GROUP ACTIONS ON ALGEBRAIC STACKS VIA BUTTERFLIES

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Abstract. We introduce an explicit method for studying actions of a group stack $\mathcal{G}$ on an algebraic stack $\mathcal{X}$. As an example, we study in detail the case where $\mathcal{X} = \mathcal{P}(n_0, \cdots, n_r)$ is a weighted projective stack over an arbitrary base S. To this end, we give an explicit description of the group stack of automorphisms of $\mathcal{P}(n_0, \cdots, n_r)$, the *weighted projective general linear 2-group* $\mathrm{PGL}(n_0, \cdots, n_r)$. As an application, we use a result of Colliot-Thélène to show that for every linear algebraic group G over an arbitrary base field k (assumed to be reductive if $\mathrm{char}(k) > 0$) such that $\mathrm{Pic}(G) = 0$, every action of G on $\mathcal{P}(n_0, \cdots, n_r)$ lifts to a linear action of G on $\mathbb{A}^{r+1}$.

1. Introduction

The aim of this work is to propose a concrete method for studying group actions on algebraic stacks. Of course, in its full generality this problem could already be very difficult in the case of schemes. The case of stacks has yet an additional layer of difficulty due to the fact that stacks have two types of symmetries: 1-symmetries (i.e., self-equivalences) and 2-symmetries (i.e., 2-morphisms between self-equivalences).

Studying actions of a group stack $\mathcal{G}$ on a stack $\mathcal{X}$ can be divided into two subproblems. One, which is of geometric nature, is to understand the two types of symmetries alluded to above; these can be packaged in a group stack $\mathcal{A}ut\mathcal{X}$. The other, which is of homotopy theoretic nature, is to get a hold of morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$. Here, a morphism $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ means a weak monoidal functor; two morphisms $f, g \colon \mathcal{G} \to \mathcal{A}ut\mathcal{X}$ that are related by a monoidal transformation $\varphi \colon f \Rightarrow g$ should be regarded as giving rise to the "same" action.

Therefore, to study actions of $\mathcal{G}$ on $\mathcal{X}$ one needs to understand the group stack $\mathcal{A}ut\mathcal{X}$, the morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$, and also the transformations between such morphisms. Our proposed method, uses techniques from 2-group theory to tackle these problems. It consists of two steps:

1) finding suitable crossed module models for $\mathcal{A}ut\mathcal{X}$ and $\mathcal{G}$;
2) using butterflies [No3, AlNo1] to give a geometric description of morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ and monoidal transformations between them.

Finding a 'suitable' crossed module model for $\mathcal{A}ut\mathcal{X}$ may not always be easy, but we can go about it by choosing a suitable 'symmetric enough' atlas $X \to \mathcal{X}$. This can be used to find an approximation of $\mathcal{A}ut\mathcal{X}$ (Proposition 6.2), and if we are lucky (e.g., when $\mathcal{X} = \mathcal{P}(n_0, \cdots, n_r)$) it gives us the whole $\mathcal{A}ut\mathcal{X}$.

Once crossed module models for $\mathcal{G}$ and $\mathcal{A}ut\mathcal{X}$ are found, the butterfly method reduces the action problem to standard problems about group homomorphisms and group extensions, which can be tackled using techniques from group theory.

Organization of the paper

Sections §3–§5 are devoted to setting up the basic homotopy theory of 2-group actions and using butterflies to formulate our strategy for studying actions. To illustrate our method, in the subsequent sections we apply these ideas to study group actions on weighted projective stacks. In §6 we define weighted projective general linear 2-groups $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ and prove (see Theorem 6.3) that they model $\mathcal{A}ut\mathcal{P}(n_0, \cdots, n_r)$; we prove this over any base scheme S, generalizing the case $S = \operatorname{Spec}\mathbb{C}$ proved in [BeNo]:

Theorem 1.1. *Let $\mathcal{A}ut\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ be the group stack of automorphisms of the weighted projective stack $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ relative to an arbitrary base scheme S. Then, there is a natural equivalence of group stacks*

$$\mathcal{PGL}_S(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}_S(n_0, n_1, \cdots, n_r).$$

Here, $\mathcal{PGL}_S(n_0, n_1, \cdots, n_r)$ stands for the group stack associated to the crossed module $\mathrm{PGL}_S(n_0, \cdots, n_r)$ (see §6 for the defnition).

We analyze the structure of $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ in detail in §7. In Theorem 7.7 we make explicit the structure of $\mathrm{PGL}(n_0, \cdots, n_r) = [\mathbb{G}_m \to G]$ by writing G as a semidirect product of a reductive part (product of general linear groups) and a unipotent part (successive semidirect product of linear affine groups).

In light of the two step approach discussed above, Theorems 6.3 and 7.7 enable us to study actions of group schemes (or group stacks, for that matter) on weighted projective stacks in an explicit manner. This is discussed in §9. We classify actions of a group scheme G on $\mathcal{P}_S(n_0, n_1, \cdots, n_r)$ in terms of certain central extensions of G by the multiplicative group $\mathbb{G}_m$. We also describe the stack structure of the corresponding quotient (2-)stacks. As a consequence, we obtain the following (see Theorem 9.1).

Theorem 1.2. *Let k be a field and G a connected linear algebraic group over k, assumed to be reductive if $\mathrm{char}(k) > 0$. Let $\mathcal{X} = \mathcal{P}(n_0, n_1, \cdots, n_r)$ be a wighted projective stack over k. Suppose that $\mathrm{Pic}(G) = 0$. Then, every action of G on $\mathcal{X}$ lifts to a linear action of G on $\mathbb{A}^{r+1}$.*

In a forthcoming paper, we use the results of this paper (more specifically, Theorem 6.3), together with the results of [No2], to give a complete classification, and explicit construction, of twisted forms of weighted projective stacks; these are the weighted analogues of Brauer-Severi varieties.

Contents

2. Notation and terminology

Our notation for 2-groups and crossed modules is that of [No1] and [No3], to which the reader is referred to for more on 2-group theory relevant to this work. In particular, we use mathfrak letters $\mathfrak{G}$, $\mathfrak{H}$ for 2-groups or crossed modules. By a weak 2-group we mean a strict monoidal category $\mathfrak{G}$ with weak inverses (Definition 3.1). If the inverses are also strict, we call $\mathfrak{G}$ a strict 2-group.

By a stack we mean a presheaf of groupoids (and not a category fibered in groupoids) over a Grothendieck site that satisfies the decent condition. We use mathcal letters $\mathcal{X}$, $\mathcal{Y}$,... for stacks.

Given a presheaf of groupoids $\mathcal{X}$ over a site, its stackification is denoted by $\mathcal{X}^a$. We use the same notation for the sheafification of a presheaf of sets (or groups).

The m-dimensional general linear group scheme over Spec R is denoted by $\mathrm{GL}(m, R)$. When $R = \mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. The corresponding projectivized general linear group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ for a weighted projective general linear 2-group (§6) because in the latter case we always assume $r \geq 1$.

3. Review of 2-groups and crossed modules

A *strict 2-group* is a group object in the category of groupoids. Equivalently, a strict 2-group is a strict monoidal groupoid $\mathfrak{G}$ in which every object has a strict inverse; that is, multiplication by an object induces an isomorphism from $\mathfrak{G}$ onto itself. A morphism of 2-groups is, by definition, a strict monoidal functor.

The weak 2-groups we will encounter in this paper are less weak than the ones discussed in [No1, No3]. We hope that this change in terminology is not too confusing for the reader.

Definition 3.1. A *weak 2-group* is a strict monoidal groupoid $\mathfrak{G}$ in which multiplication by an object induces an equivalence of categories from $\mathfrak{G}$ to itself. By a morphism of weak 2-groups we mean a strict monoidal functor. By a *weak morphism* we mean a weak monoidal functor. (We will not encounter weak morphisms until later sections.)

The set of isomorphism classes of objects in a 2-group $\mathfrak{G}$ is denoted by $\pi_0\mathfrak{G}$; this is a group. The automorphism group of the identity object $1 \in \mathrm{Ob}\,\mathfrak{G}$ is denoted by $\pi_1\mathfrak{G}$; this is an abelian group.

Weak 2-groups and strict monoidal functors between them form a category **W2Gp** which contains the category **2Gp** of strict 2-groups as a full subcategory.[1] Morphisms in **W2Gp** induce group homomorphisms on π_0 and π_1. In other words, we have functors $\pi_0, \pi_1 \colon \mathbf{W2Gp} \to \mathbf{Gp}$; the functor π_1 indeed lands in the full subcategory of abelian groups. A morphism between weak 2-groups is called an *equivalence* if the induced homomorphisms on π_0 and π_1 are isomorphisms. Note that an equivalence may not have an inverse.

The following lemma is straightforward.

Lemma 3.2. *Let $f \colon \mathfrak{H} \to \mathfrak{G}$ be a morphism of weak 2-groups. Then f, viewed as a morphism of underlying groupoids, is fully faithful if and only if $\pi_0 f \colon \pi_0\mathfrak{H} \to \pi_0\mathfrak{G}$ is injective and $\pi_1 f \colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$ is an isomorphism. It is an equivalence of groupoids if and only if both $\pi_0 f$ and $\pi_1 f$ are isomorphisms.*

A *crossed module* $\mathfrak{G} = [\partial \colon G_1 \to G_0]$ consists of a pair of groups G_0 and G_1, a group homomorphism $\partial \colon G_1 \to G_0$, and a (right) action of G_0 on G_1, denoted $-^a$. This action lifts the conjugation action of G_0 on the image of ∂ and descends the conjugation action of G_1 on itself. In other words, the following axioms are satisfied:

- $\forall \beta \in G_1, \forall a \in G_0,\ \partial(\beta^a) = a^{-1}\partial(\beta)a$;
- $\forall \alpha, \beta \in G_1,\ \beta^{\partial(\alpha)} = \alpha^{-1}\beta\alpha$.

It is easy to see that the kernel of ∂ is a central (in particular abelian) subgroup of G_1; we denote this abelian group by $\pi_1\mathfrak{G}$. The image of ∂ is always a normal subgroup of G_0; we denote the cokernel of ∂ by $\pi_0\mathfrak{G}$. A morphism of crossed modules is a pair of group homomorphisms which commute with the ∂ maps and respect the actions. Such a morphism induces group homomorphisms on π_0 and π_1.

Crossed modules and morphisms between them form a category, which we denote by **XMod**. We have functors $\pi_0, \pi_1 \colon \mathbf{XMod} \to \mathbf{Gp}$; the functor π_1 indeed lands in the full subcategory of abelian groups. A morphism in **XMod** is said to be an *equivalence* if it induces isomorphisms on π_0 and π_1. Note that an equivalence may not have an inverse.

There is a well-known natural equivalence of categories $\mathbf{2Gp} \cong \mathbf{XMod}$; see [No1], §3.3. This equivalence respects the functors π_0 and π_1. This way, we can think of a crossed module as a strict 2-group, and vice versa. For this reason, we may sometimes abuse terminology and use the term (strict) 2-group for an object which is actually a crossed module; we hope that this will not cause any confusion. Note that **W2Gp** contains **2Gp** as a full subcategory.

4. 2-groups over a site and group stacks

First a few words on terminology. For us a stack is presheaf of groupoids over a Grothendieck site (and not a category fibered in groupoids) that satisfies the descent condition. This may be somewhat unusual for algebraic geometers who are used to categories fibered in groupoids, but it makes the exposition simpler. Of course, it is standard that this point of view is equivalent to the one via categories fibered in groupoids. Just to recall how this equivalence works, to any category fibered in groupoids $\mathcal{X}$ one can associate a presheaf $\underline{\mathcal{X}}$ of groupoids over C which is defined

[1] Both **W2Gp** and **2Gp** are *2-categories* but we will ignore the 2-morphisms for the time being and only look at the underlying 1-category.

as follows. By definition, $\underline{\mathcal{X}}$ is the presheaf that assigns to an object $U \in \mathsf{C}$ the groupoids $\underline{\mathcal{X}}(U) := \mathrm{Hom}(\underline{U}, \mathcal{X})$, where $\underline{U}$ stands for the presheaf of sets represented by U and Hom is computed in the category of stacks over C. Conversely, to any presheaf of groupoids one associates a category fibered in groupoids defined via the Grothendieck construction. For more on this we refer the reader to [Ho], especially §5.2.

4.1. **Presheaves of weak 2-groups over a site.** Let C be a Grothendieck site. Let $\mathbf{W2Gp_C}$ be the category of presheaves of weak 2-groups over C; that is, the category of contravariant functors from C to $\mathbf{W2Gp}$. We define $\mathbf{2Gp_C}$ and $\mathbf{XMod_C}$ analogously. There is a natural equivalence of categories $\mathbf{2Gp_C} \simeq \mathbf{XMod_C}$. In particular, we can think of a presheaf of crossed modules as a presheaf of (strict) 2-groups. Note that $\mathbf{W2Gp_C}$ contains $\mathbf{2Gp_C}$ as a full subcategory.

Let $\mathcal{X}$ be a presheaf of groupoids over C. To $\mathcal{X}$ we associate a presheaf of weak 2-groups $\mathcal{A}ut\mathcal{X} \in \mathbf{W2Gp_C}$ which parametrizes auto-equivalences of $\mathcal{X}$. By definition, $\mathcal{A}ut\mathcal{X}$ is the functor that associates to an object U in C the weak 2-group of self-equivalences of $\mathcal{X}_U$, where $\mathcal{X}_U$ is the restriction of $\mathcal{X}$ to the comma category C_U. (The 'comma category', or the 'over category', C_U is the category of objects in C over U.) Notice that in the case where $\mathcal{X}$ is a stack, $\mathcal{A}ut\mathcal{X}$, viewed as a presheaf of groupoids, is also a stack. Indeed, $\mathcal{A}ut\mathcal{X}$ is almost a group object in the category of stacks over C. To be more precise, $\mathcal{A}ut\mathcal{X}$ is a group stack in the sense of Definition 4.1 below.

Let $\underline{\mathfrak{G}} \in \mathbf{W2Gp_C}$ be a presheaf of weak 2-groups on C. We define $\pi_0^{pre}\underline{\mathfrak{G}}$ to be the presheaf $U \mapsto \pi_0\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_0\underline{\mathfrak{G}}$ to be the sheaf associated to $\pi_0^{pre}\underline{\mathfrak{G}}$. Similarly, $\pi_1^{pre}\underline{\mathfrak{G}}$ is defined to be the presheaf $U \mapsto \pi_1\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_1\underline{\mathfrak{G}}$ to be the sheaf associated to $\pi_1^{pre}\underline{\mathfrak{G}}$.

We define $\pi_0\underline{\mathfrak{G}}$ and $\pi_1\underline{\mathfrak{G}}$ for a presheaf of crossed modules $\underline{\mathfrak{G}} \in \mathbf{XMod_C}$ in a similar manner. The equivalence of categories between $\mathbf{2Gp_C}$ and $\mathbf{XMod_C}$ respects π_0^{pre}, π_0, π_1^{pre} and π_1. Lemma 3.2 remains valid in this setting if instead of π_0 and π_1 we use π_0^{pre} and π_1^{pre}.

4.2. **Group stacks over C.** We recall the definition of a group stack from [Bre]. We modify Breen's definition by assuming that our group stacks are strictly associative and have strict units. This is all we will need because the group stack $\mathcal{A}ut\mathcal{X}$ of self-equivalences of a stack $\mathcal{X}$ (indeed, any presheaf of groupoids $\mathcal{X}$) has this property, and that is all we are concerned with in this paper.

Definition 4.1 ([Bre], page 19)**.** Let C be a Grothendieck site. By a *group stack* over C we mean a stack $\mathcal{G}$ that is a strict monoid object in the category of stacks over C and for which weak inverses exist. By a *morphism* of group stacks we mean a strict monoidal functor. That is, a morphism of stacks that strictly respects the monoidal structures. By a *weak morphism* we mean a weak monoidal functor.

The condition on existence of weak inverses means that for every $U \in \mathrm{Ob}\,\mathsf{C}$ and every object a in the groupoid $\mathcal{G}(U)$, multiplication by a induces an equivalence of categories from $\mathcal{G}(U)$ to itself (or equivalently, an equivalence of stacks from $\mathcal{X}_U$ to itself). This condition is equivalent to saying that, for every $U \in \mathrm{Ob}\,\mathsf{C}$, $\mathcal{X}(U)$ is a weak 2-group. More concisely, it is equivalent to

$$\mathcal{G} \times \mathcal{G} \stackrel{(pr,mult)}{\longrightarrow} \mathcal{G} \times \mathcal{G}$$

being an equivalence of stacks.

Remark 4.2. It is well known that a weak group stack can always be strictified to a strict one. So, theoretically speaking, the strictness of monoidal structure in Definition 4.1 is not restrictive. However, given fixed (strict) group stacks $\mathcal{G}$ and $\mathcal{H}$, strict morphisms $\mathcal{H} \to \mathcal{G}$ are *not* adequate. We will see in the subsequent sections that when studying group actions on stacks we can not avoid weak morphisms. In this section, however, we will only discuss strict morphisms.

Let $\mathbf{grSt}_{\mathsf{C}}$ be the category of group stacks and strict morphisms between them (Definition 4.1); this is naturally a full subcategory of $\mathbf{W2Gp}_{\mathsf{C}}$. There are natural functors

$$\mathbf{W2Gp}_{\mathsf{C}} \to \mathbf{grSt}_{\mathsf{C}} \ \text{ and } \ \mathbf{XMod}_{\mathsf{C}} \to \mathbf{grSt}_{\mathsf{C}}.$$

The former is the stackification functor that sends a presheaf of groupoids to its associated stack; note that since the stackification functor preserves products, we can carry over the monoidal structure from a presheaf of groupoids to its stackification. The latter functor is obtained from the former by precomposing with the natural fully faithful functor $\mathbf{XMod}_{\mathsf{C}} \to \mathbf{W2Gp}_{\mathsf{C}}$ (see the beginning of §4.1). Given a presheaf of crossed modules $[\partial \colon \underline{G_1} \to \underline{G_0}]$, the associated group stack has as underlying stack the quotient stack $[\underline{G_0}/\underline{G_1}]$, where $\underline{G_1}$ acts on $\underline{G_0}$ by multiplication on the right (via ∂).

Definition 4.3. Let $\mathcal{X}$ be a presheaf of groupoids over C. We define $\pi^{pre}\mathcal{X}$ to be the presheaf that sends an object U in C to the set of isomorphism classes in $\mathcal{X}(U)$. We denote the sheaf associated to $\pi^{pre}\mathcal{X}$ by $\pi\mathcal{X}$. For a global section e of $\mathcal{X}$, we define $\underline{\mathrm{Aut}}_{\mathcal{X}}(e)$ to be sheaf associated to the presheaf that sends an object U in C to the group of automorphisms, in the groupoid $\mathcal{X}(U)$, of the object e_U; note that when $\mathcal{X}$ is a stack this presheaf is already a sheaf and no sheafification is needed.

Note that when $\mathcal{G}$ is the underlying presheaf of groupoids of a presheaf of weak 2-groups $\underline{\mathfrak{G}} \in \mathbf{W2Gp}_{\mathsf{C}}$, then $\pi_0\underline{\mathfrak{G}} = \pi\mathcal{G}$ and $\pi_1\underline{\mathfrak{G}} = \underline{\mathrm{Aut}}_{\mathcal{G}}(e)$, where e is the identity section of $\mathcal{G}$.

4.3. **Equivalences of group stacks.** There are two ways of defining the notion of equivalence between group stacks. One way is to regard them as stacks and use the usual notion of equivalence of stacks. The other is to regard them as presheaves of weak 2-groups and use π_0 and π_1 (see §4.1). The next lemma shows that these two definitions agree.

Lemma 4.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be group stacks, and let $f \colon \mathcal{H} \to \mathcal{G}$ be a morphism of group stacks. Then, the following are equivalent:*

- (*i*) *f is an equivalence of stacks;*
- (*ii*) *The induced maps $\pi_0 f \colon \pi_0^{pre}\mathcal{H} \to \pi_0^{pre}\mathcal{G}$ and $\pi_1 f \colon \pi_1^{pre}\mathcal{H} \to \pi_1^{pre}\mathcal{G}$ are isomorphisms of presheaves of groups;*
- (*iii*) *The induced maps $\pi_0 f \colon \pi_0\mathcal{H} \to \pi_0\mathcal{G}$ and $\pi_1 f \colon \pi_1\mathcal{H} \to \pi_1\mathcal{G}$ are isomorphisms of sheaves of groups.*

Proof. The only non-trivial implication is $(iii) \Rightarrow (ii)$. In the proof we will use the following standard fact from closed model category theory.

> **Theorem** ([Hi], Theorem 3.2.13)**.** Let $\mathcal{M}$ be a closed model category, L a localizaing class of morphisms in $\mathcal{M}$, and $\mathcal{M}_L$ the localized model category. Let $\mathcal{X}$ and $\mathcal{Y}$ be fibrant objects (i.e., L-local objects) in $\mathcal{M}_L$, and let $f \colon \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathcal{M}$ that is a weak

equivalence in the localized model structure $\mathcal{M}_L$ (that is, f is an L-local weak equivalence). Then, f is a weak equivalence in $\mathcal{M}$.

We will apply the above theorem with $\mathcal{M}$ being the model structure on the category $\mathbf{Gpd}_{\mathsf{C}}$ of presheaves of groupoids on C in which weak equivalences are morphisms that induce isomorphisms (of presheaves of groups) on π_0^{pre} and π_1^{pre}, and fibrations are objectwise. We take L to be the class of hypercovers. The weak equivalences in the localized model structure will then be the ones inducing isomorphism (of sheaves of groups) on π_0 and π_1. The main reference for this is [Ho].

Let us now prove $(iii) \Rightarrow (ii)$. It is shown in [Ho] that $\mathcal{G}$ and $\mathcal{H}$ are L-local objects (see §5.2 and §7.3 of [ibid.]). By hypothesis, f induces isomorphisms (of sheaves) on π_0 and π_1, so it is a weak equivalence in the localized model structure. Therefore, since $\mathcal{G}$ and $\mathcal{H}$ are L-local, f is already a weak equivalence in the non-localized model structure. This exactly means that $\pi_0 f\colon \pi_0^{pre}\mathcal{H} \to \pi_0^{pre}\mathcal{G}$ and $\pi_1 f\colon \pi_1^{pre}\mathcal{H} \to \pi_1^{pre}\mathcal{G}$ are isomorphisms of presheaves. □

Lemma 4.5. *Let $\mathcal{X}$ be a presheaf of groupoids over C and $\varphi\colon \mathcal{X} \to \mathcal{X}^a$ its stackification. Then, we have the following (see Definition 4.3 for notation):*

- *(i) The induced morphism $\pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is an isomorphism of sheaves of sets;*
- *(ii) For every global section e of $\mathcal{X}$, the natural map $\underline{\mathrm{Aut}}_{\mathcal{X}}(e) \to \underline{\mathrm{Aut}}_{\mathcal{X}^a}(e)$ is an isomorphism of sheaves of groups.*

Proof. This is a simple sheaf theory exercise. We include the proof of (i). Proof of (ii) is similar.

First we prove that $\pi\varphi\colon \pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is injective. Let $U \in \mathrm{Ob}\,\mathsf{C}$, and let x, y be element in $\pi\mathcal{X}(U)$ such that $\pi\varphi(x) = \pi\varphi(y)$. We have to show that $x = y$. By passing to a cover of U, we may assume x and y lift to objects $\bar{x}$ and $\bar{y}$ in $\mathcal{X}(U)$. We will show that there is an open cover of U over which $\bar{x}$ and $\bar{y}$ become isomorphic. Since $\varphi(\bar{x})$ and $\varphi(\bar{y})$ become equal in $\pi(\mathcal{X}^a)$, there is a cover $\{U_i\}$ of U such that there is an isomorphism $\alpha_i\colon \varphi(\bar{x}|_{U_i}) \xrightarrow{\sim} \varphi(\bar{y}|_{U_i})$ in the groupoid $\mathcal{X}^a(U_i)$, for every i. By replacing $\{U_i\}$ with a finer cover, we may assume that α_i come from $\mathcal{X}(U_i)$. (More precisely, $\alpha_i = \varphi(\beta_i)$, where β_i is a morphism in the groupoid $\mathcal{X}(U_i)$.) This implies that, for every i, $\bar{x}|_{U_i}$ and $\bar{y}|_{U_i}$ are isomorphic as objects of the groupoid $\mathcal{X}(U_i)$. This is exactly what we wanted to prove.

Having proved the injectivity, to prove the surjectivity it is enough to show that every object x in $\pi(\mathcal{X}^a)(U)$ is in the image of φ, possibly after replacing U by an open cover. By choosing an appropriate cover, we may assume x lifts to $\mathcal{X}^a(U)$. Since $\mathcal{X}^a$ is the stackification of $\mathcal{X}$, we may assume, after refining our cover, that x is in the image of $\mathcal{X}(U) \to \mathcal{X}^a(U)$. The claim is now immediate. □

Lemma 4.6. *Let $\underline{\mathfrak{G}} = [\underline{G_1} \to \underline{G_0}]$ be a presheaf of crossed modules, and let $\mathcal{G} = [\underline{G_0}/\underline{G_1}]$ be the corresponding group stack. Then, we have natural isomorphisms of sheaves of groups $\pi_i\underline{\mathfrak{G}} \xrightarrow{\sim} \pi_i\mathcal{G}$, $i = 1, 2$.*

Proof. Apply Lemma 4.5. □

5. Actions of group stacks

In this section we present an interpretation of an action of a group stack on a stack in terms of butterflies. We begin with the definition of an action.

Definition 5.1. Let $\mathcal{X}$ be a stack and $\mathcal{G}$ a group stack. By an *action* of $\mathcal{G}$ on $\mathcal{X}$ we mean a weak morphism $f\colon \mathcal{G} \to \mathcal{A}ut\mathcal{X}$. We say two actions f and f' are *equivalent* if there is a monoidal transformation $\varphi\colon f \to f'$.

In the case where G is a group (over the base site), it is easy to see that our definition of action is equivalent to Definitions 1.3.(i) of [Ro]. A monoidal transformation $\varphi\colon f \to f'$ between two such actions is the same as the structure of a *morphism of G-groupoids* (in the sense of Definitions 1.3.(ii) of [Ro]) on the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$, where the source and the target are endowed with the G-groupoid structures coming from the actions f and f', respectively.

5.1. **Formulation in terms of crossed modules and butterflies.** Butterflies were introduced in [No3] as a convenient way of encoding weak morphisms between 2-groups (rather, crossed modules representing the 2-groups). The theory was further extended in [AlNo1] to the relative case (over a Grothendieck site). We will use this theory to translate problems about 2-group actions on stacks to certain group extension problems.

We begin by recalling the definition of a butterfly (see [No3], Definition 8.1 and [AlNo1], §4.1.3).

Definition 5.2. Let $\mathfrak{G} = [\varphi\colon G_1 \to G_0]$ and $\mathfrak{H} = [\psi\colon H_1 \to H_0]$ be crossed modules. By a *butterfly* from $\mathfrak{H}$ to $\mathfrak{G}$ we mean a commutative diagram of groups

$$\begin{array}{ccccc} H_1 & & & & G_1 \\ & \overset{\kappa}{\searrow} & & \overset{\iota}{\swarrow} & \\ {\scriptstyle\psi}\downarrow & & E & & \downarrow{\scriptstyle\varphi} \\ & \underset{\sigma}{\swarrow} & & \underset{\rho}{\searrow} & \\ H_0 & & & & G_0 \end{array}$$

in which both diagonal sequences are complexes, and the NE-SW sequence, that is, $G_1 \to E \to H_0$, is short exact. We require that ρ and σ satisfy the following compatibility with actions. For every $x \in E$, $\alpha \in G_1$, and $\beta \in H_1$,

$$\iota(\alpha^{\rho(x)}) = x^{-1}\iota(\alpha)x, \quad \kappa(\beta^{\sigma(x)}) = x^{-1}\kappa(\beta)x.$$

A *morphism* between two butterflies $(E, \rho, \sigma, \iota, \kappa)$ and $(E', \rho', \sigma', \iota', \kappa')$ is a morphism $f\colon E \to E'$ commuting with all four maps (it is easy to see that such an f is necessarily an isomorphism). We define $\mathrm{B}(\mathfrak{H}, \mathfrak{G})$ to be the groupoid of butterflies from $\mathfrak{H}$ to $\mathfrak{G}$.

This definition is justified by the following result (see [No3] and [AlNo1]).

Theorem 5.3. *Let $\mathfrak{G} = [G_1 \to G_0]$ and $\mathfrak{H} = [H_1 \to H_0]$ be crossed modules of sheaves of groups over the cite C. Let $\mathcal{G} = [G_0/G_1]$ and $\mathcal{H} = [H_0/H_1]$ be the corresponding quotient group stacks. Then, there is a natural equivalence of groupoids*

$$\mathrm{B}(\mathfrak{H}, \mathfrak{G}) \cong \mathrm{Hom}_{\mathrm{weak}}(\mathcal{H}, \mathcal{G}).$$

Here, the right hand side stands for the groupoid whose objects are weak morphisms of group stacks and whose morphisms are monoidal transformations.

The above result can be interpreted as follows. A butterfly as in the theorem gives rise to a canonical zigzag in $\mathbf{XMod}_{\mathsf{C}}$

$$\mathfrak{H} \overset{\sim}{\longleftarrow} \mathfrak{E} \to \mathfrak{G},$$

where $\mathfrak{E} = [H_1 \times G_1 \xrightarrow{\kappa \cdot \iota} E]$. After passing to the associated stacks, it gives rise to a zigzag in $\mathbf{grSt}_{\mathsf{C}}$

$$\mathcal{H} \xleftarrow{\sim} \mathcal{E} \to \mathcal{G},$$

which after inverting the left map (as a weak morphism), results in a weak morphism $\mathcal{H} \to \mathcal{G}$. It follows from this description of a butterfly that π_0 and π_1 are functorial with respect to butterflies. Furthermore, the equivalence of Theorem 5.3 respects π_0 and π_1 (see Lemma 4.6).

When $[G_1 \to G_0]$ is a crossed module model for the group stack $\mathcal{A}ut\mathcal{X}$ of auto-equivalences of a stack $\mathcal{X}$, then it follows from Theorem 5.3 that an action of $\mathcal{H} = [H_0/H_1]$ on $\mathcal{X}$ is the same thing as an isomorphism class of a butterfly as in Definition 5.2. In other words, to give an action of $\mathcal{H}$ on $\mathcal{X}$, we need to find an extension E of H_0 by G_1, together with group homomorphisms $\kappa\colon H_1 \to E$ and $\rho\colon E \to G_0$ satisfying the conditions of Definition 5.2. This summarizes our strategy for studying group actions on stacks. To show its usefulness, in the subsequent sections we will apply this method to the case where $\mathcal{X} = \mathcal{P}_S(n_0, n_1, \cdots, n_r)$ is a weighted projective stack over a base scheme S.

6. Weighted projective general linear 2-groups $PGL(n_0, n_1, ..., n_r)$

In this section we introduce weighted projective general linear 2-group schemes and prove that they model self-equivalences of weighted projective stacks (Theorem 6.3).

We begin by some general observations about automorphism 2-groups of quotient stacks. From now on, we assume that $\mathsf{C} = \mathbf{Sch}_S$ is the big site of schemes over a base scheme S, endowed with a subcanonical topology (say, étale, Zariski, fppf, fpqc, etc.).

6.1. **Automorphism 2-group of a quotient stack.** We define a *crossed module in S-schemes* $[\partial\colon G_1 \to G_0]$ to be a pair of S-group schemes G_0 and G_1, an S-group scheme homomorphism $\partial\colon G_1 \to G_0$, and a (right) action of G_0 on G_1 satisfying the axioms of a crossed module. These are precisely the representable objects in $\mathbf{XMod}_{\mathbf{Sch}_S}$; in other words, a crossed module in schemes $[\partial\colon G_1 \to G_0]$ gives rise to a presheaf of crossed modules

$$U \mapsto [\partial(U)\colon G_1(U) \to G_0(U)].$$

We often abuse terminology and call a crossed module in schemes over S simply a *strict 2-group scheme over S*.

The following two propositions generalize Lemma 8.2 of [BeNo].

Proposition 6.1. *Let S be a base scheme. Let A be an abelian affine group scheme over S acting on a S-scheme X, and let $\mathcal{X} = [X/A]$ be the quotient stack. Let G be those automorphisms of X which commute with the A action; this is a sheaf of groups on $\mathbf{Sch}_S$. We have the following:*

(i) *With the trivial action of G on A, the natural map $\varphi\colon A \to G$ becomes a crossed modules in $\mathbf{Sch}_S$-schemes.*
(ii) *Let $\mathcal{G}$ be the group stack associated to $[\varphi\colon A \to G]$. Then, there is a natural morphism of group stacks $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$. Furthermore, this morphism induces an isomorphism of sheaves of groups $\pi_1\mathcal{G} \xrightarrow{\sim} \pi_1(\mathcal{A}ut\mathcal{X})$.*

Proof. Part (i) is straightforward, because φ maps A to the center of G. Let $\underline{\mathfrak{G}}$ denote the presheaf of 2-groups associated to $[\varphi\colon A \to G]$. To prove part (ii), it is enough to construct a morphism of presheaves of 2-groups $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ and show that it has the required properties. Stackification of this map gives us the desired map (Lemma 4.6).

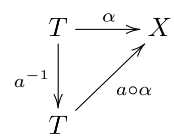
Let us construct the morphism $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$. We give the effect of this morphism on the sections over S. Since everything commutes with base change, the same construction works for every $U \to S$ in the site $\mathbf{Sch}_S$ and gives rise to the desired morphism. of presheaves.

To define $\underline{\mathfrak{G}}(S) \to \mathcal{A}ut\mathcal{X}(S)$, recall the explicit description of the S-points of the quotient stack $[X/A]$:

$$\mathrm{Ob}[X/A](S) = \left\{ \begin{array}{ccl} (T,\alpha) & | & T \text{ an } A\text{-torsor over } S \\ & & \alpha\colon T \to X \text{ an } A\text{-map} \end{array} \right\}$$

$$\mathrm{Mor}[X/H](S)((T,\alpha),(T',\alpha')) = \{f\colon T \to T' \text{ an } A\text{-torsor map s.t. } \alpha' \circ f = \alpha\}$$

Any element of $g \in G(S)$ induces an automorphism of $\mathcal{X}$ relative to S (keep the same torsor T and compose α with the action of g on X). Also, for any element $a \in A(S)$, there is a natural 2-isomorphism from the identity automorphism of $\mathcal{X}$ to the automorphism induced by $\varphi(a) \in G(S)$ (which is by definition the same as the action of a). It is given by the multiplication action of a^{-1} on the torsor T (remember that A is abelian) which makes the following triangle commute



Interpreted in the language of 2-groups, this gives a morphism of 2-groups $\underline{\mathfrak{G}}(S) \to \mathcal{A}ut\mathcal{X}(S)$.

To prove that $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ induces an isomorphism on π_1, we show that, for every $U \to S$ in the site $\mathbf{Sch}_S$, the morphism of 2-groups $\underline{\mathfrak{G}}(U) \to \mathcal{A}ut\mathcal{X}(U)$ induces an isomorphism on π_1. Again, we may assume that $U = S$. We know that the group of 2-isomorphisms from the identity automorphism of $\mathcal{X}$ to itself is naturally isomorphic to the group of global sections of the inertia stack of $\mathcal{X}$. In the case $\mathcal{X} = [X/A]$, this is naturally isomorphic to the group of elements of $A(S)$ which act trivially on X. Note that this group is naturally isomorphic to $\pi_1\underline{\mathfrak{G}}(S)$. Therefore, the map $\underline{\mathfrak{G}}(S) \to \mathcal{A}ut\mathcal{X}(S)$ induces an isomorphism on π_1. □

Proposition 6.2. *Notation being as in Proposition 6.1, assume that $\mathcal{X}$ is a proper Deligne-Mumford stack over S, and that $X \to S$ has geometrically connected and reduced fibers. Also, assume that A fits in an extension*

$$0 \to A_0 \to A \to A/A_0 \to 0$$

where A/A_0 is finite over S and A_0 has geometrically connected fibers (this is automatics, for example, in the case where A is smooth and the number of its geometric connected components is a locally constant function on S). Then, $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ is fully faithful (as a morphism of presheaves of groupoids). In particular, the induced map $\pi_0\mathcal{G} \to \pi_0(\mathcal{A}ut\mathcal{X})$ of sheaves of groups is injective.

Proof. As in Proposition 6.1, let $\underline{\mathfrak{G}}$ denote the presheaf of 2-groups associated to $[\varphi\colon A \to G]$. We need to show that, for every $U \to S$ in the site $\mathbf{Sch}_S$, $\underline{\mathfrak{G}}(U) \to \mathcal{A}ut\mathcal{X}(U)$ is fully faithful; since $\mathcal{A}ut\mathcal{X}$ is a stack, it would then follow that the stackified morphism $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ is also fully faithful.

We may assume that $U = S$. By Proposition 6.1.(ii) and Lemma 3.2, it is enough to prove that if the action of $g \in G(S)$ on $\mathcal{X}$ is 2-isomorphic to the identity, then g is of the form $\varphi(a)$, for some $a \in A(S)$. Let us fix such a 2-isomorphism. The effect of this 2-isomorphism on the A-torsor on X corresponding the point $X \to [X/A]$, viewed as an object in the groupoid $[X/A](X)$ of X-points of $[X/A]$, is given by an A-torsor map $F\colon A \times_S X \to A \times_S X$ which makes the following A-equivariant triangle commute:

$$\begin{array}{ccc} A \times_S X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle F}\uparrow & \nearrow_{g\circ\mu} & \\ A \times_S X & & \end{array}$$

Here, $A \times_S X$ is the trivial A-torsor on X and μ is the action of A on X.

Precomposing F with the canonical section $X \to A \times_S X$ (corresponding to the identity element of A) and then projecting onto the first factor, we obtain a map $f\colon X \to A$ relative to S. The proposition follows from the following.

Claim. The map f is constant, in the sense that it factors through an S-point $a\colon S \to A$ of A. Furthermore, the effect of a on X (induced from the action of A on X) is the same as the effect of g on X.

Let us prove the claim. It follows from the commutativity of the above diagram that, for any point x in X, the effect of g on x is the same as the effect of $f(x)$ on x.[2] In other words, $f(x)g^{-1}$ leaves x fixed. Applying this to ax instead of x, and using the fact that a and $f(x)g^{-1}$ commute, we find that $f(ax)g^{-1}$ also leaves x fixed, for every $a \in A$. This implies that, for any point x of X, and any $a \in A$, the element $r(a,x) := f(ax)f(x)^{-1}$ leaves x fixed. Therefore, the map $\rho\colon A \times_S X \to A \times_S X$, $\rho(a,x) := (r(a,x),x)$ factors through the stabilizer group scheme $\tau\colon \Sigma \to X$. Thus, we have a commutative triangle

$$\begin{array}{ccc} A \times_S X & \xrightarrow{\quad \rho \quad} & \Sigma \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle \tau} & \\ & X & \end{array}$$

Now, consider the short exact sequence

$$0 \to A_0 \to A \to A/A_0 \to 0,$$

where A_0 is a group scheme over S with geometrically connected fibers and A/A_0 is finite over S. Since $\tau\colon \Sigma \to X$ has discrete fibers (because $\mathcal{X}$ is Deligne-Mumford) the restriction of ρ to $A_0 \times_S X$ factors through the identity section. Hence, for every $a \in A_0$ and $x \in X$ (over the same point in S), $r(a,x) = f(ax)f(x)^{-1}$ is the identity element of A. This implies that $f\colon X \to A$ is A_0-equivariant (for the trivial action of A_0 on A). So, we obtain an induced map $\lambda\colon [X/A_0] \to A$ (relative to S). Since $[X/A_0]$ is finite over $[X/A]$, and $[X/A]$ is proper over S, the structure map

[2] When we say a "point" of X we mean a scheme T over S and a morphism $T \to X$ relative to S.

$\pi\colon [X/A_0] \to S$ is proper. From our assumptions we have that π has geometrically connected and reduced fibers. Base change then implies that $\pi_*\mathcal{O}_{[X/A_0]} = \mathcal{O}_S$. Since A is affine over S, it follows that λ is constant, i.e., factors through a section $a\colon S \to A$. Since $f\colon X \to A$ factors through λ, it also factors through a. By construction, the effect of a on X is the same as the effect of g on X, which is what we wanted to prove. □

6.2. **Weighted projective general linear 2-groups.** Since the construction of the weighted projective stacks, and also of the weighted projective general linear 2-group schemes, commutes with base change, we can work over $\mathbb{Z}$. We begin with some notation. We denote the multiplicative group scheme over $\operatorname{Spec}\mathbb{Z}$ by $\mathbb{G}_{m,\mathbb{Z}}$, or simply $\mathbb{G}_m$. The affine $(r+1)$-space over a base scheme S is denoted by $\mathbb{A}_S^{r+1}$; when the base scheme is $\operatorname{Spec} R$ it is denoted by $\mathbb{A}_R^{r+1}$, and when the base scheme is $\operatorname{Spec}\mathbb{Z}$ simply by $\mathbb{A}^{r+1}$. Since r will be fixed throughout this section, we will usually denote $\mathbb{A}_S^{r+1} - \{0\}$ by $\mathbb{U}_S$. We will abbreviate $\mathbb{U}_{\operatorname{Spec} R}$ and $\mathbb{U}_{\operatorname{Spec}\mathbb{Z}}$ to $\mathbb{U}_R$ and $\mathbb{U}$, respectively. We fix a Grothendieck topology on $\mathbf{Sch}_S$ that is not coarser than Zariski.

Let $n_0, n_1, \cdots, n_r$ be a sequence of positive integers, and consider the weight $(n_0, n_1, \cdots, n_r)$ action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$. (That is, for every scheme T, an element $t \in \mathbb{G}_m(T)$ acts on $\mathbb{U}_T$ by multiplication by $(t^{n_0}, t^{n_1}, \cdots, t^{n_r})$.) The quotient stack of this action is called the *weighted projective stack* of weight $(n_0, n_1, \cdots, n_r)$ and is denoted by $\mathcal{P}_{\mathbb{Z}}(n_0, n_1, \cdots, n_r)$, or simply by $\mathcal{P}(n_0, n_1, \cdots, n_r)$. The **weighted projective general linear 2-group scheme** $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ is defined to be the 2-group scheme associated to the crossed module

$$[\partial\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}],$$

where $G_{n_0,n_1,\cdots,n_r}$ is the group scheme, over $\mathbb{Z}$, of all $\mathbb{G}_m$-equivariant (for the above weighted action) automorphisms of $\mathbb{U}$. More precisely, the T-points of $G_{n_0,n_1,\cdots,n_r}$ are automorphisms

$$f\colon \mathbb{U}_T \to \mathbb{U}_T$$

that commute with the $\mathbb{G}_m$-action. The homomorphism $\partial\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}$ is the one induced from the $\mathbb{G}_m$-action itself. We take the action of $G_{n_0,n_1,\cdots,n_r}$ on $\mathbb{G}_m$ to be trivial. The associated group stack is denoted by $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$, and is called the *projective general linear* group stack of weight $(n_0, n_1, \cdots, n_r)$.

The following theorem says that a weighted projective general linear 2-group scheme is a model for the group stack of self-equivalences of the corresponding weighted projective stack. A special case of this theorem (namely, the case where the base scheme is $\mathbb{C}$) was proved in ([BeNo], Theorem 8.1). We briefly sketch how the proof in [ibid.] can be modified to cover the general case.

Theorem 6.3. *Let $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ be the group stack of automorphisms of the weighted projective stack $\mathcal{P}(n_0, n_1, \cdots, n_r)$. Then, the natural map*

$$\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$$

is an equivalence of group stacks. In particular, we have isomorphisms of sheaves of groups

$$\pi_0\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_0\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_0 \mathrm{PGL}(n_0, n_1, \cdots, n_r),$$

$$\pi_1\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_1\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_1 \mathrm{PGL}(n_0, n_1, \cdots, n_r) \cong \mu_d,$$

where $d = \gcd(n_0, n_1, \cdots, n_r)$ *and* μ_d *stands for the multiplicative group scheme of* d^{th} *roots of unity.*

In order to prove our main result (Theorem 6.3) we need the following result about line bundles on weighted projective stacks. For more details on this, the reader is referred to [No4]. More general results about Picard stacks of algebraic stacks can be found in [Bro].

Proposition 6.4. *Let* $\mathcal{P} = \mathcal{P}_S(n_0, n_1, \cdots, n_r)$*, where* $S = \operatorname{Spec} R$ *is the spectrum of a local ring. Then every line bundle on* $\mathcal{P}$ *is of the form* $\mathcal{O}(d)$ *for some* $d \in \mathbb{Z}$*.*

Proof. In the proof we use stack versions of Grothendieck's base change and semi-continuity results ([Ha], III. Theorem 12.11). We will assume that R is Noetherian.

In the case where R is a field, the assertion is easy to prove using the fact that the Picard group of $\mathcal{P}$ is isomorphic to the Weil divisor class group. To prove the general case, let x be the closed point of $S = \operatorname{Spec} R$. Let $\mathcal{L}$ be a line bundle on $\mathcal{P}$. After twisting with on appropriate $\mathcal{O}(d)$, we may assume $\mathcal{L}_x \cong \mathcal{O}$. We will show that $\mathcal{L}$ is trivial. We have $H^1(\mathcal{P}_x, \mathcal{L}_x) = H^1(\mathcal{P}_x, \mathcal{O}_x) = 0$. Hence, by semicontinuity, $H^1(\mathcal{P}_y, \mathcal{L}_y) = 0$ for every point y of S. Base change implies that $R^1 f_*(\mathcal{L}) = 0$, and that $R^0 f_*(\mathcal{L}) = f_*(\mathcal{L})$ is locally free (necessarily of rank 1). Therefore, $f_*(\mathcal{L})$ is free of rank 1 and, by base change, $H^0(\mathcal{P}_y, \mathcal{L}_y)$ is 1-dimensional as a $k(y)$-vector space, for every y in S. In fact, this is true for every tensor power $\mathcal{L}^{\otimes n}$, $n \in \mathbb{Z}$. So, $\mathcal{L}_y$ is trivial for every y in S. (Note that, when k is a field, $\dim_k H^0(\mathcal{P}_k(n_0, n_1, \cdots, n_r), \mathcal{O}(d))$ is equal to the number of solutions of the equation $a_1 n_0 + a_2 n_1 + \cdots + a_r n_r = d$ in non-negative integers a_i.)

Now let s be a generating section of $f_*(\mathcal{L}) \cong R$. It follows that $f^*(s)$ is a generating section of $\mathcal{L}$. So $\mathcal{L}$ is trivial. □

Proof of Theorem 6.3. We apply Propositions 6.1 and 6.2 with $S = \operatorname{Spec} \mathbb{Z}$, $X = \mathbb{A}^{r+1} - \{0\}$, and $H = \mathbb{G}_m$. This implies that

$$\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$$

is a fully faithful morphism of stacks. That is, for every scheme U, the morphism of groupoids

$$\mathcal{PGL}(n_0, n_1, \cdots, n_r)(U) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)(U)$$

is fully faithful. All that is left to show is that it is essentially surjective. Since $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$ and $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ are both stacks, it is enough to prove this for $U = \operatorname{Spec} R$, where R is a local ring. In this case, we know by Proposition 6.4 that $\operatorname{Pic} \mathcal{P}(n_0, n_1, \cdots, n_r) \cong \mathbb{Z}$. We can now proceed exactly as in ([BeNo], Theorem 8.1).

The isomorphisms stated at the end of the theorem follow from Lemma 4.4 and Lemma 4.6. □

7. Structure of $PGL(n_0, n_1, ..., n_r)$

In this section we give detailed information about the structure of the group $G_{n_0,n_1,\cdots,n_r}$. We show that, as a group scheme over an arbitrary base, it splits as a semi-direct product of a reductive group scheme and a unipotent group scheme. The reductive part is a product of a copies of the general linear groups. The unipotent part is a successive semi-direct product of vector groups; see Theorem 7.7.

Throughout this section, the action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$ means the weight $(n_0, n_1, \cdots, n_r)$ action. To shorten the notation, we denote the group $G_{n_0,n_1,\cdots,n_r}$ by G. The rank m general linear group scheme over $\operatorname{Spec} R$ is denoted by $\mathrm{GL}(m, R)$. When $R = \mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. We always assume $r \geq 1$. The corresponding projectivized group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ for a weighted projective general linear 2-group as in the latter case we have at least two variables.

We begin with a simple lemma.

Lemma 7.1. *Let R be an arbitrary ring, and let f be a global section of the structure sheaf of $\mathbb{U}_R = \mathbb{A}^{r+1}_R - \{0\}$, $r \geq 1$. Then f extends uniquely to a global section of $\mathbb{A}^{r+1}_R$.*

Proof. Let $U_i = \operatorname{Spec} R[x_0, \cdots, x_r, x_i^{-1}]$ and consider the covering $\mathbb{U}_R = \cup_{i=1}^n U_i$. We show that the restriction $f_i := f|_{U_i}$ is a polynomial for every i. To see this, observe that, except possibly for x_i, all variables occur with positive powers in f_i. To show that x_i also occurs with a positive power, pick some $j \neq i$ and use the fact that x_i occurs with a positive power in $f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$.

Therefore, for every i, f_i actually lies in $R[x_0, \cdots, x_r, x_i^{-1}]$. Since $f_j|_{U_i} = f_i|_{U_j}$, it is obvious that all f_i are actually the same and provide the desired extension of f to $\mathbb{U}_R$. $\square$

From now on, we will use a slightly different notation with indices. Namely, we assume that the weights are $m_1 < m_2 < \cdots < m_t$, with each m_i appearing exactly $r_i \geq 1$ times in the weight sequence (so in the previous notation we would have $r + 1 = r_1 + \cdots + r_t$). We denote the corresponding projective general linear 2-group by $\mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t)$. We use the coordinates x^i_j, $1 \leq i \leq t$, $1 \leq j \leq r_i$, for $\mathbb{A}^{r+1}$. We think of x^i_j as a variable of degree m_i. We will usually abbreviate the sequence $x^i_1, \cdots, x^i_{r_i}$ to $\mathbf{x}^i$. Similarly, a sequence $F^i_1, \cdots, F^i_{r_i}$ of polynomials is abbreviated to $\mathbf{F}^i$.

Let R be a ring. The following proposition tells us how a $\mathbb{G}_{m,R}$-equivariant automorphisms of $\mathbb{U}_R$ looks like.

Proposition 7.2. *Let $F\colon \mathbb{U}_R \to \mathbb{U}_R$ be a $\mathbb{G}_m$-equivariant map. Then F is of the form $(\mathbf{F}^i)_{1 \leq i \leq t}$, where for every i, each component $F^i_j \in R[x^i_j; 1 \leq i \leq t, 1 \leq j \leq r_i]$ of $\mathbf{F}^i$ is a weighted homogeneous polynomial of weight m_i.*

Proof. The fact that components of F are polynomial follows from Lemma 7.1. The statement about homogeneity of F^i_j is a simple exercise in polynomial algebra and is left to the reader. $\square$

In the above proposition, each F^i_j can be written in the form $F^i_j = L^i_j + P^i_j$, where L^i_j is linear in the variables $x^i_1, \cdots, x^i_{r_i}$, and P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. Let $L_F := (\mathbf{L}^i)_{1 \leq i \leq t}$ be the linear part of F. It is again a $\mathbb{G}_m$-equivariant endomorphism of $\mathbb{U}$.

Proposition 7.3. *Let F be as in the Proposition 7.2. The assignment $F \mapsto L_F$ respects composition of endomorphisms. In particular, if F is an automorphism, then so is L_F.*

Proof. This follows from direct calculation, or, alternatively, by using the fact that L_F is simply the derivative of F at the origin. $\square$

Corollary 7.4. *There is a natural split homomorphism*

$$\phi\colon G \to \mathrm{GL}(r_1) \times \mathrm{GL}(r_2) \times \cdots \times \mathrm{GL}(r_t).$$

Next we give some information about the structure of the kernel U of ϕ. It consists of endomorphisms $F = (F^i_j)_{i,j}$, where F^i_j has the form

$$F^i_j = x^i_j + P^i_j.$$

Here, P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. Indeed, it is easily seen that, for an arbitrary choice of the polynomials P^i_j, the resulting endomorphism F is automatically invertible. So, to give such an $F \in U$ is equivalent to giving an arbitrary collection of polynomials $\{P^i_j\}_{1\leq i\leq t, 1\leq j\leq r_i}$ such that each P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. So, from now on we switch the notation and denote such an element of U by $(P^i_j)_{i,j}$.

Proposition 7.5. *For each $1 \leq a \leq t$, let $U_a \subseteq U$ be the set of those endomorphisms $F = (P^i_j)_{i,j}$ for which $P^i_j = 0$ whenever $i \neq a$. Let K_a denote the set of monomials of degree m_a in variables x^i_j, $i < a$, and let k_a be the cardinality of K_a. (In other words, k_a is the number of solutions of the equation*

$$\sum_{i=1}^{a-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_a$$

in non-negative integers $z_{i,j}$.) Then we have the following:

- (*i*) *U_a is a subgroup of U and is canonically isomorphic to the vector group scheme $\mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a k_a}$. (Note: U_1 is trivial.)*
- (*ii*) *If $a < b$, then U_a normalizes U_b.*
- (*iii*) *The groups U_a, $1 \leq i \leq t$, generate U and we have $U_a \cap U_b = \{1\}$ if $a \neq b$.*

Proof of (i). The action of $(P^i_j)_{i,j} \in U_a$ on $\mathbb{A}^{r+1}$ is given by

$$(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) \longmapsto (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t).$$

So, if $\mathbb{A}^{K_a}$ stands for the vector group scheme on the basis K_a, there is a canonical isomorphism

$$U_a \cong \bigoplus_{i=1}^{r_a} \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}.$$

Proof of (*ii*). Let $G = (Q^i_j)_{i,j}$ be an element in U_a and $F = (P^i_j)_{i,j}$ an element in U_b. By (*i*), the inverse of G is $G^{-1} = (-Q^i_j)_{i,j}$. Let us analyze the effect of the composite $G \circ F \circ G^{-1}$ on $\mathbb{A}^{r+1}$:

$$\begin{array}{rcl}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) & \stackrel{G^{-1}}{\longmapsto} & (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) \\ & \stackrel{F}{\longmapsto} & (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t) \\ & \stackrel{G}{\longmapsto} & (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t).\end{array}$$

Here the polynomial R^b_k, $1 \leq k \leq r_b$, is obtained from P^b_k by substituting the variables x^a_j with the polynomial $x^a_j - Q^a_j$.

Proof of (*iii*). Easy. □

Part (ii) implies that each U_a acts by conjugation on each of U_{a+1}, $U_{a+2}, \cdots$, U_t.[3] To fix the notation, in what follows we let the conjugate of an automorphism f by an automorphism g to be $g \circ f \circ g^{-1}$.

Notation. Let $\{U_a\}_{a=1}^t$ be a family of subgroups of a group U which satisfies the following properties: 1) Each U_a normalizes every U_b with $a < b$; 2) No two distinct U_a intersect; 3) The U_a generate U. In this case, we say that U is a successive semi-direct product of the U_a, and use the notation $U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1$.

The following is an immediate corollary of Proposition 7.5.

Corollary 7.6. *There is a natural decomposition of U as a semi-direct product*

$$U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1,$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ is the group introduced in Proposition 7.5. (Note that U_1 is trivial.)

In the next theorem we use the notation $\mathbb{A}^m$ for two things. One that has already appeared is the affine group scheme of dimension m. When there is a group scheme G involved, we also use the notation $\mathbb{A}^m$ for the trivial representation of G on $\mathbb{A}^m$.

Theorem 7.7. *There is a natural decomposition of G as a semi-direct product*

$$G \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes \big(\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t)\big),$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ and k_a is as in Proposition 7.5. (Note that U_1 is trivial.) Furthermore, for every $1 \leq a \leq t$, the action of $\mathrm{GL}(r_a)$ leaves each U_b invariant. We also have the following:

- (i) *When $a > b$ the induced action of $\mathrm{GL}(r_a)$ on U_b is trivial.*
- (ii) *When $a = b$ the induced action of $\mathrm{GL}(r_a)$ on U_a is naturally isomorphic to the representation $\rho \otimes \mathbb{A}^{K_a}$, where ρ is the standard representation of $\mathrm{GL}(r_a)$ and K_a is as in Proposition 7.5. (Recall that U_a is canonically isomorphic to $\mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}$.)*
- (iii) *When $a < b$ the action of $\mathrm{GL}(r_a)$ on U_b is naturally isomorphic to the representation*

$$\bigoplus_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{r_b d_l} \otimes \hat{\rho}^{\otimes l}.$$

Here $\hat{\rho}$ stands for the inverse transpose of ρ, and d_l is the number of monomials of degree m_b in variables x_j^i, $i < b$, $i \neq a$; so d_l also depends on a and b. (In other words, d_l is the number of solutions of the equation

$$\sum_{\substack{i=1 \\ i \neq a}}^{b-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_b - l m_a$$

in non-negative integers $z_{i,j}$.)

Proof. Let $g \in \mathrm{GL}(r_a)$ and $F \in U_b$. As in the proof of Proposition 7.5.i, we analyze the effect of the composite $g \circ F \circ g^{-1}$ on $\mathbb{A}^{r+1}$. The element $g \in \mathrm{GL}(r_a)$ acts on $\mathbb{A}^{r+1}$ as follows: it leaves every component x_i^j invariant if $i \neq a$ and on the coordinates $x_1^a, \cdots, x_{r_a}^a$ it acts linearly (like the action of an $r_a \times r_a$ matrix on a column vector).

[3] All group actions in this section are assumed to be on the left.

Proof of (i). The effect of $g \in \mathrm{GL}(r_a)$ only involves the variables $x^a_1, \cdots, x^a_{r_a}$ and does not see any other variable, whereas the effect of $F \in U_b$ only involves the variables x^i_j, $i \leq b$. Since $b < a$, these two are independent of each other. That is, F and g commute.

Proof of (ii). Assume $F = (P^i_j)_{i,j}$; so $P^i_j = 0$ if $i \neq a$. The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{aligned}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) &\stackrel{g^{-1}}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^t)\\ &\stackrel{F}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{y}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t)\\ &\stackrel{g}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{Q}^a, \cdots, \mathbf{x}^t).\end{aligned}$$

Here, y^a_j is the linear combination of $x^a_1, \cdots, x^a_{r_a}$, the coefficients being the entries of the j^{th} row of the matrix g^{-1}. Similarly, Q^a_j is the linear combination of $P^a_1, \cdots, P^a_{r_a}$, coefficients being the entries of the j^{th} row of the matrix g.

Proof of (iii). Assume $F = (P^i_j)_{i,j}$; so $P^i_j = 0$ if $i \neq b$. Let $\mathbf{y}^a$ be as in (ii). The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{aligned}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) &\stackrel{g^{-1}}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t)\\ &\stackrel{F}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t)\\ &\stackrel{g}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t).\end{aligned}$$

Here the polynomials R^b_k, $1 \leq k \leq r_b$, are obtained from P^b_k by substituting the variable x^a_j with y^a_j.

Let λ be the representation of $\mathrm{GL}(r_a)$ on the space V of homogenous polynomials of degree m_b which acts as follows: it takes a polynomial $P \in V$ and substitutes the variables x^a_j, $1 \leq j \leq r_a$, with y^a_j. From the description above, we see that the representation of $\mathrm{GL}(r_a)$ on U_b is a direct sum of r_b copies of λ. We will show that

$$\lambda \cong \bigoplus_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{d_l} \otimes \hat{\rho}^{\otimes l}.$$

To obtain the above decomposition, simply note that a polynomial in V can be uniquely written in the form

$$\sum_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} S_l T_l,$$

where T_l is a homogenous polynomial of degree $l m_a$ in variables $x^a_1, \cdots, x^a_{r_a}$, and S_l is a homogenous polynomial of degree $m_b - l m_a$ in the rest of the variables. The action of $\mathrm{GL}(r_a)$ leaves S_l intact and acts on T_l by the l^{th} power of the inverse transpose of the standard representation. □

The actions of various pieces in the above semi-direct product decomposition, though explicit, are tedious to write down, except for small values of t. We give some examples in §8.

Let us denote $U_t \rtimes \cdots \rtimes U_2 \rtimes U_1$ by U and define the crossed module

$$\mathrm{PGL}(n_0, n_1, \cdots, n_r)_{red} := [\partial\colon \mathbb{G}_m \to \mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t)],$$

where the k^{th} factor of $\partial(\lambda)$ is the size r_k scalar matrix λ^{m_k}. Theorem 7.7 then implies that

$$\mathrm{PGL}(n_0, n_1, \cdots, n_r) \cong U \rtimes \mathrm{PGL}(n_0, n_1, \cdots, n_r)_{red}.$$

We think of U as the unipotent radical and $\mathrm{PGL}(n_0, n_1, \cdots, n_r)_{red}$ as the reductive part of $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$.

Remark 7.8. It is perhaps useful to put the above result in the general context of algebraic group theory. Recall that every algebraic group G over a *field* fits in a short exact sequence

$$1 \to U \to G \to G_{red} \to 1,$$

where U is the unipotent radical of G and G_{red} is reductive. The sequence is not split in general. In our case, the group scheme $G_{n_0,n_1,\cdots,n_r}$ admits such a short sequence over an arbitrary base and, furthermore, the sequence is split.

The general theory of unipotent groups tells us that any unipotent group over a *perfect field* admits a filtration whose graded pieces are vector groups. This filtration splits, but only in the category of schemes (i.e., the splitting maps may not be group homomorphisms). In our case, however, the group scheme U admits such a filtration over an arbitrary base. Furthermore, the filtration is split group theoretically.

8. Some examples

In this section we look at some explicit examples of weighted projective general linear 2-groups.

Example 8.1. *Weight sequence* $m < n, m \nmid n$. In this case we have $t = 2$, and $r_1 = r_2 = 1$ and $k_1 = 0$. So $G \cong \mathbb{G}_m \times \mathbb{G}_m$.

Example 8.2. *Weight sequence* $m < n, m \mid n$. In this case we have $t = 2$, $r_1 = r_2 = 1$, and $k_1 = 1$. So we have

$$G \cong \mathbb{A} \rtimes (\mathbb{G}_m \times \mathbb{G}_m).$$

The action of an element $(\lambda_1, \lambda_2) \in \mathbb{G}_m \times \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by

$$(\lambda_1, \lambda_2) \cdot a = \lambda_2 \lambda_1^{-\frac{n}{m}} a.$$

More explicitly, an element in G is map of the form

$$(x, y) \mapsto (\lambda_1 x, \lambda_2 y + a x^{\frac{n}{m}}).$$

Note the similarity with the group of 2×2 lower-triangular matrices.

Example 8.3. *Weight sequence* $n = m$. We obviously have $G \cong \mathrm{GL}(2)$.

Example 8.4. *Weight sequence* $1, 2, 3$. First we determine U. A typical element in U is of the form

$$(x, y, z) \mapsto (x, y + ax^2, z + bx^3 + cxy).$$

We have $U_2 = \mathbb{A}$ and $U_3 = \mathbb{A}^2$. The action of an element $a \in U_2$ on an element $(b, c) \in U_3$ is given by $(b - ac, c)$. That is, a acts on $U_3 = \mathbb{A}^2$ by the matrix

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 2} \rtimes \mathbb{A}$. Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 2} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a, b, c) \in U$ is given by $(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-3}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c)$.

Example 8.5. *Weight sequence* 1, 2, 4. An element in U has the general form

$$(x, y, z) \mapsto (x, y + ax^2, z + bx^4 + cx^2y + dy^2).$$

We have $U_2 = \mathbb{A}$ and $U_3 = \mathbb{A}^3$. The action of an element $a \in U_2$ on an element $(b, c, d) \in U_3$ is given by the matrix

$$\begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 3} \rtimes \mathbb{A}$.

Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 3} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a, b, c, d) \in U$ is given by

$$(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-4}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c, \lambda_2^{-2}\lambda_3 d).$$

Next we look at $\mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t)$. Recall that, as a crossed module, this is given by $[\partial\colon \mathbb{G}_m \to G]$, where ∂ is the obvious map coming from the action of $\mathbb{G}_m$ on $\mathbb{A}^{r+1}$, and the action of G on $\mathbb{G}_m$ is the trivial one.

Observe that the map ∂ factors though the component $\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t)$ of G. So, let us define L to be the cokernel of the following map:

$$\mathbb{G}_m \xrightarrow{(\overbrace{\lambda^{m_1}, \dots, \lambda^{m_1}}^{r_1}, \dots\dots, \overbrace{\lambda^{m_t}, \dots, \lambda^{m_t}}^{r_t})} \mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t).$$

From Theorem 7.7 we immediately obtain the following.

Proposition 8.6. *Let L be the group defined in the previous paragraph, and let $k = \gcd(m_1, \cdots, m_t)$. We have natural isomorphisms of group schemes*

$$\begin{aligned} \pi_0 \mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t) &\cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes L, \\ \pi_1 \mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t) &\cong \mu_k. \end{aligned}$$

Our final result is that, if all weights are distinct (that is, $r_i = 1$), then the corresponding projective general linear 2-group is split.

Proposition 8.7. *Let $\{m_1, \cdots, m_t\}$ be distinct positive integers, and consider the projective general linear 2-group $\mathrm{PGL}(m_1, m_2, \cdots, m_t)$. Then, the projection map $G \to \pi_0 \mathrm{PGL}(m_1, m_2, \cdots, m_t)$ splits. In particular, $\mathrm{PGL}(m_1, m_2, \cdots, m_t)$ is split. That is, it is completely classified by its homotopy group schemes:*

$$\begin{aligned} \pi_0 \mathrm{PGL}(m_1, \cdots, m_t) &\cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes (\mathbb{G}_m)^{t-1}, \\ \pi_1 \mathrm{PGL}(m_1, \cdots, m_t) &\cong \mu_k. \end{aligned}$$

Proof. By Theorem 7.7 and Proposition 8.6 we know that $G \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes (\mathbb{G}_m)^t$ and $\pi_0 \mathrm{PGL}(m_1, m_2, \cdots, m_t) \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes L$, where L is the cokernel of the map

$$\alpha\colon \mathbb{G}_m \xrightarrow{(\lambda^{m_1}, \cdots, \lambda^{m_t})} (\mathbb{G}_m)^t.$$

So it is enough to show that the image of μ is a direct factor. Note that if we divide all the m_i by their greatest common divisor, the image of α does not change.

So, we may assume $\gcd(m_1, \cdots, m_t) = 1$. Let M be a $t \times t$ integer matrix whose determinant is 1 and whose first column is $(m_1, \cdots, m_t)$. The matrix M gives rise to an isomorphism $\mu\colon (\mathbb{G}_m)^t \to (\mathbb{G}_m)^t$ whose restriction to the subgroup $\mathbb{G}_m \times \{1\}^{t-1} \cong \mathbb{G}_m$ is naturally identified with α. The subgroup $\mu(\{1\} \times (\mathbb{G}_m)^{t-1}) \subset (\mathbb{G}_m)^t$ is the desired complement of the image of α. $\square$

Corollary 8.8. *Let m, n be distinct positive integers, and let $k = \gcd(m, n)$. Then $\mathrm{PGL}(m, n)$ is a split 2-group. That is, it is classified by its homotopy groups:*

$$\begin{aligned} \pi_0 \,\mathrm{PGL}(m,n) &\cong \begin{cases} \mathbb{G}_m, & \text{if } m < n, m \nmid n \\ \mathbb{A} \rtimes \mathbb{G}_m, & \text{if } m < n, m \mid n \end{cases} \\ \pi_1 \,\mathrm{PGL}(m,n) &\cong \mu_k. \end{aligned}$$

(In the case $m \mid n$, the action of $\mathbb{G}_m$ on $\mathbb{A}$ in the cross product $\mathbb{A} \rtimes \mathbb{G}_m$ is simply the multiplication action.)

Proof. Everything is clear, except perhaps a clarification is in order regarding the parenthesized statement. Observe that the $\mathbb{G}_m$ appearing in the cross product $\mathbb{A} \rtimes \mathbb{G}_m$ is indeed the cokernel of the map

$$\alpha\colon \mathbb{G}_m \xrightarrow{(\lambda^m, \lambda^n)} (\mathbb{G}_m)^2,$$

which is naturally identified with the subgroup $\{1\} \times \mathbb{G}_m \subset (\mathbb{G}_m)^2$. Therefore, by the formula of Example 8.2, the action of an element $\lambda \in \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by λa. $\square$

Finally, for the sake of completeness, we include the following.

Proposition 8.9. *The 2-group $\mathrm{PGL}(k, k, \cdots, k)$, k appearing t times, is given by the following crossed module:*

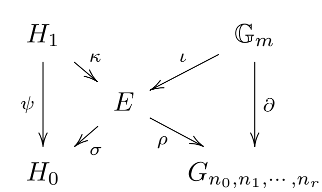
$$[\mathbb{G}_m \xrightarrow{(\lambda^k, \cdots, \lambda^k)} \mathrm{GL}(t)].$$

We have $\pi_0 \,\mathrm{PGL}(k, \cdots, k) \cong \mathrm{PGL}(t)$ and $\pi_1 \,\mathrm{PGL}(k, \cdots, k) \cong \mu_k$. In particular, $\mathrm{PGL}(1, 1, \cdots, 1)$, 1 appearing t times, is equivalent to the group scheme $\mathrm{PGL}(t)$.

9. 2-group actions on weighted projective stacks

In this section we combine the method developed in §3–§5 with the results about the structure of $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ to study 2-group actions on a weighted projective stack $\mathcal{P}(n_0, n_1, \cdots, n_r)$. The goal is to illustrate how one can classify 2-group actions using butterflies and how to describe the corresponding quotient 2-stacks. Below, all group scheme are assumed to be flat and of finite presentation over a fixed base S.

Let $\mathcal{H}$ be a group stack and $[\psi\colon H_1 \to H_0]$ a presentation for it as a crossed module in schemes. By Theorem 5.3, to give an action of $\mathcal{H}$ on $\mathcal{P}(n_0, n_1, \cdots, n_r)$ is equivalent to giving a butterfly diagram



In other words, to give an action of $\mathcal{H}$ on $\mathcal{X}$, we need to find a central extension

$$1 \to \mathbb{G}_m \to E \to H_0 \to 1$$

of H_0 by $\mathbb{G}_m$, together with

- a lift $\kappa\colon H_1 \to E$ of ψ to E such that $\kappa(\beta^{\sigma(x)}) = x^{-1}\kappa(\beta)x$, for every $\beta \in H_1$ and $x \in E$;
- an extension of the weighted action of $\mathbb{G}_m$ to a linear action of E on $\mathbb{A}^{r+1}$ which is trivial on the image of κ.

The following result is more or less immediate from the above description of a group action.

Theorem 9.1. *Let k be a field and H a connected linear algebraic group over k, assumed to be reductive if* $\operatorname{char} k > 0$. *Let $\mathcal{X} = \mathcal{P}(n_0, n_1, \cdots, n_r)$ be a wighted projective stack over k. Suppose that* $\operatorname{Pic}(H) = 0$. *Then, every action of H on $\mathcal{X}$ lifts to an action of H on $\mathbb{A}^{r+1}$ via a homomorphism $H \to G_{n_0,n_1,\cdots,n_r}$.*

Proof. An action of H on $\mathcal{X}$ lifts to $\mathbb{A}^{r+1}$ if and only if the corresponding butterfly is equivalent to a strict one. By ([AlNo1], Proposition 4.5.3) this is equivalent to the central extension

$$1 \to \mathbb{G}_m \to E \to H \to 1$$

being split. By ([C-T], Corollary 5.7) such central extensions are classified by $\operatorname{Pic}(H)$, which in our case is assumed to be trivial. Any choice of a splitting amounts to a lift of the action of H to $\mathbb{A}^{r+1}$. □

This result is essentially saying that the obstruction to lifting the H-action from $\mathcal{P}(n_0, n_1, \cdots, n_r)$ to $\mathbb{A}^{r+1}$ lies in $\operatorname{Pic}(H)$.

9.1. **Description of the quotient 2-stack.** Given an action of a group stack $\mathcal{H}$ on the weighted projective stack $\mathcal{P}(n_0, n_1, \cdots, n_r)$, we can use the associated butterfly to get information about the quotient 2-stack $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]$. First, notice that $[\kappa\colon H_1 \to E]$ is also a crossed module in schemes. Denote the associated group stack by $\mathcal{H}'$. It acts on $\mathcal{P}(n_0, n_1, \cdots, n_r)$ via ρ. We have

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}] \cong [(\mathbb{A}^{r+1} - \{0\})/\mathcal{H}'].$$

Note that the right hand side is the quotient stack of the action of a group stack on an honest scheme, namely, $\mathbb{A}^{r+1} - \{0\}$. It is easy to describe its 2-stack structure by looking at cohomologies of the NW-SE sequence

$$H_1 \xrightarrow{\kappa} E \xrightarrow{\rho} G_{n_0,n_1,\cdots,n_r}$$

of the butterfly. Set

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_1 := [(\mathbb{A}^{r+1} - \{0\})/\operatorname{coker} \kappa],$$

and

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_0 := [(\mathbb{A}^{r+1} - \{0\})/\operatorname{im} \rho].$$

(Here, $\operatorname{coker} \kappa$ and $\operatorname{im} \rho$ are the sheaf theoretic cokernel and image of the corresponding maps which we assume are representable.) Then, $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_1$ is the best approximation of $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]$ by a 1-stack, in the sense that it is obtained by killing off the 2-automorphisms of the 2-stack $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]$. More precisely, there is a natural map

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}] \to [\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_1$$

making the former a 2-gerbe over the latter for the 2-group $[\ker\kappa \to 1]$. Similarly, $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_0$ is an orbifold (i.e., a Deligne-Mumford stack which is generically a scheme) obtained by quotienting out the generic 1-automorphisms. More precisely, there is a natural map

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_1 \to [\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]_0$$

making the former a gerbe over the latter for the group $\ker\rho/\operatorname{im}\kappa$ (namely, the middle cohomology of the NW-SE sequence).

It follows that the quotient 2-stack $[\mathcal{P}(n_0, n_1, \cdots, n_r)/\mathcal{H}]$ is equivalent to a 1-stack if and only if κ is injective; it is an orbifold if and only if the NW-SE sequence is left exact.

Example 9.2. Suppose that $\mathcal{H}$ is an honest group scheme and denote it by H. Then, to give an action of H on $\mathcal{P}(n_0, n_1, \cdots, n_r)$ is equivalent to giving a central extension

$$1 \to \mathbb{G}_m \to E \to H \to 1$$

of H by $\mathbb{G}_m$, together with a linear action of E on $\mathbb{A}^{r+1}$ extending the weighted action of $\mathbb{G}_m$. We have

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/H] \cong [(\mathbb{A}^{r+1} - \{0\})/E],$$

which is an honest 1-stack.

Example 9.3. Suppose that $\mathcal{H}$ is the group stack associated to $[A \to 1]$, where A is an abelian group scheme. We rename $\mathcal{H}$ to $A[1]$. Then, to give an action of $A[1]$ on $\mathcal{P}(n_0, n_1, \cdots, n_r)$ is equivalent to giving a character $\kappa\colon A \to \mu_d \subset \mathbb{G}_m$, where d is the greatest common divisor of $(n_0, n_1, \cdots, n_r)$. The quotient 2-stack $[\mathcal{P}(n_0, n_1, \cdots, n_r)/A[1]]$ is a 1-stack if and only if κ is injective. Assume this to be the case and identify A with the corresponding subgroup of μ_d. Then, roughly speaking, the quotient stack $[\mathcal{P}(n_0, n_1, \cdots, n_r)/A[1]]$ is obtained by killing the A in $\mu_d \subseteq I_x$ at every inertia group I_x of $\mathcal{P}(n_0, n_1, \cdots, n_r)$. (Note that the generic inertia group of $\mathcal{P}(n_0, n_1, \cdots, n_r)$ is μ_d.) For example, if the base is an algebraically closed field of characteristic prime to d, then

$$[\mathcal{P}(n_0, n_1, \cdots, n_r)/A[1]] \cong \mathcal{P}(\frac{n_0}{a}, \frac{n_1}{a}, \cdots, \frac{n_r}{a}),$$

where a is the order of A.

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