 1.2.2 is a stack. This is because given a collection of continuous maps $f_i \colon U_i \to X$ on an open cover $\{U_i\}$ of T which are equal over the intersections $U_i \cap U_j$, we can uniquely glue them to a continuous map $f \colon T \to X$.

Note that the cocycle condition over triple intersections does not appear in Example 1.5.2. The reason for this is that the fiber groupoids $\mathcal{X}(U)$ are equivalent to sets. That is, if there is a morphisms between two objects in $\mathcal{X}(U)$, then it is necessarily unique. In fact, the functor $T \mapsto \mathcal{X}(T)$ is a sheaf of sets on Top.

Remark 1.6. A stack $\mathcal{X}$ over T whose fibers $\mathcal{X}(T)$ are sets gives rise to a sheaf of sets $T \mapsto \mathcal{X}(T)$ on T. Conversely, every sheaf of sets on T gives rise, via the Grothendieck construction, to a stack over T whose fibers are sets. The conclusion is that, stack over T whose fibers are sets are the same as sheaves of sets on T; see Example 1.5.**2**.

In view of Example 1.5.2 (and Lemma 1.4), the descent condition for a stack $\mathcal{X}$ can be interpreted as follows. Let T be a topological space, and let $\{U_i\}$ be an open cover of T. Assume we are given morphisms $f_i \colon U_i \to \mathcal{X}$, together with 2-isomorphisms $\varphi_{ij} \colon f_j|_{U_i \cap U_j} \Rightarrow f_i|_{U_i \cap U_j}$ satisfying the cocycle condition $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$. (This should be thought of as saying that φ_{ij} are "identifying f_i and f_j along $U_i \cap U_j$ in a coherent way.") Then, we can glue f_i to a global morphism $f \colon T \to \mathcal{X}$ whose restriction $f|_{U_i}$ to U_i is identified to f_i via a 2-isomorphism $\varphi_i \colon f|_{U_i} \Rightarrow f_i$. Furthermore, two such f and f' are 2-isomorphic via a unique $\Psi \colon f \to f'$ which intertwines φ_i and φ'_i.

1.4. **The stack associated to a category fibered in groupoids.** To any category $\mathcal{X}$ fibered in groupoids over T there is associated a stack $\mathcal{X}^+$ over T called the *stackification* of $\mathcal{X}$. This gives rise to a 2-functor from $\mathfrak{Fib}_{\mathsf{T}}$ to $\mathfrak{St}_{\mathsf{T}}$ which is left adjoint to the inclusion of $\mathfrak{St}_{\mathsf{T}}$ in $\mathfrak{Fib}_{\mathsf{T}}$. The construction of the stackification functor is rather messy and we avoid giving its details here.

When restricted to categories fibered in sets (i.e., presheaves of sets), the stackification functor coincides with the usual sheafification functor. This follows from the universal property of a left adjoint.

1.5. **Quotient stacks.** To any topological groupoid $\mathbb{X} = [X_1 \rightrightarrows X_0]$ one can associate a stack $[X_0/X_1]$ called the *quotient stack* of $\mathbb{X}$. A quick definition for the quotient stack is as follows. We define $[X_0/X_1]$ to be the stack associated to the (fibered category associated to the) presheaf of groupoids

$$T \mapsto [X_1(T) \rightrightarrows X_0(T)].$$

The quotient stack $[X_0/X_1]$ comes equipped with a natural quotient morphism $X_0 \to [X_0/X_1]$. The quotient morphism is an epimorphism. A morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of stacks is called an *epimorphism*, if it is an epimorphism in the sheaf-theoretic sense. That is, if every base extension $f_T \colon T \times_{\mathcal{Y}} \mathcal{X} \to T$ of f over a topological space T admits local sections.

Since we have not defined the stack associated to a category fibered in groupoids, we give an alternative description of $[X_0/X_1]$ in terms of principal bundles. We only discuss the case when $\mathbb{X}$ is the action groupoid $[X \times G \rightrightarrows X]$ of a topological group G acting on a topological space X (for the general case see [No1], § 12). In this case, the quotient stack is denoted by $[X/G]$.

For a topological space T, the groupoid $[X/G](T)$ of T-points of $[X/G]$ is the groupoid of pairs (P, φ), where P is a principal G-bundle over T, and $\varphi\colon P \to X$ is a G-equivariant map. The morphisms in $[X/G](T)$ are G-equivariant morphisms $f\colon P' \to P$ such that $\varphi' = \varphi \circ f$.

It is easy to verify that $[X/G]$ is a stack. When X is a point, the quotient stack $[*/G]$ coincides with $\mathcal{B}G$ of Example 1.2.**1**.

2. Topological stacks

We review some basic facts about topological stacks. More details can be found in [No1].

2.1. **Stacks over Top.** Throughout this section, by a stack we mean a stack over the site Top of topological spaces endowed with the open-cover Grothendieck topology. We list some basic facts about stacks.

1. The 2-category $\mathfrak{St}_{\mathsf{Top}}$ is a full sub 2-category of $\mathfrak{Fib}_{\mathsf{Top}}$. In particular, its 2-morphisms are invertible. Thus, given two stacks $\mathcal{X}$ and $\mathcal{Y}$, we have a *groupoid* $\mathrm{Hom}(\mathcal{Y}, \mathcal{X})$ of morphisms between them.

Although in practice one may really be interested only in the category of stacks obtained by identifying 2-isomorphic morphisms, the 2-category structure can not be ignored. For example, when we talk about *fiber products of stacks*, we exclusively mean the 2-fiber product in the 2-category of stacks.

2. The 2-category $\mathfrak{St}_{\mathsf{Top}}$, viewed as a sub 2-category of $\mathfrak{Fib}_{\mathsf{Top}}$, is closed under fiber products and inner-homs. In particular, we can talk about *mapping stacks* $\mathbf{Hom}(\mathcal{Y}, \mathcal{X})$. We have a natural equivalence of groupoids

$$\mathbf{Hom}(\mathcal{Y}, \mathcal{X})(*) \cong \mathrm{Hom}(\mathcal{Y}, \mathcal{X}),$$

where $*$ is a point.

3. The category Top of topological spaces embeds fully faithfully in $\mathfrak{St}_{\mathsf{Top}}$. This means that, given two topological spaces X and Y, viewed as stacks via the functor they represent, the hom-groupoid $\mathrm{Hom}(X, Y)$ is equivalent to a set, and this set is in natural bijection with the set of continuous functions from X to Y. This way, we can think of a topological space as a stack.

This embedding preserves the closed cartesian structure on Top. This means that fiber products of spaces get sent to 2-fiber products of the corresponding stacks, and the mapping spaces (with the compact-open topology) get sent to mapping stacks.[1]

4. The embedding of the Top in $\mathfrak{St}_{\mathsf{Top}}$ admits a left adjoint. That is, to every stack $\mathcal{X}$ one can associate a topological space $\mathcal{X}_{mod}$, together with a natural map $\pi\colon \mathcal{X} \to \mathcal{X}_{mod}$ which is universal among maps from $\mathcal{X}$ to topological spaces. (That is, every map from $\mathcal{X}$ to a topological space T factors uniquely through π.) See ([No1], §4.3) for more details.

The space $\mathcal{X}_{mod}$ is called the *coarse moduli space* of $\mathcal{X}$ and can be thought of as the "underlying space" of $\mathcal{X}$.

[1]To be precise, the last statement is true if we restrict to the site of compactly generated spaces.

The underlying set of $\mathcal{X}_{mod}$ is the set of isomorphism classes of the groupoid $\mathcal{X}(*)$, where $*$ stands for a point. In other words, the points of $\mathcal{X}_{mod}$ are the 2-isomorphism classes of *points* of $\mathcal{X}$, where by a point of $\mathcal{X}$ we mean a morphism $x\colon * \to \mathcal{X}$.

5. To a point $x\colon * \to \mathcal{X}$ of a stack $\mathcal{X}$ there is associated a group I_x, called the *inertia group* of $\mathcal{X}$ at x. By definition, I_x is the group of 2-isomorphisms from the point x to itself. An element in I_x is sometimes referred to as a *ghost* or *hidden loop*; see ([No1], §10). In the case where $\mathcal{X}$ is the quotient stack of a groupoid $[X_1 \rightrightarrows X_0]$, the stabilizer group of a point x in $\mathcal{X}$ is isomorphic to the stabilizer group of a lift $\tilde{x} \in X_0$ of x.

The groups I_x assemble into a stack $\mathcal{I}\mathcal{X} \to \mathcal{X}$ over $\mathcal{X}$ called the *inertia stack*. The inertia stack is defined by the following 2-fiber square

$$\begin{array}{ccc} \mathcal{I}\mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathcal{X} & \underset{\Delta}{\longrightarrow} & \mathcal{X} \times \mathcal{X} \end{array}$$

The map $\mathcal{I}\mathcal{X} \to \mathcal{X}$ is representable in the sense of §2.2 and makes $\mathcal{I}\mathcal{X}$ into a group stack over $\mathcal{X}$.

2.2. **Representable morphisms of stacks.** A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called **representable** if for every map $T \to \mathcal{Y}$ from a topological space T, the fiber product $T \times_{\mathcal{Y}} \mathcal{X}$ is a topological space. Roughly speaking, this is saying that the fibers of f are topological spaces.

Any property $\mathbf{P}$ of morphisms of topological spaces which is invariant under base change can be defined for an arbitrary representable morphism of stacks. More precisely, we say that a representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is $\mathbf{P}$, if for every map $T \to \mathcal{Y}$ from a topological space T, the base extension $f_T\colon T \times_{\mathcal{Y}} \mathcal{X} \to T$ is $\mathbf{P}$ as a map of topological spaces; see ([No1], §4.1).

This way we can talk about *embeddings (closed, open, locally closed, or arbitrary) of stacks, proper morphisms, finite morphisms*, and so on.

2.3. **Topological stacks.** A **topological** stack ([No1], Definition 7.1) is a stack $\mathcal{X}$ over Top which admits a representable epimorphism $p\colon X \to \mathcal{X}$ from a topological space X.[2] Such a morphism is called an *atlas* for $\mathcal{X}$. It follows that every morphism $T \to \mathcal{X}$ from a topological space T to a topological stack is representable ([No1], Corollary 7.3).

Every quotient stack $[X_0/X_1]$ of a topological groupoid is a topological stack, and the quotient map $X_0 \to [X_0/X_1]$ is an atlas for it. Conversely, given an atlas $X \to \mathcal{X}$ for a topological stack $\mathcal{X}$, $\mathcal{X}$ is equivalent to the quotient stack of the topological groupoid $[\mathrm{pr}_1, \mathrm{pr}_2\colon X \times_{\mathcal{X}} X \rightrightarrows X]$.

We list some basic facts about topological stacks.

1. Topological stacks form a full sub 2-category of $\mathfrak{St}_{\mathsf{Top}}$.

2. The 2-category $\mathfrak{TopSt}_{\mathsf{Top}}$ of topological stacks is closed under fiber products. It, however, does not seem to be closed under inner-homs. That is, it does not seem to be the case in general that the mapping stack $\mathbf{Hom}(\mathcal{Y}, \mathcal{X})$ of two topological stacks $\mathcal{X}$ and $\mathcal{Y}$ is a topological stack. This *is* the case,

[2]We use a different terminology here than [No1]. What we call a *topological* stack here is called *pretopological* in [ibid.].

however, if $\mathcal{Y}$ is the quotient stack of a groupoid $[Y_1 \rightrightarrows Y_0]$ such that Y_0 and Y_1 are compact topological spaces ([No3], Theorem 4.2). If Y_0 and Y_1 are only locally compact, the mapping stack $\mathbf{Hom}(\mathcal{Y}, \mathcal{X})$ (which we do not know if it is topological anymore) still admits a classifying space in the sense of §2.5. (The be precise, for these results to be true one needs to restrict to the site of compactly generated topological spaces.)

3. Let $\mathcal{X} = [X_0/X_1]$ be the quotient stack of a topological groupoid $[X_1 \rightrightarrows X_0]$. Then, the coarse moduli space of $\mathcal{X}$ is naturally homeomorphic to the coarse quotient space of the groupoid $[X_1 \rightrightarrows X_0]$. In particular, the coarse moduli space of the quotient stack $[X/G]$ is the orbit space X/G of the action of G on X. The coarse moduli space of the classifying stack $[*/G]$ of G is just a single point.

4. For a point $x\colon * \to \mathcal{X}$ of a topological stack $\mathcal{X}$, the inertia group I_x is naturally a topological group. The inertia stack $\mathcal{I}\mathcal{X}$ is a topological stack, and the natural map $\mathcal{I}\mathcal{X} \to \mathcal{X}$ is representable. It makes $\mathcal{I}\mathcal{X}$ a group stack over $\mathcal{X}$.

2.4. **Examples of topological stacks.** There are several general classes of topological stacks.

1. *Topological spaces.* The category of topological spaces is a full sub 2-category of the $\mathfrak{TopSt}_{\mathsf{Top}}$. Therefore, every topological space X can be regarded as a topological stack. Its coarse moduli space, as well as its inertia stack, are equal to X itself. In particular, the inertia group I_x of every point x on X is trivial. It is not true in general that a topological stack whose inertia groups are all trivial is a topological space.

2. *Orbifolds.* Every orbifold $\mathcal{X}$ is a topological stack which is locally isomorphic to a stack of the form $[U/G]$, with G a finite group acting on a manifold U. The coarse moduli space of $\mathcal{X}$ is the underlying topological space of $\mathcal{X}$ (so, locally it is U/G). The inertia group of a point x is the orbifold group (or the stabilizer group) of x. The inertia stack of $\mathcal{X}$ is the stack of twisted sectors of $\mathcal{X}$.

3. *Complexes-of-groups.* Every complex-of-group $\mathcal{X}$ gives rise naturally to a topological stack (which we denote again by $\mathcal{X}$).[3] The stack $\mathcal{X}$ is locally of the form U/G, where U is a polyhedral complex and G is a finite group acting on U preserving the polyhedral structure. The coarse moduli space of $\mathcal{X}$ is the underlying topological space of $\mathcal{X}$. The inertia group of a point x on $\mathcal{X}$ is isomorphic to the group attached to the lowest dimensional cell containing x. The inertia stack of $\mathcal{X}$ is again a complex-of-groups.

4. *Artin stacks over* $\mathbb{C}$. To any Artin stacks $\mathcal{X}$ of finite type over $\mathbb{C}$ one can associate a natural topological stack $\mathcal{X}^{top}$ ([No1], §20). The coarse moduli space of $\mathcal{X}^{top}$ is the underlying topological space of the coarse moduli space of $\mathcal{X}$ (if it exists). The inertia group of a point x in $\mathcal{X}^{top}$ is the underlying group of the algebraic inertia group of x. The inertia stack of $\mathcal{X}^{top}$ is the underlying topological stack of the algebraic inertia stack of $\mathcal{X}$, that is, $(\mathcal{I}\mathcal{X})^{top}$.

5. *Foliated manifolds.* To a foliated manifold $(M, \mathcal{F})$ we can associate two *leaf stacks*. One, let us denote it by $[M/\mathcal{F}]_h$, is the quotient stack of the

[3]The case of graphs-of-groups is discussed in [No1], §19.5

holonomy groupoid of the foliation (this is a Lie groupoid). The other, $[M/\mathcal{F}]_m$, is the quotient stack of the monodromy groupoid of the foliation (this is also a Lie groupoid). The coarse moduli spaces of both $[M/\mathcal{F}]_h$ and $[M/\mathcal{F}]_m$ are equal to the (naive) leaf space $M/\mathcal{F}$ of the foliation. The inertia group at a point of $[M/\mathcal{F}]_h$ is isomorphic to the holonomy group of the corresponding leaf in M. The inertia group at a point of $[M/\mathcal{F}]_m$ is isomorphic to the fundamental group of the corresponding leaf in M. There is a natural morphism of stacks $[M/\mathcal{F}]_m \to [M/\mathcal{F}]_h$ which induces the identity map on the coarse moduli spaces. The induced map on the inertia groups at a given point is the surjection map from the monodromy group to the holonomy group of the corresponding leaf.

I do not have a clear understanding of the structure nor the significance of the inertia stacks of either of these stacks.

6. *Gerbes.* Let G be a topological group, and let $\mathcal{X}$ be a G-gerbe over a topological space X. Then, $\mathcal{X}$ is a topological stack which is locally (on X) isomorphic (non-canonically) to $U \times BG$. The coarse moduli space of $\mathcal{X}$ is X and the inertia group at every point of $\mathcal{X}$ is (non-canonically) isomorphic to G. The inertia stack of $\mathcal{X}$ is in general no longer a gerbe on X. (For a description of the inertia stack of BG see [No1], Corollary 19.20.)

2.5. **Classifying space of a topological stack.** Let $\mathcal{X}$ be a stack over Top. By a **classifying space** for $\mathcal{X}$ we mean a topological space X together with a morphism $\varphi\colon X \to \mathcal{X}$ of stacks such that φ is a *universal weak equivalence*. The latter means that, for every morphisms $T \to \mathcal{X}$ from a topological space T, the base extension $\varphi_T\colon T \times_{\mathcal{X}} X \to T$ is a weak homotopy equivalence of topological spaces.

Theorem 2.1 ([No2], Theorem 6.2)**.** *Every topological stack $\mathcal{X}$ admits a classifying space.*

The classifying space X is unique up to a weak equivalence over $\mathcal{X}$ (which is itself unique up to weak equivalence). In particular, X calculates the weak homotopy type of $\mathcal{X}$. The significance of the map $\varphi\colon X \to \mathcal{X}$ is that it allows one to transport homotopy theoretic data from $\mathcal{X}$ to X.

Example 2.2. In the case where $\mathcal{X} = [X/G]$ is the quotient stack of a topological group action, the Borel construction $X \times_G EG$ is a classifying space for $\mathcal{X}$.

Remark 2.3. The classifying space $\varphi\colon X \to \mathcal{X}$ constructed in [No2], Theorem 6.2 has the stronger property than being a universal weak equivalence: whenever T is paracompact, the map $\varphi_T\colon T \times_{\mathcal{X}} X \to T$ admits a section and a fiberwise deformation retraction of $T \times_{\mathcal{X}} X$ onto the image of that section.

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