FIBRATIONS OF TOPOLOGICAL STACKS

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Abstract. In this note we define fibrations of topological stacks and establish their main properties. When restricted to topological spaces, our notion of fibration coincides with the classical one. We prove various standard results about fibrations (long exact sequence for homotopy groups, Leray-Serre and Eilenberg-Moore spectral sequences, etc.). We prove various criteria for a morphism of topological stacks to be a fibration, and use these to produce examples of fibrations. We prove that every morphism of topological stacks factors through a fibration and construct the homotopy fiber of a morphism of topological stacks. As an immediate consequence of the machinery we develop, we also prove van Kampen's theorem for fundamental groups of topological stacks.

Contents

1. Introduction

In this note we define fibrations of topological stacks and establish their main properties. Our theory generalizes the classical theory in the sense that when restricted to ordinary topological spaces our notion(s) of fibration reduce to the usual one(s) for topological spaces.

We prove some standard results for fibrations of topological stacks, e.g., the long exact sequence for homotopy groups (Theorem 5.2), the Leray-Serre spectral sequence (Theorems 7.6 and 7.7) and the Eilenberg-Moore spectral sequence (Theorem 8.1). We also prove that every morphism of topological stacks factors through a fibration (Theorem 6.1). We use this to define the homotopy fiber $\mathrm{hFib}(f)$ of a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks (see §6).

van Kampen's theorem for the fundamental groups of topological stacks is also an immediate corollary of the results we prove in this paper (Theorem 5.10 and Corollary 5.11).

Since our fibrations are not assumed to be representable, it is often not easy to check whether a given morphism of stacks is a fibration straight from the definition. We prove the following useful local criterion for a map f to be a weak Serre fibration;[1] see Theorem 3.19.

Theorem 1.1. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. If p is locally a weak Hurewicz fibration then it is a weak Serre fibration.*

In §4 we provide some general classes of examples of fibrations of stacks. Throughout the paper, we also prove various results which can be used to produce new fibrations out of old ones. This way we can produce plenty of examples of fibrations of topological stacks.

The results of this paper, though of more or less of standard nature, are not straightforward. The difficulty being that colimits are not well-behaved in the 2-category of topological stacks; for instance, gluing continuous maps along closed subsets is not always possible. For this reason, the usual methods for proving lifting properties (which involve building up continuous maps by attaching cells or extending maps from smaller subsets by inductive steps) often can not be applied to stacks.

The main technical input which allows us to circumvent this difficulty is the theory of classifying spaces for topological stacks developed in [No2] combined with Dold's results on fibrations [Do].

To reduce the burden of terminology we have opted to state the results of this paper only for topological stacks. There is, however, a more general class of stacks, called paratopological stacks, to which these results can be generalized – the proofs will be identical.

Paratopological stacks have a major advantage over topological stacks: they accommodate a larger class of mapping stacks (see [No4], Theorem 1.1), while essentially enjoying all the important features of topological stacks. This is especially

[1] For most practical purposes (e.g, constructing spectral sequences or the fiber homotopy exact sequence) weak fibrations are as good as fibrations.

important as mapping stacks provide a wealth of examples of fibrations (§4.1). The reader can consult [No2] and [No4] for more on paratopological stacks.

2. Review of homotopy theory of topological stacks

In this section, we recall some basic facts and definitions from [No1] and [No2]. For a quick introduction to topological stacks the reader may also consult [No5]. Our terminology is different from that of [No1] in that what is called a *pretopological* stack in [ibid.] is called a *topological* stack here.

2.1. **Topological stacks.** To fix our theory of topological stacks, the first thing to do is to fix the base Grothendieck site T. The main two candidates are the category Top of all topological spaces (with the open-cover topology) and the category CGTop of compactly generated (Hausdorff) topological spaces (with the open-cover topology). The latter behaves better for the purpose of having a theory of fibrations (as in [Wh]), so throughout the text our base Grothendieck site will be $\mathsf{T} := \mathsf{CGTop}$.

By a *topological stack* we mean a category $\mathcal{X}$ fibered in groupoids over T which is equivalent to the quotient stack of a topological groupoid $\mathbb{X} = [R \rightrightarrows X]$, with R and X topological spaces. The reader who is not comfortable with fibered categories may pretend that $\mathcal{X}$ is a presheaf of groupoids over T. For a topological space $T \in \mathsf{T}$, the value of $\mathcal{X}$ at T, or synonymously, the *groupoid of T-points of* $\mathcal{X}$, is denoted by $\mathcal{X}(T)$.

Topological stacks form a 2-category which is closed under 2-fiber products. For every two topological stacks $\mathcal{X}$ and $\mathcal{Y}$, a *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ between them is a morphism of the underlying fibered categories (or underlying presheaves of groupoids, if you wish). Given morphisms $f, g\colon \mathcal{X} \to \mathcal{Y}$, a *2-morphism* $\varphi\colon f \Rightarrow g$ is a natural transformation relative to T. If $h\colon \mathcal{Y} \to \mathcal{Z}$ is another morphism of stacks, we denote the induced 2-morphism $h \circ f \Rightarrow h \circ g$ by $h \circ \varphi$ or $h\varphi$. We use the multiplicative notation $\varphi\psi$ for the composition of 2-morphisms $\varphi\colon f \Rightarrow g$ and $\psi\colon g \Rightarrow h$. By *equivalence* of stacks we mean equivalence in the 2-categorical sense. The *2-fiber product* (or *fiber product* for short) $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ of stacks $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{Z}$ is defined by

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(T) = \mathcal{X}(T) \times_{\mathcal{Z}(T)} \mathcal{Y}(T), \quad \forall T \in \mathsf{T},$$

where the right hand side is the 2-categorical fiber product of groupoids.

Via Yoneda we identify T with a full subcategory of the 2-category of topological stacks. That is, we identify $T \in \mathsf{T}$ with the functor it represents, namely, $\mathrm{Hom}(-, T)$; note that this is a set-valued functor, but we can regard a set as a groupoid in which the only morphisms are the identities. The Yoneda embedding preserves fiber products. If a stack $\mathcal{X}$ is in the essential image of the Yoneda functor (i.e., is equivalent to $\mathrm{Hom}(-, T)$ for some $T \in \mathsf{T}$) we often abuse terminology and say that $\mathcal{X}$ is a topological space.

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks is called *representable* if for every morphism $Y \to \mathcal{Y}$ from a topological space Y, the fiber product $Y \times_{\mathcal{Y}} \mathcal{X}$ is (equivalent to) a topological space. It turns out that, for every topological stack $\mathcal{X}$, every morphism $f\colon X \to \mathcal{X}$, with X a topological space, is representable.

2.2. **Hurewicz and Serre topological stacks.**

Definition 2.1. A topological stack $\mathcal{X}$ is **Hurewicz** (respectively, **Serre**) if it admits a presentation $\mathbb{X} = [R \rightrightarrows X]$ by a topological groupoid in which the source (hence also the target) map $s\colon R \to X$ is locally (on source and target) a Hurewicz (respectively, Serre) fibration. That is, for every $a \in R$, there exists an open neighborhood $U \subseteq R$ of a and $V \subseteq X$ of $f(a)$ such that the restriction of $s|_U : U \to V$ is a Hurewicz (respectively, Serre) fibration.

Hurewicz (respectively, Serre) topological stacks from a full sub 2-category of the 2-category of topological stacks which is closed under 2-fiber products and contains the category of topological spaces.

Definition 2.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. We say that f is **Hurewicz** (respectively, **Serre**) if for every map $T \to \mathcal{Y}$ from a topological space T, the fiber product $T \times_{\mathcal{Y}} \mathcal{X}$ is a Hurewicz (respectively, Serre) topological stack (Definition 2.1).

Lemma 2.3. *Every representable morphism of topological stacks is Hurewicz and every Hurewicz morphism of topological stacks is Serre.*

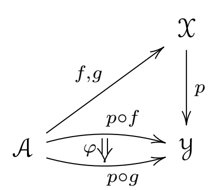
Proof. Trivial. □

Lemma 2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. If $\mathcal{X}$ is Hurewicz (respectively, Serre) then f is Hurewicz (respectively, Serre).*

Proof. Let $X \to \mathcal{X}$ be a chart for $\mathcal{X}$ such that the corresponding groupoid presentation $[R \rightrightarrows X]$, $R = X \times_{\mathcal{X}} X$, satisfies the condition of Definition 2.1. Then, for every topological space T mapping to $\mathcal{Y}$, the pullback groupoid $[R_T \rightrightarrows X_T]$ is a presentation for $\mathcal{X}_T = T \times_{\mathcal{Y}} \mathcal{X}$ which satisfies the condition of Definition 2.1. Here, the pullback groupoid is defined by $X_T = T \times_{\mathcal{Y}} X$ and $R_T = T \times_{\mathcal{Y}} R$. □

2.3. **Homotopy between maps.**

Definition 2.5. Let $p\colon \mathcal{X} \to \mathcal{Y}$ and $f, g\colon \mathcal{A} \to \mathcal{X}$ be morphisms of topological stacks. Let $\varphi\colon p \circ f \Rightarrow p \circ g$ be a 2-morphism



A **homotopy** from f to g relative to φ is a quadruple $(H, \epsilon_0, \epsilon_1, \psi)$ as follows:

- A map $H\colon I \times \mathcal{A} \to \mathcal{X}$, where I stands for the interval $[0, 1]$.
- A pair of 2-morphisms $\epsilon_0\colon f \Rightarrow H_0$ and $\epsilon_1\colon H_1 \Rightarrow g$. Here H_0 and H_1 stand for the maps $\mathcal{A} \to \mathcal{Y}$ obtained by restricting H to $\{0\} \times \mathcal{A}$ and $\{1\} \times \mathcal{A}$, respectively.
- A 2-morphism $\psi\colon p \circ \tilde{f} \Rightarrow p \circ H$ such that $\psi_0 = p \circ \epsilon_0$ and $(\psi_1)(p \circ \epsilon_1) = \varphi$. Here $\tilde{f}\colon I \times \mathcal{A} \to \mathcal{X}$ stands for $f \circ \mathrm{pr}_2$.

We usually drop ϵ_0, ϵ_1 and ψ from the notation. A **ghost homotopy** from f to g is a 2-morphism $\Phi\colon f \Rightarrow g$ such that $p \circ \Phi = \varphi$. Equivalently, this means that H can be chosen to be (2-isomorphic to) a constant homotopy (namely, one factoring through the projection $\mathrm{pr}_2\colon I \times \mathcal{A} \to \mathcal{A}$).

The usual notion of homotopy (and ghost homotopy) between maps $f, g\colon \mathcal{A} \to \mathcal{X}$ corresponds to the case where $\mathcal{Y}$ is a point. Note that in this case the 2-morphisms φ and ψ are necessarily the identity 2-morphisms.

As discussed in [No1], §17 (also see §2.4 below), gluing homotopies can in general be problematics unless we make certain fibrancy conditions on the target stack. There is, however, a way to glue homotopies which is well-defined up to a homotopy of homotopies and works for arbitrary topological stacks.

Lemma 2.6. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. Let $\mathcal{A}$ be a topological stack, and consider morphisms $f_1, f_2, f_3\colon \mathcal{A} \to \mathcal{Y}$, and 2-morphisms $\varphi_{12}\colon p \circ f_1 \Rightarrow p \circ f_2$ and $\varphi_{23}\colon p \circ f_2 \Rightarrow p \circ f_3$. If f_1 is homotopic to f_2 relative to φ_{12}, and f_2 is homotopic to f_3 relative to φ_{23}, then f_1 is homotopic to f_3 relative to $\varphi_{12}\varphi_{23}$.*

Proof. Let $(H_{12}, \epsilon_0, \epsilon_1, \psi)$ be a homotopy from f_1 to f_2 relative to φ_{12}. Similarly, let $(H_{23}, \delta_0, \delta_1, \chi)$ be a homotopy from f_2 to f_3 relative to φ_{23}. We construct a homotopy H_{13} from f_1 to f_3 relative to $\varphi_{12}\varphi_{23}$. The point of the following rather unusual construction is that gluing maps along closed subsets may not be possible, but we can always glue maps along open subsets.

Define $\tilde{H}_{12}\colon [0, 2/3) \times \mathcal{A} \to \mathcal{X}$ by

$$\tilde{H}_{12}(t, a) = \begin{cases} H_{12}(3t, a), & t \le 1/3 \\ H_{12}(1, a), & 1/3 \le t < 2/3 \end{cases}$$

More precisely, $\tilde{H}_{12} = H_{12} \circ R$, where $R = (r, \mathrm{id}_{\mathcal{A}})\colon [0, 2/3) \times \mathcal{A} \to I \times \mathcal{A}$ is defined by

$$r(t) = \begin{cases} 3t, & t \le 1/3 \\ 1, & 1/3 \le t < 2/3 \end{cases}$$

Similarly, we define $\tilde{H}_{23}\colon (1/3, 1] \times \mathcal{A} \to \mathcal{X}$ by

$$\tilde{H}_{23}(t, a) = \begin{cases} H_{23}(0, a), & 1/3 < t \le 2/3 \\ H_{23}(3t-2, a), & 2/3 \le t \le 1 \end{cases}$$

We glue $\tilde{H}_{12}$ to $\tilde{H}_{23}$ along the open set $(1/3, 2/3) \times \mathcal{A}$ using the 2-isomorphism

$$\Xi\colon \tilde{H}_{12}|_{(1/3,2/3)} \Rightarrow \tilde{H}_{23}|_{(1/3,2/3)}, \quad \Xi = \tilde{\epsilon}_1\tilde{\delta}_0,$$

where $\tilde{\epsilon}_1 = \epsilon_1 \circ \mathrm{pr}_2$, with $\mathrm{pr}_2\colon (1/3, 2/3) \times \mathcal{A} \to \mathcal{A}$ being the second projection ($\tilde{\delta}_0$ is defined similarly). Denote the resulting glued map by $H\colon [0, 1] \times \mathcal{A} \to \mathcal{X}$. The quadruple $(H, \epsilon_0, \delta_1, \psi)$ provides the desired homotopy from f_1 to f_3 relative to $\varphi_{12}\varphi_{23}$. We leave it to the reader to verify that the axioms of Definition 2.5 are satisfied. □

2.4. **Pushouts in the category of topological stacks.** A downside of the 2-category of topological stacks is that in it pushouts are not well-behaved. For example, let T be a topological space which is a union of two subspaces B and C, and let $A = B \cap C$. If B and C are open, then for any stack $\mathcal{X}$, any two morphisms $B \to \mathcal{X}$ and $C \to \mathcal{X}$ which agree along A can be glued to a morphism from T to $\mathcal{X}$ (this is simply the descent condition). If, however, B and C are not open, one may not, in general, be able to glue such overlapping morphisms. As we will see below in Proposition 2.7, this is partly remedied if we impose a cofibrancy condition on the inclusions $A \subset B$ and $A \subset C$ and a fibrancy condition on $\mathcal{X}$.

Proposition 2.7. *Let $i\colon A \hookrightarrow B$ and $j\colon A \hookrightarrow C$ be embeddings of topological spaces which are (locally) cofibrations. Then, the pushout $B \vee_A C$ remains a pushout in* *the 2-category of Hurewicz topological stacks (Definition 2.1). That is, for every Hurewicz topological stack $\mathcal{X}$, the diagram*

$$\begin{array}{ccc} \mathrm{Hom}(B \vee_A C, \mathcal{X}) & \longrightarrow & \mathrm{Hom}(C, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(B, \mathcal{X}) & \longrightarrow & \mathrm{Hom}(A, \mathcal{X}) \end{array}$$

is 2-cartesian. (The arrows in the diagram are the obvious restriction maps.)

Proof. Follows from ([No1], Theorem 16.2). □

We do not know if Proposition 2.7 remains true when the maps i or j are not embeddings, or when $\mathcal{X}$ is an arbitrary topological stack. It appears though that, even when (Y, A) is a nice pair (say an inclusion of a finite CW complex into another), the quotient space Y/A may not in general have the universal property of a quotient space when viewed in the category of (Hurewicz) topological stacks.

Nevertheless, things work rather well up to homotopy, as we see in Proposition 2.8 below. To state the proposition, we make the following definition. Let $f\colon A \to B$ and $g\colon A \to C$ be continuous maps of topological spaces. We define $B\bar{\triangledown}_A C$ to be

$$B\bar{\triangledown}_A C := B \vee_{A\times\{0\}} (A \times I) \vee_{A\times\{1\}} C.$$

In other words, $B\bar{\triangledown}_A C$ is obtained by gluing the two ends of the tube $A \times I$ to B and C using f and g. For a topological stack $\mathcal{X}$, we define the restricted hom $\mathrm{Hom}^*(B\bar{\triangledown}_A C, \mathcal{X})$ via the following 2-fiber product diagram

$$\begin{array}{ccc} \mathrm{Hom}^*(B\bar{\triangledown}_A C, \mathcal{X}) & \longrightarrow & \mathrm{Hom}(B\bar{\triangledown}_A C, \mathcal{X}) \\ \downarrow & & \downarrow r \\ \mathrm{Hom}(A, \mathcal{X}) & \xrightarrow[c]{} & \mathrm{Hom}(A \times I, \mathcal{X}) \end{array}$$

Here, r is induced by the inclusion $A \times I \hookrightarrow B\bar{\triangledown}_A C$, and c is induced by the projection $A \times I \to A$. In simple terms, the restricted hom consists of maps $B\bar{\triangledown}_A C \to \mathcal{X}$ which are constant along the tube $A \times I$. If $\mathcal{X} = X$ is an honest topological space, then $\mathrm{Hom}^*(B\bar{\triangledown}_A C, X) = \mathrm{Hom}(B \vee_A C, X)$.

Proposition 2.8. *Let $f\colon A \to B$ and $g\colon A \to C$ be continuous maps of topological spaces. Then, for every topological stack $\mathcal{X}$ the diagram*

$$\begin{array}{ccc} \mathrm{Hom}^*(B\bar{\triangledown}_A C, \mathcal{X}) & \longrightarrow & \mathrm{Hom}(C, \mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(B, \mathcal{X}) & \longrightarrow & \mathrm{Hom}(A, \mathcal{X}) \end{array}$$

of groupoids is 2-cartesian. (The arrows in the diagram are the obvious restriction maps.)

Proof. The proof of this proposition is very similar to the proof of Lemma 2.6. We sketch the idea. We want to show that two maps $u\colon B \to \mathcal{X}$ and $v\colon C \to \mathcal{X}$ which are identified by a 2-isomorphism along A give rise to a map $w\colon B\bar{\vee}_A C \to \mathcal{X}$ which is constant along the tube $A \times I$. The map $u\colon B \to \mathcal{X}$ gives rise to a map

$$\tilde{u}\colon B \vee_{A\times\{0\}} (A \times [0,1)) \to \mathcal{X},$$

$\tilde{u} := u \circ p$, where $p\colon B \vee_{A\times\{0\}} (A \times [0,1)) \to B$ is the map which is the identity on B and is $f \circ \mathrm{pr}_1$ on $A \times [0,1)$. Similarly, $v\colon C \to \mathcal{X}$ gives rise to a map

$$\tilde{v}\colon (A \times (0,1]) \vee_{A\times\{1\}} C \to \mathcal{X}.$$

The two maps $\tilde{u}$ and $\tilde{v}$ are 2-isomorphic along the open subset $A \times (0,1)$ of $B\bar{\vee}_A C$, so they glue to a map $w\colon B\bar{\vee}_A C \to \mathcal{X}$. □

2.5. Shrinkable morphisms of stacks.

Definition 2.9. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is **shrinkable** if it admits a section $s\colon \mathcal{Y} \to \mathcal{X}$ (meaning that $f \circ s$ is 2-isomorphic to $\mathrm{id}_{\mathcal{Y}}$ via some $\varphi\colon \mathrm{id}_{\mathcal{Y}} \Rightarrow f \circ s$) such that there is a homotopy from $\mathrm{id}_{\mathcal{X}}$ to $s \circ f$ relative to $\varphi \circ f$ (Definition 2.5).

We say that f is **locally shrinkable** if there is an epimorphism $\mathcal{Y}' \to \mathcal{Y}$ such that the base extension $f'\colon \mathcal{X}' \to \mathcal{Y}'$ of f to $\mathcal{Y}'$ is shrinkable.

In the case where $f\colon X \to Y$ is a continuous map of topological spaces, this definition coincides with the one in [Do], §1.5. In this case, s identifies Y with the subspace $s(Y)$ of X, and there is a fiberwise deformation retraction of X onto $s(Y)$ (the subspace $s(Y) \subseteq X$ need not remain fixed during the deformation). Note that, for a general morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$, a section s of f may not be an embedding (Example 2.10) unless f is representable. All we can say in general is that the section s is representable. To see this, choose an atlas $Y \to \mathcal{Y}$ and consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

Observe that s' is the base extension of s along the epimorphism h, and that it is representable because $\mathcal{X}'$ is a topological stack. The claim now follows from the fact that being representable is local on the target (see [No1], Lemma 6.3). For an alternative proof see Lemma 6.2.

Example 2.10. Consider the quotient stack $[\mathbb{R}/\mathbb{Q}]$, where $\mathbb{R}$ is the real numbers with the standard topology and $\mathbb{Q}$ is the rational numbers with the discrete topology. The action is given by translation. The map $f\colon [\mathbb{R}/\mathbb{Q}] \to *$ has the property that no section of f is an embedding. To see this, consider, for example, the zero section $s\colon * \to [\mathbb{R}/\mathbb{Q}]$. If s were an embedding, then so would be its base extension along the projection $\mathbb{R} \to [\mathbb{R}/\mathbb{Q}]$. The latter, however, is the inclusion of $\mathbb{Q}$, with the discrete topology, in $\mathbb{R}$.

As another example, let G be a nontrivial group and take the map $f\colon [*/G] \to *$. Then, f has a section which is not an embedding, namely, the quotient map $* \to [*/G]$. In fact, in the case where $G = \mathbb{R}$, the map f is shrinkable. The deformation retraction of $[*/\mathbb{R}]$ is given by the family $r_t\colon [*/\mathbb{R}] \to [*/\mathbb{R}]$, $0 \le t \le 1$, where r_t is induced by the scaling homomorphism $\mathbb{R} \to \mathbb{R}$, $x \mapsto tx$.

Lemma 2.11. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks and $s\colon \mathcal{Y} \to \mathcal{X}$ a section for it. Suppose that f is shrinkable onto s. Let $g\colon \mathcal{Y}' \to \mathcal{Y}$ be an arbitrary morphism. Then, the base extension $f'\colon \mathcal{X}' \to \mathcal{Y}'$ of f along g is shrinkable onto s'.*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle s'}\uparrow\downarrow{\scriptstyle f'} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

where s' is the section induced by s. In particular, any base extension of a (locally) shrinkable morphism is (locally) shrinkable.

Proof. Easy. □

Lemma 2.12. *A continuous map $f\colon X \to Y$ of topological spaces is locally shrinkable if and only if there is an open cover $\{U_i\}$ of Y such that $f|_{U_i}\colon f^{-1}(U_i) \to U_i$ is shrinkable for all i. A representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is locally shrinkable if for any map $T \to \mathcal{Y}$ from a topological space T, the base extension $f_T\colon \mathcal{X} \times_Y T \to T$ is locally shrinkable.*

Proof. Easy. □

Lemma 2.13. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a locally shrinkable representable morphism of stacks, then f is a universal weak equivalence. That is, for any map $T \to \mathcal{Y}$ from a topological space T, the base extension $f_T\colon \mathcal{X} \times_Y T \to T$ is a weak equivalence of topological spaces.*

Proof. This is [No2], Lemma 5.4. □

The following lemma provides a natural class of shrinkable morphisms constructed using mapping stacks. For a discussion of mapping stacks we refer the reader to [No4]. All we need to know here about mapping stacks $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ is the functoriality in the two variables $\mathcal{X}$ and $\mathcal{Y}$ and the exponential property.

Lemma 2.14. *Let X be a topological space and $r\colon I \times X \to X$ a deformation retraction of X onto a point $x \in X$. Let $\mathcal{Y}$ be a stack, and let $c_{\mathcal{Y}}\colon \mathcal{Y} \to$ $\mathrm{Map}(X, \mathcal{Y})$ be the morphism parametrizing the constant maps from X to $\mathcal{Y}$ (more precisely, $c_{\mathcal{Y}}$ is induced from $X \to *$ by the functoriality of the mapping stack). Let $\mathrm{ev}_x\colon \mathrm{Map}(X, \mathcal{Y}) \to \mathcal{Y}$ be the evaluation map at x. Then, $c_{\mathcal{Y}}$ is a section of ev_x and ev_x is shrinkable onto $c_{\mathcal{Y}}$.*

Proof. The deformation retraction $r\colon I \times X \to X$ induces

$$r^*\colon \mathrm{Map}(X, \mathcal{Y}) \to \mathrm{Map}(I \times X, \mathcal{Y}) \cong \mathrm{Map}(I, \mathrm{Map}(X, \mathcal{Y})).$$

Using the exponential property again, this gives the desired shrinking map

$$H\colon I \times \mathrm{Map}(X, \mathcal{Y}) \to \mathrm{Map}(X, \mathcal{Y}).$$

That is, H is fiberwise homotopy from $\mathrm{id}_{\mathrm{Map}(X,\mathcal{Y})}$ to $c_{\mathcal{Y}} \circ \mathrm{ev}_x$. □

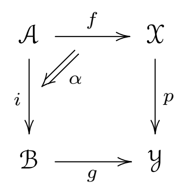
2.6. **Classifying spaces of topological stacks.** The main theorem here is the following. It will be of crucial importance for us throughout the paper.

Theorem 2.15 ([No2], Theorem 6.3)**.** *Every topological stack $\mathcal{X}$ admits an atlas $\varphi\colon X \to \mathcal{X}$ which is locally shrinkable. In particular, φ is a universal weak equivalence.*

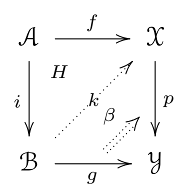
By definition ([No2], Definition 6.2), such an atlas $\varphi\colon X \to \mathcal{X}$ is a *classifying space* for $\mathcal{X}$. A classifying space X captures the homotopy theoretic information in $\mathcal{X}$ via the map φ. In the following sections we will make use of classifying spaces to reduce problems about topological stacks to ones about topological spaces.

3. Fibrations between stacks

Definition 3.1. Let $i\colon \mathcal{A} \to \mathcal{B}$ and $p\colon \mathcal{X} \to \mathcal{Y}$ be morphisms of stacks. We say that i has the **weak left lifting property** (WLLP) with respect to p, if for every 2-commutative diagram



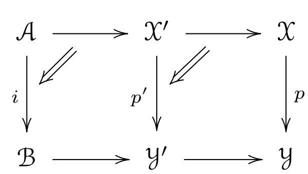
we can find a morphism $k\colon \mathcal{B} \to \mathcal{X}$, a 2-morphism $\beta\colon p \circ k \Rightarrow g$, and a fiberwise homotopy (Definition 2.5) H from f to $k \circ i$ relative to $(\alpha)(\beta \circ i)$, as in the diagram



If the homotopy H can be chosen to be a ghost homotopy (Definition 2.5), we say that i has the **left lifting property** (LLP) with respect to p. (So $p \circ H$ is a ghost homotopy from $p \circ f$ to $p \circ k \circ i$ and we have $p \circ H = (\alpha)(\beta \circ i)\colon p \circ f \Rightarrow p \circ k \circ i$.)

Definition 3.2 ([Do] §5)**.** Let $\mathcal{A}$ be a stack and $p\colon \mathcal{X} \to \mathcal{Y}$ a morphism of stacks. We say that p has the **weak covering homotopy property** (WCHP) for $\mathcal{A}$ if the inclusion $i\colon \mathcal{A} \to \mathcal{A} \times I$, $a \mapsto (a, 0)$, has WLLP with respect to p. Given a fixed class $\mathcal{T}$ of stacks (e.g., all compactly generated topological spaces, paracompact spaces, CW complexes, etc.) we say that p has the WCHP with respect to $\mathcal{T}$ if it has WCHP for all $\mathcal{A} \in \mathcal{T}$. Similar definitions can be made for **covering homotopy property** (CHP).

Lemma 3.3. *Consider the 2-commutative diagram of stacks*

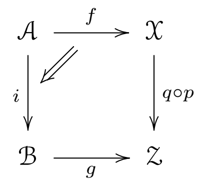


Assume that the right square is 2-cartesian. Then, the (weak) left lifting problem of i can be solved with respect to p if and only if it can be solved with respect to p'.

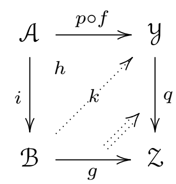
Proof. Straightforward. □

Lemma 3.4. *Let $p\colon \mathcal{X} \to \mathcal{Y}$, $q\colon \mathcal{Y} \to \mathcal{Z}$ and $i\colon \mathcal{A} \to \mathcal{B}$ be morphisms of stacks. If i has LLP with respect to p and q, then it has LLP with respect to $q \circ p$. If i has WLLP with respect to p and q, and p has WCHP with respect to $\mathcal{A}$, then i has WLLP with respect to $q \circ p$.*

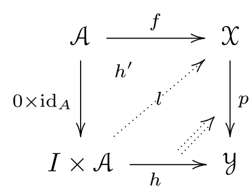
Proof. The lemma is straightforward without the 'weak' adjective. The case where the adjective 'weak' is present is less trivial, so we give more details. Consider the weak lifting problem



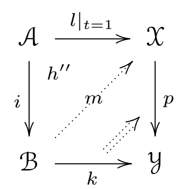
We solve it in three steps. First we solve



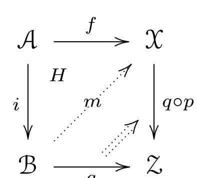
Here, h is a fiberwise homotopy from $p \circ f$ to $k \circ i$. Next, we solve



Here, h' is a fiberwise homotopy from f to $l|_{t=0}$. Finally, we solve



Here, h'' is a fiberwise homotopy from $l|_{t=1}$ to $m \circ i$. The solution to our original problem would then be



where H is the fiberwise homotopy from f to $m \circ i$ obtained by gluing h', l, and h'' (Lemma 2.6). □

Lemma 3.5. *Let $p\colon \mathcal{X} \to \mathcal{Y}$, $q\colon \mathcal{Y} \to \mathcal{Z}$ and $i\colon \mathcal{A} \to \mathcal{B}$ be morphisms of stacks. If $\emptyset \to \mathcal{A}$ has LLP with respect to p and i has LLP (resp., WLLP) with respect to $q \circ p$, then i has LLP (resp., WLLP) with respect to q.*

Proof. Straightforward. □

Definition 3.6. Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. We say that p is a **Hurewicz fibration**, if it has CHP for all compactly generated topological spaces. We say that p is a **Serre fibration**, if it has CHP for all finite CW complexes. (In general, we can define a $\mathcal{T}$-*fibration* to be a map which has CHP for $\mathcal{T}$.) Similarly, we can define *weak Hurewicz fibration* and *weak Serre fibration* (more generally, *weak* $\mathcal{T}$-*fibration*).

Definition 3.7. Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of stacks. We say that p is a **(weak) trivial Hurewicz fibration**, if every cofibration $i\colon A \to B$ of topological spaces has (W)LLP with respect to p. We say that p is **(weak) trivial Serre fibration** if every cellular inclusion $i\colon A \to B$ of CW complexes has (W)LLP with respect to p.

Lemma 3.8. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ and $q\colon \mathcal{Y} \to \mathcal{Z}$ be morphisms of stacks. If p and q are (weak) (trivial) Hurewicz/Serre fibrations, then so is $q \circ p$.*

Proof. Follows from Lemma 3.4. □

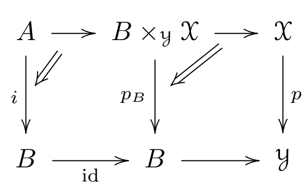
Definition 3.9. Let P be a property of morphisms of stacks which is invariant under base extension (e.g., any of the properties defined in Definitions 3.6, 3.7). We say that a morphism $p\colon \mathcal{X} \to \mathcal{Y}$ is stacks is **locally** P, if for some epimorphism $\mathcal{Y}' \to \mathcal{Y}$ the base extension $p'\colon \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is P.

Lemma 3.10. *Let P be a property of morphisms of stacks which is invariant under base extension. For example, P can be any of the following: (weak) (trivial) Hurewicz/Serre/$\mathcal{T}$ fibration. Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks and $\mathcal{Y}' \to \mathcal{Y}$ an epimorphism. If the base extension $p'\colon \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is locally P, then so is p.*

Proof. Straightforward. □

Lemma 3.11. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Let P be any of the properties (locally) (weak) (trivial) Hurewicz/Serre fibration (or, $\mathcal{T}$-fibration, for a class $\mathcal{T}$ of topological spaces). If p is P, then the base extension of p along any morphism $\mathcal{Y}' \to \mathcal{Y}$ of stacks is again P. Conversely, if the base extension of p along every morphism $B \to \mathcal{Y}$, with B a topological space, is P, then p is P. (In the case of (weak) $\mathcal{T}$-fibrations it is enough to take B in $\mathcal{T}$.)*

Proof. The proof is a simple application of Lemma 3.3. Here is how the typical argument works. Let $i\colon A \to B$ be a map for which we want to prove (W)LLP. To see that i has (W)LLP with respect to p, apply Lemma 3.3 to the following diagram



□

If a continuous map $p\colon X \to Y$ of topological spaces is a trivial Hurewicz fibration then p is shrinkable. The converse does not seem to be true, but we have the following.

Proposition 3.12. *Let $p\colon X \to Y$ be a continuous map of topological spaces. Then, the following are equivalent:*

1) *Every continuous map $i\colon A \to B$ has WLLP with respect to p;*
2) *The map p is a weak trivial Hurewicz fibration (Definition 3.7);*
3) *The map p is a weak Hurewicz fibration and a homotopy equivalence;*
4) *The map p is shrinkable.*

Proof. The implication 1) ⇒ 2) is obvious.

Proof of 2) ⇒ 3). Let $\mathrm{Cyl}(p) = (X \times [0,1]) \coprod_p Y$ be the mapping cylinder of p, and consider the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle i}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathrm{Cyl}(p) & \xrightarrow[g]{} & Y \end{array}$$

Here, $i\colon X \to \mathrm{Cyl}(p)$ is the natural inclusion of X in $\mathrm{Cyl}(p)$, and $g\colon \mathrm{Cyl}(p) \to Y$ is defined by $g(x,t) = p(x)$ and $g(y) = y$, for $x \in X$ and $y \in Y$. Since i is a Hurewicz cofibration, we can find a lift k in the diagram. If we set $q = k|_Y : Y \to X$, it follows that $p \circ q = \mathrm{id}_Y$ and $q \circ p$ is homotopic to id_X. This proves that p is a homotopy equivalence.

To see that p is a weak Hurewicz fibration, observe that the $t = 0$ inclusion $A \to A \times [0,1]$ is a Hurewicz cofibration.

Proof of 3) ⇒ 4). This is [Do], Corollary 6.2.

Proof of 4) ⇒ 1). Consider a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

where i is an arbitrary continuous map. Since p is shrinkable, it admits a section $s\colon Y \to X$. Set $k = s \circ g$. This makes the lower triangle commutative. The fiberwise deformation retraction of X onto $s(X)$ provides a fiberwise homotopy between f and $k \circ i$. □

Corollary 3.13. *A representable locally shrinkable map $p\colon \mathcal{X} \to \mathcal{Y}$ of stacks is locally a weak trivial Hurewicz fibration (Definition 3.9).*

Proof. By Lemma 3.11, we are reduced to the case where X and Y are spaces. The claim follows from Proposition 3.12. □

Lemma 3.14. *Let $p\colon X \to Y$ be a locally shrinkable map of topological spaces. If Y is paracompact, then p is shrinkable. In particular, every map $\emptyset \to A$, with A a paracompact topological space, has LLP with respect to every locally shrinkable representable morphism $p\colon \mathcal{X} \to \mathcal{Y}$ of topological stacks.*

Proof. Follows from [Do], §2.1. □

Proposition 3.15 (Hurewicz Uniformization Theorem)**.** *Let $p\colon X \to Y$ be locally a (weak) (trivial) Hurewicz fibration of topological spaces, with Y paracompact. Then, p is a (weak) (trivial) Hurewicz fibration.*

Proof. Without the 'trivial' adjective this follows from [Do], Theorems 4.8 and 5.12. If p is locally a weak trivial Hurewicz fibration, then it is locally shrinkable (Proposition 3.12), hence shrinkable (Lemma 3.14), and so a weak trivial Hurewicz fibration (Proposition 3.12).

If p is locally a trivial Hurewicz fibration, then, as we explained in the beginning of the proof, it follows from Dold's result that p is a Hurewicz fibration . On the other hand, we just showed that p is a weak trivial Hurewicz fibrations. Therefore, by Proposition 3.12, p is a homotopy equivalence. This proves that p is a trivial Hurewicz fibration. □

In practice, it is much easier to check that a given morphism of stacks is *locally* a fibration. However, for applications (such as the homotopy exact sequence of a fibration) we need to have a global fibration (at least a weak one). Proposition 3.16 and Theorem 3.19 provide local criteria for a morphism of stacks to be a (weak) Serre fibration.

Proposition 3.16. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of topological stacks. If p is locally a (weak) (trivial) Hurewicz fibration, then the base extension of p along any map $Y \to \mathcal{Y}$ from a paracompact topological space Y is a (weak) (trivial) Hurewicz fibration. In particular, p is a (weak) (trivial) Serre fibration.*

Proof. By Lemma 3.11, it is enough to prove the statement after base extending p along an arbitrary map $Y \to \mathcal{Y}$ from a topological space Y. So we are reduced to the case where $p\colon X \to Y$ is a continuous map of topological spaces.

To show that p is a (weak) (trivial) Serre fibration it is enough to check that the base extension $p_K\colon K \times_Y X \to K$ of p along any morphism $g\colon K \to Y$, with K a finite CW complex, is a (weak) (trivial) Serre fibration. Since being locally a (weak) Hurewicz fibration is invariant under base change, p_K is locally a (weak) (trivial) Hurewicz fibration. Since K is paracompact, p_K is indeed a (weak) (trivial) Hurewicz fibration (Proposition 3.15), hence also a (weak) (trivial) Serre fibration. □

Corollary 3.17. *Let $\mathcal{X}$ be a topological stack and $\varphi\colon X \to \mathcal{X}$ a classifying space for $\mathcal{X}$ as in Theorem 2.15. Then φ is a weak trivial Serre fibration.*

Lemma 3.18. *Let $p\colon X \to \mathcal{Y}$ and $q\colon \mathcal{Y} \to \mathcal{Z}$ be morphisms of topological stacks, with X a topological space. Suppose that p is locally shrinkable and $q \circ p$ is a weak (trivial) Serre fibration. Then q is a weak (trivial) Serre fibration.*

Proof. Follows from Lemmas 3.5 and 3.14. □

We now come to the main result of this section. It strengthens Proposition 3.16 by removing the representability condition on p, at the cost of having to add the adjective 'weak'.

Theorem 3.19. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be an arbitrary morphism of topological stacks. If p is locally a weak (trivial) Hurewicz fibration then it is a weak (trivial) Serre fibration.*

Proof. Let $\varphi\colon X \to \mathcal{X}$ be a classifying space for $\mathcal{X}$ as in Theorem 2.15. Since φ is locally shrinkable, it is locally a weak trivial Hurewicz fibration (Proposition 3.12). By Lemma 3.8, $p \circ \varphi$ is also locally a weak (trivial) Hurewicz fibration. By Proposition 3.16, $p \circ \varphi$ is a weak (trivial) Serre fibration. The claim follows from Lemma 3.18. $\square$

We also have the following result about trivial Serre fibrations whose proof we pospone to §5.2.

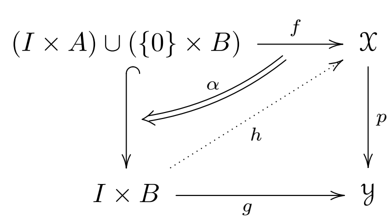
Proposition 3.20. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a Serre morphism of topological stacks (Definition 2.2). Then, p is a trivial Serre fibration if and only if it is a Serre fibration and a weak equivalence (i.e., induces isomorphisms on all homotopy groups).*

3.1. **Lifting property of fibrations with respect to cofibrations.**

Proposition 3.21. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. Suppose that p is a Hurewicz (respectively, Serre) morphism (Definition 2.2). If p is a Hurewicz (respectively, Serre) fibration, then every trivial cofibration (respectively, cellular inclusion of finite CW complexes inducing isomorphisms on all π_n) $i\colon A \to B$ has LLP with respect to p. Here, by a trivial cofibration we mean a DR pair (B, A) of topological spaces in the sense of* [Wh].

Proof. Using the usual base extension trick, we may assume that $\mathcal{Y} = Y$ is a topological space and $\mathcal{X}$ is a Hurewicz (respectively, Serre) topological stack. The proof of Theorem 7.16 of [Wh] now applies. $\square$

Proposition 3.22. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. Suppose that p is a Hurewicz (respectively, Serre) morphism (Definition 2.2) and a Hurewicz (respectively, Serre) fibration. Then, for every cofibration $i\colon A \to B$ of topological spaces, every homotopy lifting extension problem*

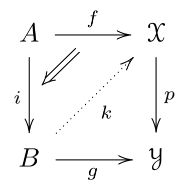


has a solution $h\colon I \times B \to \mathcal{X}$.

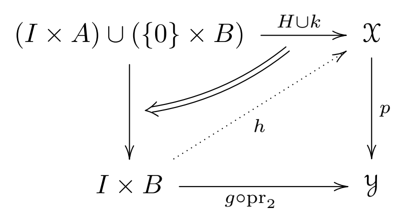
Proof. Use Proposition 3.21 and repeat the proof of [Wh], Theorem 7.16. $\square$

Corollary 3.23. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a Hurewicz (respectively, Serre) morphism of topological stacks. If p is a Hurewicz (respectively, Serre) fibration and a weak trivial Hurewicz (respectively, Serre) fibration (Definition 3.7), then it is a trivial Hurewicz (respectively, Serre) fibration.*

Proof. Consider a lifting problem



Since p is a weak trivial fibration, there is a lift $k\colon B \to \mathcal{X}$ together with a fiberwise homotopy H from $f' := k \circ i$ to f. By Proposition 3.22, we can solve the homotopy lifting extension problem



Letting $f := h|_{\{1\}\times B}$ we find a solution to our original lifting problem. □

3.2. **Summary.** For the convenience of the reader we highlight some of the main ideas and results about fibrations discussed in this section. These can be divided into three categories.

The first set of results establish the functoriality of fibrations (e.g., invariance under composition and base change). These are completely straightforward and their proofs are easy. For example, see Lemmas 3.8, 3.10 and 3.11.

The second set of results provide local criteria for fibrations. The main tool here is the Hurewicz Uniformization Theorem (see Proposition 3.15). Of course, already in the case of topological spaces, the situation is not ideal: if $f\colon X \to Y$ is locally, on Y, a Serre fibration, we do not know if f is a Serre fibration; if f is locally a Hurewicz fibration then all we can say is that its base extension to a paracompact Y' is a Hurewicz fibration. Proposition 3.16 shows that the same thing is true for representable morphisms of topological stacks; in particular, a classifying space $\varphi\colon X \to \mathcal{X}$ in the sense of Theorem 2.15 is a weak trivial Serre fibration (Corollary 3.17). For arbitrary morphisms of topological stacks, Theorem 3.19 establishes the same result in the case of 'weak fibrations.'

The third set of results generalize standard facts about fibrations of topological spaces to the case of topological stacks. For example, Proposition 3.20 shows that a Serre fibration that is also a weak equivalence is a trivial Serre fibration (i.e., has the RLP with respect to CW inclusions). The proof is in fact quite difficult and is deferred to §5.2 (see Proposition 5.4). The similar statement is, of course not true for Hurewicz fibrations, even in the classical case. In Corollary 3.23 we show, however, that if f is a Hurewicz (resp., Serre) fibration that is also a weak trivial Hurewicz (resp., Serre) fibration, and furthermore it satisfies the property of Defintion 2.2, then it is a trivial Hurewicz (resp., Serre) fibration.

One final remark we would like to make is the special attention we have given to the adjective 'weak' in the statements of some of the results in this section. On the one hand, weak fibrations have most of the nice properties of fibrations (e.g., homotopy exact sequneces and spectral sequences – see §5, §7 and §8). On the other hand, weak fibrations are more aboundant than fibrations, especially in the world of stacks: every classifying space $\varphi\colon X \to \mathcal{X}$ is a weak trivial Serre fibration; also, being a weak Serre fibration can be checked locally (Theorem 3.19). Ideally, we wish both these facts to be true without the 'weak' adjective as well, but we have not been able to prove or disprove this.

4. Examples of fibrations

In this section we discuss a few general classes of examples of fibrations of stacks. Of course, a trivial class of examples is the ones coming from topological spaces, because any notion of fibration we have defined for morphisms of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ coincides with the corresponding classical notion when $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces.

Another tirival class of examples is the projections maps.

Lemma 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary stacks. Then, the projection $\mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is a Hurewicz fibration.*

Proof. Trivial. □

4.1. Fibrations induced by mapping stacks.

Proposition 4.2. *Let $A \to B$ be a cofibration of topological spaces, and let $\mathcal{X}$ be a stack. Then the induced morphism*

$$\mathrm{Map}(B, \mathcal{X}) \to \mathrm{Map}(A, \mathcal{X})$$

is a Hurewicz fibration.

Proof. This is Theorem 7.8 of [Wh]. The same proof works. □

Example 4.3. Let $L\mathcal{X}$ and $P\mathcal{X}$ be the loop and path stacks of a stack $\mathcal{X}$. Then, the time t evaluation maps $\mathrm{ev}_t\colon L\mathcal{X} \to \mathcal{X}$ and $\mathrm{ev}_t\colon P\mathcal{X} \to \mathcal{X}$ are Hurewicz fibrations. Also, the map $(\mathrm{ev}_0, \mathrm{ev}_1)\colon P\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a Hurewicz fibration.

Proposition 4.4. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a (weak) (trivial) Hurewicz (respectively, Serre) fibration of stacks. Then, for any topological space (respectively, CW complex) Z the induced map*

$$\mathrm{Map}(Z, \mathcal{X}) \to \mathrm{Map}(Z, \mathcal{Y})$$

is a (weak) (trivial) Hurewicz (respectively, Serre) fibration.

Proof. This is Theorem 7.10 of [Wh]. The same proof works. □

4.2. **Quotient stacks.** Let $\mathcal{X}$ be a topological stack and $[R \rightrightarrows X]$ a topological groupoid presentation for it. If $s\colon R \to X$ is a (weak) (trivial) Hurewicz/Serre fibration, then the quotient map $p\colon X \to \mathcal{X}$ is locally a (weak) (trivial) Hurewicz/Serre fibration. This follows from the 2-cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow{t} & X \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & \mathcal{X} \end{array}$$

and Lemma 3.10.

In particular, by Proposition 3.15, if s is a (weak) (trivial) Hurewicz fibration, then $p\colon X \to \mathcal{X}$ is a (weak) (trivial) Serre fibration whose base extension along any map $T \to \mathcal{X}$ from a paracompact T is a (weak) (trivial) Hurewicz fibration.

Example 4.5. The above discussion applies to the case $\mathcal{X} = [X/G]$, where G is a topological group acting on a topological space X. Therefore, the quotient map $p\colon X \to [X/G]$ is Serre fibration (with fiber G) which becomes a Hurewicz fibration upon base extension along any map $T \to [X/G]$ from a paracompact T. If the group G is contractible, then p is a trivial Serre fibration whose base extension p_T is a trivial Hurewicz fibration for T paracompact.

4.3. **Covering maps.** Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a covering morphism of stacks in the sense of [No1], §18. Then p is a Hurewicz fibration. This is true thanks to Lemma 3.11.

4.4. **Gerbes.** Another source of examples of weak Serre fibrations are gerbes. We will not need the general definition of a gerbe here (for this see, e.g., [Mo] Definition 3.1), so we only discuss a special class, namely G-gerbes. Here, $G \to Y$ stands for a bundle of group (over a topological space Y). The simplest example of a G-gerbe over Y is the *trivial gerbe* (or the *classifying gerbe*) which is nothing but $B_Y G := [Y/G]$; here G acts trivially on Y. More generally, a G-gerbe over Y is a stack $\mathcal{X} \to Y$ over Y which is locally (on Y) equivalent to a trivial gerbe.

Proposition 4.6. *Let Y be a topological space and $G \to Y$ a locally trivial bundle of topological groups over Y. Let $\mathcal{X}$ be a G-gerbe over Y. Then, the structure map $p\colon \mathcal{X} \to Y$ is locally a Hurewicz fibration, hence, in particular, a weak Serre fibration.*

Proof. Note that we can work locally on Y, so we may assume that $G \to Y$ is a trivial bundle, i.e., is of the form $H \times X \to X$ for some topological group H. By further shrinking Y, we may also assume that $\mathcal{X} = [Y/H]$, for the trivial action of H on Y. But in this case $\mathcal{X} = [Y/H] = [*/H] \times Y$, and the map $p\colon \mathcal{X} \to Y$ is simply the second projection. In this case, by Lemma 4.1, p is a Hurewicz fibration. The last statement follows from Theorem 3.19. ☐

In the discussion above the assumption on Y being a topological space is not essential. Due to the local nature of the notion of a gerbe, and also that of a local Hurewicz fibration, everything makes sense, and remains valid, when Y itself is a topological stack.

5. Long exact sequence of homotopy groups

5.1. **Homotopy groups of topological stacks.** There are at least two ways to define homotopy groups of a pointed topological stack $(\mathcal{X}, x)$. One is discussed in [No1], §17. It is the standard definition

$$\pi_n(\mathcal{X}, x) := [(S^n, *), (\mathcal{X}, x)]$$

in terms of homotopy classes of pointed maps. For this definition to make sense, it is argued in *loc. cit.* that $\mathcal{X}$ needs to be a Serre topological stack (Definition 2.1). However, it was pointed out to me by A. Henriques that this definition makes sense for all topological stacks thanks to Lemma 2.6.

The second definition for the homotopy groups makes use of a classifying space $\varphi\colon X \to \mathcal{X}$ for $\mathcal{X}$ (see §2.6). More precisely, we define

$$\pi_n(\mathcal{X}, x) := \pi_n(X, \tilde{x}),$$

where $\tilde{x}$ is a lift of x to X. It is shown in ([No2], §10) that this is well-defined up to canonical isomorphism. In fact, it is shown in ([No2], Theorem 10.5) that this definition is equivalent to the previous definition whenever $\mathcal{X}$ is Serre. Thanks to Lemma 2.6, the assumption on $\mathcal{X}$ being Serre is redundant and the two definitions are indeed equivalent for all topological stacks $\mathcal{X}$, as we will see in Corollary 5.3.

Lemma 5.1. *Let $A \hookrightarrow B$ be a cofibration of topological spaces. Let $a \in A$ a point. Let $(\mathcal{X}, x)$ be a pointed topological stack. Then, the natural map*

$$[(B/A, a), (\mathcal{X}, x)] \to [(B, A, a), (\mathcal{X}, x, x)]$$

is a bijection. Here, $[-,-]$ stands for homotopy classes of maps (of pairs or triples).

Proof. The point is that, given a map of triples $(B, A, a) \to (\mathcal{X}, x, x)$, instead of trying to produce a map $B/A \to \mathcal{X}$, which may actually not exist, we can construct a map of triples $(CA \vee_A B, CA, a) \to (\mathcal{X}, x, x)$, uniquely up to homotopy. (Here, CA stands for the cone of A.) This is true thanks to Proposition 2.8. It follows that the natural map

$$[(CA \vee_A B, CA, a), (\mathcal{X}, x, x)] \to [(B, A, a), (\mathcal{X}, x, x)]$$

is a bijection.

Since $A \hookrightarrow B$ is a cofibration, the quotient map $(CA \vee_A B, CA, a) \to (B/A, a, a)$ is a homotopy equivalence of triples. So,

$$[(B/A, a, a), (\mathcal{X}, x, x)] \cong [(CA \vee_A B, CA, a), (\mathcal{X}, x, x)] \cong [(B/A, a), (\mathcal{X}, x)].$$

This completes the proof. ☐

5.2. **Homotopy exact sequence of a fibration.** In this subsection we prove that the two definitions given in §5.1 for homotopy groups of topological stacks agree and they give rise to exact sequences for homotopy groups whenever we have a weak Serre fibration.

Theorem 5.2. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a weak Serre fibration of topological stacks. Let $x\colon * \to \mathcal{X}$ be a point in $\mathcal{X}$, and $\mathcal{F} := * \times_{\mathcal{Y}} \mathcal{X}$ the fiber of p over $y := p(x)$. Then, there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_{n+1}(\mathcal{Y}, y) \to \pi_n(\mathcal{F}, x) \to \pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{Y}, y) \to \pi_{n-1}(\mathcal{F}, x) \to \cdots.$$

Here, $\pi_n(\mathcal{X}, x)$ stands for either of the two definitions of homotopy groups given in §5.1.

Proof. For the first definition of π_n (using homotopy classes of maps from S^n) the classical proof carries over, except that one has to be careful that to give a map of pairs $(\mathbb{D}^n, \partial\mathbb{D}^n) \to (\mathcal{X}, x)$ is not the same thing as giving a pointed map $(\mathbb{D}^n/\partial\mathbb{D}^n, *) \to (\mathcal{X}, x)$. This, however, is fine if we work up to homotopy, thanks to Lemma 5.1.

For the second definition of π_n we can assume, by making a base extension along a universal weak equivalence $Y \to \mathcal{Y}$, that $\mathcal{Y} = Y$ is a topological space. Choose a map $X \to \mathcal{X}$ which is a universal weak equivalence and a weak Serre fibration (see Theorem 2.15 and Corollary 3.17). It is enough to prove the statement for the composite map $p'\colon X \to Y$. But p' is a weak Serre fibration of topological spaces by Lemma 3.8, and the claim in this case is standard. ☐

Corollary 5.3. *The two definitions for $\pi_n(\mathcal{X}, x)$ given in §5.1 coincide.*

Proof. Use Corollary 3.17 and Theorem 5.2. □

Proposition 5.4. *Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a Serre morphism of topological stacks (Definition 2.2). Then, p is a trivial Serre fibration if and only if it is a Serre fibration and a weak equivalence (i.e., induces isomorphisms on all homotopy groups).*

Proof. Assume that p is a Serre fibration and a weak equivalence. Pick a classifying space $\varphi\colon X \to \mathcal{X}$ for X. By Corollary 3.17, $f := p\circ\varphi$ is a weak Serre fibration and a weak equivalence. Therefore, the base extension f_T of f along an arbitrary map $T \to \mathcal{Y}$ from a topological space T is weak Serre fibration of topological spaces and has contractible fibers (by Theorem 5.2). Thus, f_T is a weak trivial Serre fibration. Since $T \to \mathcal{Y}$ is arbitrary, it follows from Lemma 3.11 that $f = p\circ\varphi$ is a weak trivial Serre fibration. Lemma 3.18 implies that p itself is a weak trivial Serre fibration. Since p is also a Serre fibration, it follows from Corollary 3.23 that it is a trivial Serre fibration.

To prove the converse, first we consider the case where p is representable. To show that p induces isomorphisms on homotopy groups, it is enough to show that the fibers of p have vanishing homotopy groups (Theorem 5.2). But this is obvious because, by Lemma 3.11, the base extension of p along every map $y\colon * \to \mathcal{Y}$ is a trivial Serre fibration.

Now, let p be arbitrary. Pick a classifying space $\varphi\colon X \to \mathcal{X}$ for X. It follows from Corollary 3.17 and Lemma 3.8 that $p\circ\varphi$ is a weak trivial Serre fibration. Therefore, by the representable case, $p\circ\varphi$ is a weak equivalence. Since φ is also a weak equivalence, it follows that p is a weak equivalence. □

5.3. **Some examples of long exact sequences.** Let us work out the long exact sequences coming from the examples discussed in §4.

Example 5.5. Let $\mathcal{X}$ be a topological stack and $x\colon * \to \mathcal{X}$ a point in it. Let $P_*\mathcal{X}$ be the stack of paths in $\mathcal{X}$ initiating at x, and consider $\mathrm{ev}_1\colon P_*\mathcal{X} \to \mathcal{X}$. Equivalently, ev_1 is the base extension of the fibration $(\mathrm{ev}_0, \mathrm{ev}_1)\colon P\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ (see Example 4.3) along the map $(x, \mathrm{id}_{\mathcal{X}})\colon \mathcal{X} \to \mathcal{X}\times\mathcal{X}$. The fiber of $\mathrm{ev}_1\colon P_*\mathcal{X} \to \mathcal{X}$ over x is the based loop stack $\Omega_x\mathcal{X}$, that is, we have a fiber sequence

$$\Omega_x\mathcal{X} \to P_*\mathcal{X} \to \mathcal{X}.$$

Notice that $P_*\mathcal{X}$, being a fiber of the shrinkable map (Lemma 2.14) $\mathrm{ev}_0\colon P\mathcal{X} \to \mathcal{X}$, is a contractible stack (Lemma 2.11). Therefore, by looking at the long exact sequence of the above fibration, we find that $\pi_n(\Omega_x\mathcal{X}) \cong \pi_{n+1}\mathcal{X}$, for $n \geq 0$.

Example 5.6. Let $\mathcal{X} = [X/R]$ be the quotient stack of a topological groupoid $[R \rightrightarrows X]$ such that $s\colon R \to X$ is a Hurewicz fibration. Suppose for simplicity that X is connected. Let x be a point of X and F the fiber of s over x. Then, we have a fiber sequence

$$F \to X \to \mathcal{X}.$$

The corresponding long exact sequence of homotopy groups is

$$\cdots \to \pi_{n+1}\mathcal{X} \to \pi_n F \to \pi_n X \to \pi_n\mathcal{X} \to \pi_{n-1}F \to \cdots .$$

In the special case where $\mathcal{X} = [X/G]$, this gives

$$\cdots \to \pi_{n+1}\mathcal{X} \to \pi_n G \to \pi_n X \to \pi_n\mathcal{X} \to \pi_{n-1}G \to \cdots .$$

The map $\pi_n G \to \pi_n X$ is induced by the inclusion of the orbit $G\cdot x \hookrightarrow X$.

Example 5.7. Let $p\colon \mathcal{X} \to \mathcal{Y}$ be a covering map of topological stacks. Then, since the fiber F of p is discrete, the long exact sequence of homotopy groups implies that $\pi_n\mathcal{X} \to \pi_n\mathcal{Y}$ is an isomorphism, for all $n \geq 2$. Furthermore, $\pi_1\mathcal{X} \to \pi_1\mathcal{Y}$ is injective and its cokernel is in a natural bijection with F (upon fixing base points).

Example 5.8. Let $G \to Y$ be a locally trivial bundle of groups over a topological space Y. Let $\mathcal{X}$ be a G-gerbe over Y. Then, for a point $y \in Y$, we have a fiber sequence

$$[*/H] \to \mathcal{X} \to Y,$$

where we have denoted the fiber G_y of $G \to Y$ by H. This gives rise to the long exact sequence

$$\cdots \to \pi_{n+1}Y \to \pi_{n-1}H \to \pi_n\mathcal{X} \to \pi_nY \to \pi_{n-2}H \to \pi_{n-1}\mathcal{X} \to \cdots .$$

Here, we have used the fact that $\pi_n[*/H] \cong \pi_{n-1}H$ for $n \geq 1$ and $\pi_0[*/H] = \{*\}$ (see Example 5.6).

5.4. **van Kampen's theorem for topological stacks.** Using the results of this section we find an easy proof of van Kampen's theorem for topological stacks. We first prove the groupoid version of the theorem, and then deduce the classical version for fundamental groups.

Definition 5.9. Let $\mathcal{X}$ be a topological stack. The fundamental groupoid $\Pi\mathcal{X}$ of $\mathcal{X}$ is defined in the usual way. Namely, the objects of $\Pi\mathcal{X}$ are points $x\colon * \to \mathcal{X}$. An arrow in $\Pi\mathcal{X}$ from x to x' is a path from x to x', up to a homotopy relative to the end points.

It follows from Lemma 2.6 that $\Pi\mathcal{X}$ is a groupoid. The fundamental groupoid $\Pi\mathcal{X}$ is functorial in $\mathcal{X}$. If $\varphi\colon X \to \mathcal{X}$ is a classifying space for $\mathcal{X}$, then, by Corollary 5.3, the induced map $\Pi\varphi\colon \Pi X \to \Pi\mathcal{X}$ is an equivalence of groupoids.

Theorem 5.10. *Let $\mathcal{X}$ be a topological stack and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ open substacks such that $\mathcal{U} \cup \mathcal{V} = \mathcal{X}$. Then, the diagram*

$$\begin{array}{ccc} \Pi(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & \Pi\mathcal{U} \\ \downarrow & & \downarrow \\ \Pi\mathcal{V} & \longrightarrow & \Pi\mathcal{X} \end{array}$$

is a pushout square in the category of groupoids

Proof. Choose a classifying space $\varphi\colon X \to \mathcal{X}$ for $\mathcal{X}$ and pullback every thing along φ. We obtain open subsets $U, V \subseteq X$ such that $U \cup V = X$. Since classifying spaces commute with base change, U, V and $U \cap V$ are classifying spaces for $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$, respectively. Since a classifying space induces an equivalence between fundamental groupoids, the claim follows from the space version of the van Kampen theorem for the covering $U \cup V = X$. ☐

Corollary 5.11. *Let $\mathcal{X}$ be a topological stack and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ open substacks such that $\mathcal{U} \cup \mathcal{V} = \mathcal{X}$. Suppose that $\mathcal{U} \cap \mathcal{V}$ is path connected, and pick a base point x in it. Then, we have a natural isomorphism*

$$\pi_1(\mathcal{U}, x) \underset{\pi_1(\mathcal{U}\cap\mathcal{V},x)}{*} \pi_1(\mathcal{V}, x) \cong \pi_1(\mathcal{X}, x).$$

Proof. As in the case of topological spaces, the result follows easily from Theorem 5.10. □

5.5. **Homotopy groups of the coarse moduli space.** Any topological stack $\mathcal{X}$ has a coarse moduli space $\mathcal{X}_{mod}$ which is an honest topological space, and there is a natural map $f\colon \mathcal{X} \to \mathcal{X}_{mod}$ (see [No1], §4.3). For example, when $\mathcal{X}$ is the quotient stack $[X/R]$ of a topological groupoid $[R \rightrightarrows X]$, then $\mathcal{X}_{mod}$ is simply the coarse quotient X/R. When $\mathcal{X} = [X/G]$, this is just X/G.

We have induced maps

$$\pi_n(\mathcal{X}, x) \to \pi_n(\mathcal{X}_{mod}, x)$$

on homotopies. For $n = 0$ this is a bijection. For $n \geq 1$, these homomorphisms are in general far from being isomorphisms. Except in the case where $\mathcal{X}$ is a G-gerbe over $\mathcal{X}_{mod}$ (as in §4.4), the relation between $\pi_n(\mathcal{X}, x)$ and $\pi_n(\mathcal{X}_{mod}, x)$ is unclear. In the case $n = 1$, however, the map $f_*\colon \pi_1(\mathcal{X}, x) \to \pi_1(\mathcal{X}_{mod}, x)$ is quite interesting and there is an explicit description for it. More precisely, under a certain mild condition on $\mathcal{X}$, $\pi_1(\mathcal{X}_{mod}, x)$ is obtained from $\pi_1(\mathcal{X}, x)$ by killing the images of all inertia groups of $\mathcal{X}$. Combined with the computations of §5.3, this leads to interesting formulas for the fundamental groups of coarse quotients of topological groupoids. For more on this, we refer the reader to [No3].

6. Homotopy fiber of a morphism of topological stacks

In this section, we prove that every morphism of stacks has a natural fibrant replacement (Theorem 6.1). We then use this to define the homotopy fiber of a morphism of (topological) stacks (Definition 6.5).

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. Set $\tilde{\mathcal{X}} := \mathcal{X} \times_{f,\mathcal{Y},\mathrm{ev}_0} P\mathcal{Y}$, where $P\mathcal{Y} = \mathrm{Map}([0,1], \mathcal{Y})$ is the path stack of $\mathcal{X}$, and $\mathrm{ev}_0\colon P\mathcal{Y} \to \mathcal{Y}$ is the time $t = 0$ evaluation map ([No4], §5.2). Note that if $\mathcal{X}$ and $\mathcal{Y}$ are topological stacks, then so is $\tilde{\mathcal{X}}$. We define $p_f\colon \tilde{\mathcal{X}} \to \mathcal{Y}$ to be the composition $\mathrm{ev}_1 \circ \mathrm{pr}_2$, and $i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$ to be the map whose first and second components are $\mathrm{id}_{\mathcal{X}}$ and $c_{\mathcal{Y}} \circ f$, respectively. Here, $c_{\mathcal{Y}}\colon \mathcal{Y} \to P\mathcal{Y}$ is the map parametrizing the constant paths (i.e., $c_{\mathcal{Y}}$ is the map induced from $[0,1] \to *$ by the functoriality of the mapping stack).

Theorem 6.1. *Notation being as above, we have a factorization $f = p_f \circ i_f$,*

$$\mathcal{X} \underset{r_f}{\overset{i_f}{\rightleftarrows}} \tilde{\mathcal{X}} \xrightarrow{p_f} \mathcal{Y},$$

such that:

i) *the map $p_f\colon \tilde{\mathcal{X}} \to \mathcal{Y}$ is a Hurewicz fibration;*
ii) *the map $r_f\colon \tilde{\mathcal{X}} \to \mathcal{X}$ is shrinkable (Definition 2.9) onto the section $i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$. Here, $r_f\colon \mathcal{X} \times_{f,\mathcal{Y},\mathrm{ev}_0} P\mathcal{Y} \to \mathcal{X}$ is the first projection map.*

Proof. To prove (i), note that $\tilde{\mathcal{X}}$ sits in the following 2-cartesian diagram:

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \longrightarrow & P\mathcal{Y} \\ {\scriptstyle (r_f, p_f)}\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow[f \times \mathrm{id}_Y]{} & \mathcal{Y} \times \mathcal{Y} \end{array}$$

We saw in §4.1 that $(\mathrm{ev}_0, \mathrm{ev}_1) \colon P\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is a Hurewicz fibration. Hence, $(r_f, p_f) \colon \tilde{\mathcal{X}} \to \mathcal{X} \times \mathcal{Y}$ is a Hurewicz fibration. It follows from Lemma 4.1 that $p_f \colon \tilde{\mathcal{X}} \to \mathcal{Y}$ is also a Hurewicz fibration.

Let us now prove (ii). By Lemma 2.14, $\mathrm{ev}_0 \colon P\mathcal{Y} \to \mathcal{Y}$ is shrinkable onto $c_{\mathcal{Y}} \colon \mathcal{Y} \to P\mathcal{Y}$. It follows from the 2-cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \longrightarrow & P\mathcal{Y} \\ {\scriptstyle i_f}\uparrow\downarrow{\scriptstyle r_f} & & {\scriptstyle c_{\mathcal{Y}}}\uparrow\downarrow{\scriptstyle ev_0} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array}$$

and Lemma 2.11 that $r_f \colon \tilde{\mathcal{X}} \to \mathcal{X}$ is shrinkable onto $i_f \colon \mathcal{X} \to \tilde{\mathcal{X}}$. □

Contrary to the classical case, the map $i_f \colon \mathcal{X} \to \tilde{\mathcal{X}}$ is not necessarily an embedding. This is because $c_{\mathcal{Y}} \colon \mathcal{Y} \to P\mathcal{Y}$ is not necessarily an embedding. Proposition 6.3 explains what goes wrong.

Lemma 6.2. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ and $g \colon \mathcal{Y} \to \mathcal{Z}$ be morphisms of stacks. Suppose that $\mathcal{Y}$ has representable diagonal. If $g \circ f$ is representable, then so is f.*

Proof. First consider the 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{(\mathrm{id}_{\mathcal{X}}, f)} & \mathcal{X} \times \mathcal{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f, \mathrm{id}_{\mathcal{Y}})} \\ \mathcal{Y} & \xrightarrow[\Delta]{} & \mathcal{Y} \times \mathcal{Y} \end{array}$$

Since Δ is representable, it follows that $(\mathrm{id}_{\mathcal{X}}, f)$ is also representable. Now consider the 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{\mathrm{pr}_2} & \mathcal{Y} \\ {\scriptstyle (\mathrm{id}_{\mathcal{X}}, f)}\uparrow\downarrow & \nearrow{\scriptstyle f} & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow[g \circ f]{} & \mathcal{Z} \end{array}$$

Since f is the composition of two representable morphisms $(\mathrm{id}_{\mathcal{X}}, f)$ and pr_2, it is representable itself. □

Proposition 6.3. *Let X be a topological space and $\mathcal{Y}$ a stack with representable diagonal (e.g., a topological stack). Let $c_{\mathcal{Y}} \colon \mathcal{Y} \to \mathrm{Map}(X, \mathcal{Y})$ be the map parametrizing the constant maps. Then, $c_{\mathcal{Y}}$ is representable. If X is connected, then, for every point y in $\mathcal{Y}$, the fiber of $c_{\mathcal{Y}}$ over the point in $\mathrm{Map}(X, \mathcal{Y})$ corresponding to the constant map at y is homeomorphic to the space $\mathrm{Map}_*(X, I_y)$ of pointed maps from X to the inertia group I_y (where we have fixed a base point in X). (Note that, since $\mathcal{Y}$ has representable diagonal, the inertia stack is representable over $\mathcal{Y}$, so the groups I_y are naturally topological groups.)*

Proof. By ([No4], Lemma 4.1), $\mathrm{Map}(X, \mathcal{Y})$ has representable diagonal. Since $c_{\mathcal{Y}}$ has a left inverse, it follows from Lemma 6.2 that $c_{\mathcal{Y}}$ is representable.

By simply writing down the definition of the fiber of a moprhism over a point, it follows that the fiber of $c_{\mathcal{Y}}$ over the point in $\mathrm{Map}(X, \mathcal{Y})$ corresponding to the constant map y is equal to the quotient of $\mathrm{Map}(X, I_y)$ by the subgroup of constant maps $X \to I_y$. This quotient is homeomorphic to $\mathrm{Map}_*(X, I_y)$. □

Corollary 6.4. *The map* $i_f\colon \mathcal{X} \to \tilde{\mathcal{X}}$ *of Theorem 6.1 is representable.*

Proof. Follows immediately from Proposition 6.3. □

Definition 6.5. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Z} \to \mathcal{Y}$ be morphisms of stacks. We define the **homotopy fiber product** of $\mathcal{X}$ and $\mathcal{Z}$ over $\mathcal{Y}$ to be

$$\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z} := \tilde{\mathcal{X}} \times_{p_f, \mathcal{Y}, g} \mathcal{Z} = \mathcal{X} \times_{f, \mathcal{Y}, \mathrm{ev}_0} P\mathcal{Y} \times_{\mathrm{ev}_1, \mathcal{Y}, g} \mathcal{Z},$$

where p_f is as in Theorem 6.1. For a point $y\colon * \to \mathcal{Y}$ in $\mathcal{Y}$, we define the **homotopy fiber** of f over y to be

$$\mathrm{hFib}_y(f) := \tilde{\mathcal{X}} \times_{p_f, \mathcal{Y}, y} *.$$

Since the 2-category of topological stacks is closed under fiber products, the homotopy fiber product of topological stacks (and, in particular, the homotopy fiber of a morphism of topological stacks) is again a topological stack. There is a natural map

$$j_y\colon \mathrm{Fib}_y(f) \to \mathrm{hFib}_y(f),$$

where $\mathrm{Fib}_y(f) := \mathcal{X} \times_{f, \mathcal{Y}, y} *$ is the fiber of f over y. It follows from Theorem 5.2 and the observation in the next paragraph that if f is a weak Serre fibration, then j_y is a weak equivalence.

Since p_f is a Hurewicz (hence Serre) fibration, Theorem 5.2 gives a long exact sequence

$$\cdots \to \pi_{n+1}\mathcal{X} \to \pi_{n+1}\mathcal{Y} \to \pi_n\, \mathrm{hFib}_y(f) \to \pi_n\mathcal{X} \to \pi_n\mathcal{Y} \to \pi_{n-1}\, \mathrm{hFib}_y(f) \to \cdots$$

of homotopy groups. Here, we have fixed a base point $x\colon * \to \mathcal{X}$ in $\mathcal{X}$, and set $y = f(x)$. (Note that x gives rise to a base point in $\mathrm{Fib}_y(f)$, and also to one in $\mathrm{hFib}_y(f)$ through the map j_y.)

The construction of the fibration replacement (Theorem 6.1) is functorial in the map f, in the sense that, given a 2-commutative square

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

we get an induced map on the corresponding replacements, 2-commuting with all the relevant data. It follows that the homotopy fiber $\mathrm{hFib}_y(f)$ is also functorial (upon fixing a base point y' in $\mathcal{Y}'$, and letting y be its image in $\mathcal{Y}$). In particular, the resulting long exact sequences for f and f' are also functorial.

The following lemma follows immediately from the existence of the long exact sequence for the homotopy groups of a fibration.

Lemma 6.6. *In the 2-commutative square above, if the vertical maps are weak equivalences, then so is the induced map* $\mathrm{hFib}_y(f') \to \mathrm{hFib}_y(f)$ *on homotopy fibers.*

More generally, the homotopy fiber product $\mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$ is functorial in the diagram

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

in the sense that if we have a morphism of diagrams given by maps $u\colon \mathcal{X}' \to \mathcal{X}$, $v\colon \mathcal{Y}' \to \mathcal{Y}$ and $w\colon \mathcal{Z}' \to \mathcal{Z}$, then we have an induced map

$$\mathcal{X}' \times^h_{\mathcal{Y}'} \mathcal{Z}' \to \mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$$

on the homotopy fiber products.

Lemma 6.7. *In the above situation, if u, v and w are weak equivalences, then so is the induced map $\mathcal{X}' \times^h_{\mathcal{Y}'} \mathcal{Z}' \to \mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z}$.*

Proof. Fix a point y in $\mathcal{Y}$. Let $q\colon \mathcal{X} \times^h_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ be $g \circ \mathrm{pr}_2$. Using a standard argument involving composing paths (for which we will need Lemma 2.6) we can construct a pair of inverse homotopy equivalences between $\mathrm{hFib}_y(f) \times \mathrm{hFib}_y(g)$ and $\mathrm{hFib}_y(q)$. The claim now follows from Lemma 6.6 and the long exact sequence for q. □

7. Leray-Serre spectral sequence

In this section we use the results of the previous sections to construct the Leray-Serre spectral sequence for a fibration of topological stacks. As we will see shortly, there is nothing deep about the construction of the stack version of the Leray-Serre spectral sequence, as the real work has been done in the construction of the space version. All we do is to reduce the problem to the case of spaces by making careful use of classifying spaces for stacks. The same method can be used to construct stack versions of other variants of the Leray-Serre spectral sequence as well.

7.1. Local coefficients on topological stacks.

Definition 7.1. By a **system of local coefficients** on a topological stack $\mathcal{X}$ we mean a presheaf $\mathcal{G}$ of groups on the fundamental groupoid $\Pi\mathcal{X}$ (Definition 5.9). In other words, a system of coefficients is a rule which assigns to a point x in $\mathcal{X}$ a group $\mathcal{G}_x$, and to a homotopy class γ of paths from x to x' a homomorphism $\gamma^*\colon \mathcal{G}_{x'} \to \mathcal{G}_x$. We require that under this assignment composition of paths goes to composition of homomorphisms.

If $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of topological stacks and $\mathcal{G}$ a local system of coefficients on $\mathcal{X}$, we obtain a local system of coefficients $f^*\mathcal{G}$ on $\mathcal{Y}$ via the map $\Pi f\colon \Pi\mathcal{Y} \to \Pi\mathcal{X}$. If $\varphi\colon X \to \mathcal{X}$ is a classifying space for $\mathcal{X}$, this pullback construction induces a bijection (more precisely, an equivalence of categories) between local systems on $\mathcal{X}$ and those on X.

Given a local system $\mathcal{A}$ of abelian groups on a topological stack $\mathcal{X}$, we can define *homology $H_*(\mathcal{X}, \mathcal{A})$ with coefficients in $\mathcal{A}$* exactly as in ([No2], §11). Namely, we choose a classifying space $\varphi\colon X \to \mathcal{X}$ and set

$$H_*(\mathcal{X}, \mathcal{A}) := H_*(X, \varphi^*\mathcal{A}).$$

Analogously, we can define *cohomology $H^*(\mathcal{X}, \mathcal{A})$ with coefficients in $\mathcal{A}$.*

Some important examples of local systems of coefficients are discussed in the following.

Example 7.2 (Trivial local system). For every topological stack $\mathcal{X}$ and every abelian group A, the rule

$$x \mapsto A,$$

$$\gamma \mapsto \mathrm{id}_A$$

defines a *trivial* local system on $\mathcal{X}$. This is simply the constant presheaf on $\Pi\mathcal{X}$ with value A.

Example 7.3 (Cohomologies of a fibration). Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. To each point y in $\mathcal{Y}$, associate its homotopy fiber $\mathrm{hFib}_y(f)$. For simplicity of notation, we denote $\mathrm{hFib}_y(f)$ by hFib_y. Given a path γ from y_0 to y_1, we obtain a map $e_\gamma\colon \mathrm{hFib}_{y_0} \to \mathrm{hFib}_{y_1}$ defined by composing the path that appears in the definition of hFib_{y_0} with γ (Lemma 2.6). A standard argument shows that, given two composable paths γ and γ', $e_{\gamma'} \circ e_\gamma$ is homotopic to $e_{\gamma\gamma'}$. (So, in particular, e_γ is a homotopy equivalence.) This implies that the rule

$$y \mapsto H^*(\mathrm{hFib}_y, A),$$

$$\gamma \mapsto e_\gamma^*$$

is a local system of coefficients on $\mathcal{Y}$. Here, A is an abelian group and

$$e_\gamma^*\colon H^*(\mathrm{hFib}_{y_1}, A) \to H^*(\mathrm{hFib}_{y_0}, A)$$

is the induced map on cohomology. We denote this local system by $\mathcal{H}^*(\mathcal{F}, A)$, where $\mathcal{F}$ stands for a homotopy fiber of f (note the slight abuse of notation).

In the case where $A = R$ is a ring, each $H^*(\mathrm{hFib}_y, R)$ is also a ring, and the above local system is a local system of rings.

Example 7.4 (Homologies of a fibration). As we saw in Example 7.3, the map $e_\gamma\colon \mathrm{hFib}_{y_0} \to \mathrm{hFib}_{y_1}$ is a homotopy equivalence. Therefore, by inverting the direction of arrows, we find a similar local system for homology with coefficients in A. Namely,

$$y \mapsto H_*(\mathrm{hFib}_y, A),$$

$$\gamma \mapsto (e_\gamma)_*^{-1}.$$

We denote this local system by $\mathcal{H}_*(\mathcal{F}, A)$, where $\mathcal{F}$ stands for a homotopy fiber of f.

For a fixed abelian group A, it follows from the functorial properties of the homotopy fiber (see after Definition 6.5) that the local systems $\mathcal{H}^*(\mathcal{F}, A)$ and $\mathcal{H}_*(\mathcal{F}, A)$ constructed in Examples 7.3 and 7.4 are functorial with respect to 2-commutative diagrams

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{u} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow[v]{} & \mathcal{Y} \end{array}$$

Let us spell out what this means for the cohomological version. Let us call the local system associated to f by H_f^*. Then, the functoriality of H_f^* with respect to the 2-commutative square above means that we have a Πv-equivariant morphism of presheaves $\rho\colon H_f^* \to H_{f'}^*$, as in the diagram

$$\begin{array}{ccc} H_{f'}^* & \xleftarrow{\rho} & H_f^* \\ \Big| & & \Big| \\ \Pi\mathcal{Y}' & \xrightarrow[\Pi v]{} & \Pi\mathcal{Y} \end{array}$$

(More precisely, ρ is a morphism of presheaves on $\Pi\mathcal{Y}'$ from the pullback of H_f^* along Πv to $H_{f'}^*$.) It follows from Lemma 6.6 that if v and u are weak equivalences, then the above diagram is an equivalence. (More precisely, the pullback of H_f^* along Πv to $\Pi_{\mathcal{Y}}$ is equivalent to $H_{f'}^*$ via ρ.)

7.2. **Construction of the Leray-Serre spectral sequence.** We now prove the existence of the Leray-Serre spectral sequence. We begin with the following.

Lemma 7.5. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks. Then, there exists a 2-commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\psi]{} & \mathcal{Y} \end{array}$$

such that X and Y are topological spaces and φ and ψ are (universal) weak equivalences. The similar statement is true when instead of a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ we start with a diagram

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{Y} \end{array}$$

Proof. We only prove the morphism case. The proof of the diagram case is similar. Choose a classifying space $\psi\colon Y \to \mathcal{Y}$, and set $\mathcal{X}_0 = Y \times_{\mathcal{Y}} \mathcal{X}$. Take a classifying space $\varphi_0\colon X \to \mathcal{X}_0$ for $\mathcal{X}_0$, and set $\varphi := \mathrm{pr}_2 \circ \varphi_0$ and $g := \mathrm{pr}_1 \circ \varphi_0$. $\square$

We now formulate the Leray-Serre spectral sequence as in [McCl].

Theorem 7.6 (homology Leray-Serre spectral sequence). *Let A be an abelian group. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks with $\mathcal{Y}$ path-connected and $\mathcal{F} := \mathrm{hFib}(f)$ connected. Then, there is a first quadrant spectral sequence $\{E_{*,*}^r, d^r\}$ converging to $H_*(\mathcal{X}, A)$, with*

$$E_{p,q}^2 \cong H_p\big(\mathcal{Y}, \mathcal{H}_q(\mathcal{F}, A)\big),$$

the homology of $\mathcal{Y}$ with local coefficients in the homology of the homotopy fiber $\mathcal{F}$ of f (see Example 7.4). This spectral sequence is natural with respect to 2-commutative *squares*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathcal{Y} & \longrightarrow & \mathcal{Y}' \end{array}$$

If f is a weak Serre fibration with fiber $\mathcal{F}'$, then

$$E^2_{p,q} \cong H_p(\mathcal{Y}, \mathcal{H}_q(\mathcal{F}', A)).$$

Proof. Use Lemma 7.5 to replace $f\colon \mathcal{X} \to \mathcal{Y}$ by a continuous map $g\colon X \to Y$ of topological spaces. Thanks to Lemma 6.6, the Leray-Serre spectral sequence for g (see [McCl]) gives rise to the desired spectral sequence for f. The functoriality (and the fact that the resulting spectral sequence is independent of the choice of g) follows from a similar reasoning as in ([No2], §11).

The last part of the theorem follows from the fact that, when f is a weak Serre fibration, the natural map $j_y\colon \mathcal{F}' \to \mathrm{hFib}_y(f)$ is a weak equivalence (use Theorem 5.2). □

Theorem 7.7 (cohomology Leray-Serre spectral sequence). *Let R be a commutative ring with unit. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks with $\mathcal{Y}$ path-connected and $\mathcal{F} := \mathrm{hFib}(f)$ connected. Then, there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$ converging to $H^*(\mathcal{X}, R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p\big(\mathcal{Y}, \mathcal{H}^q(\mathcal{F}, R)\big),$$

the cohomology of $\mathcal{Y}$ with local coefficients in the cohomology of the homotopy fiber $\mathcal{F}$ of f (see Example 7.3). This spectral sequence is natural with respect to 2-commutative squares

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

Furthermore, the cup product $\smile$ on cohomology with local coefficients and the product $\cdot_2$ on $E_2^{*,*}$ are related by $u \cdot_2 v = (-1)^{p'q} u \smile v$, when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$. If f is a weak Serre fibration with fiber $\mathcal{F}'$, then

$$E_2^{p,q} \cong H^p(\mathcal{Y}, \mathcal{H}^q(\mathcal{F}', R)).$$

Proof. The same proof as Theorem 7.6. □

8. Eilenberg-Moore spectral sequence

In this section we present the Eilenberg-Moore spectral sequence for fibrations of topological stacks. We only state the cohomology version and leave the homology version to the reader.

Theorem 8.1. *Let k be a commutative ring. Let $\mathcal{Y}$ be a simply-connected topological stack, and consider the diagram*

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow[g]{} & \mathcal{Y} \end{array}$$

of topological stacks. Suppose that the cohomologies of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are finitely generated over k in each dimension. Then, there is a second quadrant spectral sequence with

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(\mathcal{Y},k)}\left(H^*(\mathcal{X},k), H^*(\mathcal{Z},k)\right)$$

converging strongly to $H^*(\mathcal{P},k)$, where $\mathcal{P} := \mathcal{X} \times_{\mathcal{Y}}^h \mathcal{Z}$ is the homotopy fiber product (Definition 6.5). This spectral sequence is natural with respect to morphisms of diagrams.

Proof. The proof is similar to the proof of Theorem 7.6. First we use Lemma 7.5 to replace the given diagram by a similar diagram of classifying spaces. Then, by Lemma 6.7, the Eilenberg-Moore spectral sequence for the corresponding diagram of spaces (see [McCl][2]) gives rise to the Eilenberg-Moore spectral sequence for the original diagram. The functoriality (and the proof that the resulting spectral sequence is independent of the choice of the diagram of classifying spaces) follows from a similar reasoning as in ([No2], §11). □

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[2]Beware that [ibid.] leaves out the seemingly necessary condition on finite generation of the cohomologies. You can also consult Hatcher's ongoing book project which is available online.